After completing this chapter you should be able to

1 simplify algebraic fractions by dividing

2 divide a polynomial $f(x)$ by $(x \pm p)$

3 factorise a polynomial by using the factor theorem

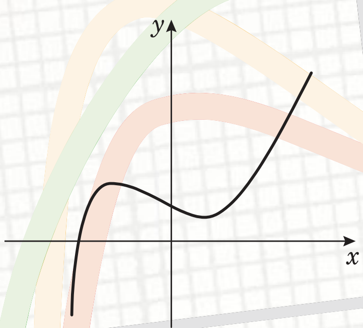
4 use the remainder theorem to find the remainder when a polynomial $f(x)$ is divided by $(ax - b)$.

You will use the above techniques to help you sketch the graphs of polynomial functions. Once you have factorised a polynomial you can find its roots and therefore where it crosses the x-axis.

1 Algebra and functions

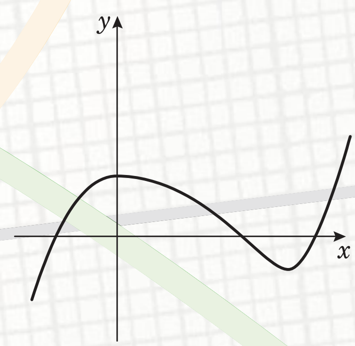
Did you know?

...that the graph of a cubic function must cross the x-axis *once* and therefore must have at least *one* root and factor



A cubic graph showing *one* root

A cubic graph showing *two* roots



A cubic graph showing *three* roots

Can you explain why a quartic graph could have 0, 1, 2, 3 *or* 4 roots?

1.1 You can simplify algebraic fractions by division.

Example 1

Simplify these fractions:

a $\dfrac{7x^4 - 2x^3 + 6x}{x}$

b $\dfrac{5x^2 - 6}{2x}$

c $\dfrac{3x^5 - 4x^2}{-3x}$

a $\dfrac{7x^4 - 2x^3 + 6x}{x}$

$= \dfrac{7x^4}{x} - \dfrac{2x^3}{x} + \dfrac{6x}{x}$

Divide each term on top of the fraction by x.

Simplify the individual fractions when possible, so that:

① $\dfrac{7x^4}{x} = 7 \times \dfrac{x^4}{x}$

$= 7 \times x^{4-1}$

$= 7x^3$

② $\dfrac{2x^3}{x} = 2 \times \dfrac{x^3}{x}$

$= 2 \times x^{3-1}$

$= 2x^2$

③ $\dfrac{6x}{x} = 6 \times \dfrac{x}{x}$

$= 6 \times x^{1-1}$

$= 6x^0$

$= 6$

Remember $x^0 = 1$.

So $\dfrac{7x^4 - 2x^3 + 6x}{x} = 7x^3 - 2x^2 + 6.$

b $\frac{5x^2 - 6}{2x}$

$= \frac{5x^2}{2x} - \frac{6}{2x}$

Divide each term on top of the fraction by $2x$.

Simplify the fractions, so that:

① $\frac{5x^2}{2x} = \frac{5}{2} \times \frac{x^2}{x}$

$= \frac{5}{2} \times x^{2-1}$

$= \frac{5}{2}x$

Remember $x^1 = x$ and $\frac{5}{2}x = \frac{5x}{2}$.

② $-\frac{6}{2x} = -\frac{3}{x}$

So $\frac{5x^2 - 6}{2x} = \frac{5x}{2} - \frac{3}{x}$.

c $\frac{3x^5 - 4x^2}{-3x}$

$= \frac{3x^5}{-3x} - \frac{4x^2}{-3x}$

Divide each term on top of the fraction by $-3x$.

Simplify the fractions, so that:

① $\frac{3x^5}{-3x} = \frac{3}{-3} \times \frac{x^5}{x}$

$= -1 \times x^{5-1}$

$= -x^4$

② $\frac{-4x^2}{-3x} = \frac{-4}{-3} \times \frac{x^2}{x}$

$= \frac{4}{3} \times x^{2-1}$

Minus divided by minus equals plus.

$= \frac{4}{3}x$

So $\frac{3x^5 - 4x^2}{-3x} = -x^4 + \frac{4x}{3}$.

$\frac{4}{3}x = \frac{4x}{3}$

Example 2

Simplify these fractions by factorising:

a $\dfrac{(x+7)(2x-1)}{(2x-1)}$ **b** $\dfrac{x^2+7x+12}{(x+3)}$ **c** $\dfrac{x^2+6x+5}{x^2+3x-10}$ **d** $\dfrac{2x^2+11x+12}{(x+3)(x+4)}$

a $\dfrac{(x+7)(2x-1)}{(2x-1)}$

$= x + 7$

Simplify by dividing the top and the bottom of the fraction by $(2x - 1)$.

b $\dfrac{x^2+7x+12}{(x+3)}$

$= \dfrac{(x+3)(x+4)}{(x+3)}$

$= x + 4$

Factorise $x^2 + 7x + 12$:
$(+3) \times (+4) = +12$
$(+3) + (+4) = +7$
So $x^2 + 7x + 12 = (x + 3)(x + 4)$.

Divide top and bottom by $(x + 3)$.

c $\dfrac{x^2+6x+5}{x^2+3x-10}$

$= \dfrac{(x+5)(x+1)}{(x+5)(x-2)}$

$= \dfrac{x+1}{x-2}$

Factorise $x^2 + 6x + 5$:
$(+5) \times (+1) = +5$
$(+5) + (+1) = +6$
So $x^2 + 6x + 5 = (x + 5)(x + 1)$.

Factorise $x^2 + 3x - 10$:
$(+5) \times (-2) = -10$
$(+5) + (-2) = +3$
So $x^2 + 3x - 10 = (x + 5)(x - 2)$.

Divide top and bottom by $(x + 5)$.

d Factorise $2x^2 + 11x + 12$:

$2 \times +12 = +24$

and $(+3) \times (+8) = +24$

$(+3) + (+8) = +11$

So $2x^2 + 11x + 12 = 2x^2 + 3x + 8x + 12$

$= x(2x + 3) + 4(2x + 3)$

$= (2x + 3)(x + 4)$.

So $\dfrac{2x^2+11x+12}{(x+3)(x+4)}$

$= \dfrac{(2x+3)(x+4)}{(x+3)(x+4)}$

$= \dfrac{2x+3}{x+3}$

Divide top and bottom by $(x + 4)$.

Exercise 1A

1 Simplify these fractions:

a $\dfrac{4x^4 + 5x^2 - 7x}{x}$

b $\dfrac{7x^8 - 5x^5 + 9x^3 + x^2}{x}$

c $\dfrac{-2x^3 + x}{x}$

d $\dfrac{-x^4 + 4x^2 + 6}{x}$

e $\dfrac{7x^5 - x^3 - 4}{x}$

f $\dfrac{8x^4 - 4x^3 + 6x}{2x}$

g $\dfrac{9x^2 - 12x^3 - 3x}{3x}$

h $\dfrac{8x^5 - 2x^3}{4x}$

i $\dfrac{7x^3 - x^4 - 2}{5x}$

j $\dfrac{-4x^2 + 6x^4 - 2x}{-2x}$

k $\dfrac{-x^8 + 9x^4 + 6}{-2x}$

l $\dfrac{-9x^9 - 6x^4 - 2}{-3x}$

2 Simplify these fractions as far as possible:

a $\dfrac{(x + 3)(x - 2)}{(x - 2)}$

b $\dfrac{(x + 4)(3x - 1)}{(3x - 1)}$

c $\dfrac{(x + 3)^2}{(x + 3)}$

d $\dfrac{x^2 + 10x + 21}{(x + 3)}$

e $\dfrac{x^2 + 9x + 20}{(x + 4)}$

f $\dfrac{x^2 + x - 12}{(x - 3)}$

g $\dfrac{x^2 + x - 20}{x^2 + 2x - 15}$

h $\dfrac{x^2 + 3x + 2}{x^2 + 5x + 4}$

i $\dfrac{x^2 + x - 12}{x^2 - 9x + 18}$

j $\dfrac{2x^2 + 7x + 6}{(x - 5)(x + 2)}$

k $\dfrac{2x^2 + 9x - 18}{(x + 6)(x + 1)}$

l $\dfrac{3x^2 - 7x + 2}{(3x - 1)(x + 2)}$

m $\dfrac{2x^2 + 3x + 1}{x^2 - x - 2}$

n $\dfrac{x^2 + 6x + 8}{3x^2 + 7x + 2}$

o $\dfrac{2x^2 - 5x - 3}{2x^2 - 9x + 9}$

1.2 You can divide a polynomial by $(x \pm p)$.

Example 3

Divide $x^3 + 2x^2 - 17x + 6$ by $(x - 3)$.

①

$$\begin{array}{r} x^2 \phantom{{}+2x^2-17x+6} \\ x-3\,\overline{)\,x^3+2x^2-17x+6} \\ \underline{x^3-3x^2} \phantom{{}-17x+6} \\ 5x^2-17x \phantom{{}+6} \end{array}$$

Start by dividing the first term of the polynomial by x, so that $x^3 \div x = x^2$.

Next multiply $(x - 3)$ by x^2, so that $x^2 \times (x - 3) = x^3 - 3x^2$.

Now subtract, so that $(x^3 + 2x^2) - (x^3 - 3x^2) = 5x^2$.

Finally copy $-17x$.

②

$$\begin{array}{r} x^2+5x \phantom{{}-17x+6} \\ x-3\,\overline{)\,x^3+2x^2-17x+6} \\ \underline{x^3-3x^2} \phantom{{}-17x+6} \\ 5x^2-17x \phantom{{}+6} \\ \underline{5x^2-15x} \phantom{{}+6} \\ -2x+6 \end{array}$$

Repeat the method. Divide $5x^2$ by x, so that $5x^2 \div x = 5x$.

Multiply $(x - 3)$ by $5x$, so that $5x \times (x - 3) = 5x^2 - 15x$.

Subtract, so that $(5x^2 - 17x) - (5x^2 - 15x) = -2x$.

Copy 6.

③

$$\begin{array}{r} x^2+5x-2 \phantom{{}+6} \\ x-3\,\overline{)\,x^3+2x^2-17x+6} \\ \underline{x^3-3x^2} \phantom{{}-17x+6} \\ 5x^2-17x \phantom{{}+6} \\ \underline{5x^2-15x} \phantom{{}+6} \\ -2x+6 \\ \underline{-2x+6} \\ 0 \end{array}$$

Repeat the method. Divide $-2x$ by x, so that $-2x \div x = -2$.

Multiply $(x - 3)$ by -2, so that $-2 \times (x - 3) = -2x + 6$.

Subtract, so that $(-2x + 6) - (-2x + 6) = 0$.

No numbers left to copy, so you have finished.

So $x^3 + 2x^2 - 17x + 6 \div (x - 3) =$
$x^2 + 5x - 2.$

This is called the quotient.

Example 4

Divide $6x^3 + 28x^2 - 7x + 15$ by $(x + 5)$.

①

$$\begin{array}{r} 6x^2 \phantom{{}-7x+15} \\ x+5\overline{)\,6x^3 + 28x^2 - 7x + 15} \\ \underline{6x^3 + 30x^2} \phantom{{}-7x+15} \\ -2x^2 - 7x \phantom{{}+15} \end{array}$$

Start by dividing the first term of the polynomial by x, so that $6x^3 \div x = 6x^2$.

Next multiply $(x + 5)$ by $6x^2$, so that $6x^2 \times (x + 5) = 6x^3 + 30x^2$.

Now subtract, so that $(6x^3 + 28x^2) - (6x^3 + 30x^2) = -2x^2$.

Finally copy $-7x$.

②

$$\begin{array}{r} 6x^2 - 2x \phantom{{}+15} \\ x+5\overline{)\,6x^3 + 28x^2 - 7x + 15} \\ \underline{6x^3 + 30x^2} \phantom{{}-7x+15} \\ -2x^2 - 7x \phantom{{}+15} \\ \underline{-2x^2 - 10x} \phantom{{}+15} \\ 3x + 15 \end{array}$$

Repeat the method. Divide $-2x^2$ by x, so that $-2x^2 \div x = -2x$.

Multiply $(x + 5)$ by $-2x$, so that $-2x \times (x + 5) = -2x^2 - 10x$.

Subtract, so that $(-2x^2 - 7x) - (-2x^2 - 10x) = 3x$.

Copy 15.

③

$$\begin{array}{r} 6x^2 - 2x + 3 \\ x+5\overline{)\,6x^3 + 28x^2 - 7x + 15} \\ \underline{6x^3 + 30x^2} \phantom{{}-7x+15} \\ -2x^2 - 7x \phantom{{}+15} \\ \underline{-2x^2 - 10x} \phantom{{}+15} \\ 3x + 15 \\ \underline{3x + 15} \\ 0 \end{array}$$

Repeat the method. Divide $3x$ by x, so that $3x \div x = 3$.

Multiply $(x + 5)$ by 3, so that $3 \times (x + 5) = 3x + 15$.

Subtract, so that $(3x + 15) - (3x + 15) = 0$.

So $6x^3 + 28x^2 - 7x + 15 \div (x + 5) =$
$6x^2 - 2x + 3$.

Example 5

Divide $-3x^4 + 8x^3 - 8x^2 + 13x - 10$ by $(x - 2)$.

①

$$\begin{array}{r} -3x^3 \\ x-2\overline{)-3x^4 + 8x^3 - 8x^2 + 13x - 10} \\ \underline{-3x^4 + 6x^3} \\ 2x^3 - 8x^2 \end{array}$$

Start by dividing the first term of the polynomial by x, so that $-3x^4 \div x = -3x^3$.

Next multiply $(x - 2)$ by $-3x^3$, so that $-3x^3 \times (x - 2) = -3x^4 + 6x^3$.

Now subtract, so that $(-3x^4 + 8x^3) - (-3x^4 + 6x^3) = 2x^3$.

Copy $-8x^2$.

②

$$\begin{array}{r} -3x^3 + 2x^2 \\ x-2\overline{)-3x^4 + 8x^3 - 8x^2 + 13x - 10} \\ \underline{-3x^4 + 6x^3} \\ 2x^3 - 8x^2 \\ \underline{2x^3 - 4x^2} \\ -4x^2 + 13x \end{array}$$

Repeat the method. Divide $2x^3$ by x, so that $2x^3 \div x = 2x^2$.

Multiply $(x - 2)$ by $2x^2$, so that $2x^2 \times (x - 2) = 2x^3 - 4x^2$.

Subtract, so that $(2x^3 - 8x^2) - (2x^3 - 4x^2) = -4x^2$.

Copy $13x$.

③

$$\begin{array}{r} -3x^3 + 2x^2 - 4x \\ x-2\overline{)-3x^4 + 8x^3 - 8x^2 + 13x - 10} \\ \underline{-3x^4 + 6x^3} \\ 2x^3 - 8x^2 \\ \underline{2x^3 - 4x^2} \\ -4x^2 + 13x \\ \underline{-4x^2 + 8x} \\ 5x - 10 \end{array}$$

Repeat the method. Divide $-4x^2$ by x, so that $-4x^2 \div x = -4x$.

Multiply $(x - 2)$ by $-4x$, so that $-4x \times (x - 2) = -4x^2 + 8x$.

Subtract, so that $(-4x^2 + 13x) - (-4x^2 + 8x) = 5x$.

Copy -10.

④

$$\begin{array}{r} -3x^3 + 2x^2 - 4x + 5 \\ x-2\overline{)-3x^4 + 8x^3 - 8x^2 + 13x - 10} \\ \underline{-3x^4 + 6x^3} \\ 2x^3 - 8x^2 \\ \underline{2x^3 - 4x^2} \\ -4x^2 + 13x \\ \underline{-4x^2 + 8x} \\ 5x - 10 \\ \underline{5x - 10} \\ 0 \end{array}$$

Repeat the method. Divide $5x$ by x, so that $5x \div x = 5$.

Multiply $(x - 2)$ by 5 so that $5 \times (x - 2) = 5x - 10$.

Subtract, so that $(5x - 10) - (5x - 10) = 0$.

So $-3x^4 + 8x^3 - 8x^2 + 13x - 10 \div (x - 2)$
$= -3x^3 + 2x^2 - 4x + 5$.

Exercise 1B

1 Divide:

a $x^3 + 6x^2 + 8x + 3$ by $(x + 1)$
b $x^3 + 10x^2 + 25x + 4$ by $(x + 4)$
c $x^3 + 7x^2 - 3x - 54$ by $(x + 6)$
d $x^3 + 9x^2 + 18x - 10$ by $(x + 5)$
e $x^3 - x^2 + x + 14$ by $(x + 2)$
f $x^3 + x^2 - 7x - 15$ by $(x - 3)$
g $x^3 - 5x^2 + 8x - 4$ by $(x - 2)$
h $x^3 - 3x^2 + 8x - 6$ by $(x - 1)$
i $x^3 - 8x^2 + 13x + 10$ by $(x - 5)$
j $x^3 - 5x^2 - 6x - 56$ by $(x - 7)$

2 Divide:

a $6x^3 + 27x^2 + 14x + 8$ by $(x + 4)$
b $4x^3 + 9x^2 - 3x - 10$ by $(x + 2)$
c $3x^3 - 10x^2 - 10x + 8$ by $(x - 4)$
d $3x^3 - 5x^2 - 4x - 24$ by $(x - 3)$
e $2x^3 + 4x^2 - 9x - 9$ by $(x + 3)$
f $2x^3 - 15x^2 + 14x + 24$ by $(x - 6)$
g $-3x^3 + 2x^2 - 2x - 7$ by $(x + 1)$
h $-2x^3 + 5x^2 + 17x - 20$ by $(x - 4)$
i $-5x^3 - 27x^2 + 23x + 30$ by $(x + 6)$
j $-4x^3 + 9x^2 - 3x + 2$ by $(x - 2)$

3 Divide:

a $x^4 + 5x^3 + 2x^2 - 7x + 2$ by $(x + 2)$
b $x^4 + 11x^3 + 25x^2 - 29x - 20$ by $(x + 5)$
c $4x^4 + 14x^3 + 3x^2 - 14x - 15$ by $(x + 3)$
d $3x^4 - 7x^3 - 23x^2 + 14x - 8$ by $(x - 4)$
e $-3x^4 + 9x^3 - 10x^2 + x + 14$ by $(x - 2)$
f $3x^5 + 17x^4 + 2x^3 - 38x^2 + 5x - 25$ by $(x + 5)$
g $6x^5 - 19x^4 + x^3 + x^2 + 13x + 6$ by $(x - 3)$
h $-5x^5 + 7x^4 + 2x^3 - 7x^2 + 10x - 7$ by $(x - 1)$
i $2x^6 - 11x^5 + 14x^4 - 16x^3 + 36x^2 - 10x - 24$ by $(x - 4)$
j $-x^6 + 4x^5 - 4x^4 + 4x^3 - 5x^2 + 7x - 3$ by $(x - 3)$

Example 6

Divide $x^3 - 3x - 2$ by $(x - 2)$.

$$
\begin{array}{r}
x^2 + 2x + 1 \\
x - 2\,\overline{)\,x^3 + 0x^2 - 3x - 2} \\
\underline{x^3 - 2x^2} \qquad\qquad \\
2x^2 - 3x \qquad \\
\underline{2x^2 - 4x} \qquad \\
x - 2 \\
\underline{x - 2} \\
0
\end{array}
$$

Use $0x^2$ so that the sum is laid out correctly.

Subtract, so that $(x^3 + 0x^2) - (x^3 - 2x^2) = 2x^2$.

The number remaining after a division is called the **remainder**. In this case the remainder = 0, so $(x - 2)$ is a **factor** of $x^3 - 3x - 2$.

So $x^3 - 3x - 2 \div (x - 2) = x^2 + 2x + 1$.

$x^2 + 2x + 1$ is called the **quotient**.

Example 7

Divide $3x^3 - 3x^2 - 4x + 4$ by $(x - 1)$.

$$
\begin{array}{r}
3x^2 \qquad\quad - 4 \\
x - 1\overline{)\,3x^3 - 3x^2 - 4x + 4} \\
\underline{3x^3 - 3x^2} \qquad\qquad \\
0 - 4x + 4 \\
\underline{-4x + 4} \\
0
\end{array}
$$

So $x^3 - 3x^2 - 4x + 4 \div (x - 1) =$
$3x^2 - 4$.

Divide $-4x$ by x, so that $-4x \div x = -4$.

Subtract, so that
$(3x^3 - 3x^2) - (3x^3 - 3x^2) = 0$.

Copy $-4x$ and 4.

Example 8

Find the remainder when $2x^3 - 5x^2 - 16x + 10$ is divided by $(x - 4)$.

$$
\begin{array}{r}
2x^2 + 3x - 4 \qquad\qquad \\
x - 4\overline{)\,2x^3 - 5x^2 - 16x + 10} \\
\underline{2x^3 - 8x^2} \qquad\qquad\qquad \\
3x^2 - 16x \qquad\quad \\
\underline{3x^2 - 12x} \qquad\quad \\
-4x + 10 \\
\underline{-4x + 16} \\
-6
\end{array}
$$

So the remainder is -6.

$(x - 4)$ is not a factor of $2x^3 - 5x^2 - 16x + 10$ as the remainder $\neq 0$.

Exercise 1C

1 Divide:

a $x^3 + x + 10$ by $(x + 2)$

b $2x^3 - 17x + 3$ by $(x + 3)$

c $-3x^3 + 50x - 8$ by $(x - 4)$

2 Divide:

a $x^3 + x^2 - 36$ by $(x - 3)$

b $2x^3 + 9x^2 + 25$ by $(x + 5)$

c $-3x^3 + 11x^2 - 20$ by $(x - 2)$

Hint for question 2:
Use $0x$.

3 Divide:

a $x^3 + 2x^2 - 5x - 10$ by $(x + 2)$

b $2x^3 - 6x^2 + 7x - 21$ by $(x - 3)$

c $-3x^3 + 21x^2 - 4x + 28$ by $(x - 7)$

4 Find the remainder when:

a $x^3 + 4x^2 - 3x + 2$ is divided by $(x + 5)$

b $3x^3 - 20x^2 + 10x + 5$ is divided by $(x - 6)$

c $-2x^3 + 3x^2 + 12x + 20$ is divided by $(x - 4)$

5 Show that when $3x^3 - 2x^2 + 4$ is divided by $(x - 1)$ the remainder is 5.

6 Show that when $3x^4 - 8x^3 + 10x^2 - 3x - 25$ is divided by $(x + 1)$ the remainder is -1.

7 Show that $(x + 4)$ is a factor of $5x^3 - 73x + 28$.

8 Simplify $\dfrac{3x^3 - 8x - 8}{x - 2}$.

9 Divide $x^3 - 1$ by $(x - 1)$.

10 Divide $x^4 - 16$ by $(x + 2)$.

Hint for question 8: Divide $3x^3 - 8x - 8$ by $(x - 2)$.

Hint for question 9: Use $0x^2$ and $0x$.

1.3 You can factorise a polynomial by using the factor theorem: If f(x) is a polynomial and f(p) = 0, then $x - p$ is a factor of f(x).

Example 9

Show that $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$ by:

a algebraic division

b the factor theorem

a

$$\begin{array}{r} x^2 + 3x + 2 \\ x - 2\overline{)x^3 + x^2 - 4x - 4} \\ \underline{x^3 - 2x^2} \qquad\qquad \\ 3x^2 - 4x \quad\;\; \\ \underline{3x^2 - 6x} \quad\;\; \\ 2x - 4 \\ \underline{2x - 4} \\ 0 \end{array}$$

So $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.

Divide $x^3 + x^2 - 4x - 4$ by $(x - 2)$.

The remainder = 0, so $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.

b $f(x) = x^3 + x^2 - 4x - 4$

Write the polynomial as a function.

$f(2) = (2)^3 + (2)^2 - 4(2) - 4$

Substitute $x = 2$ into the polynomial.

$= 8 + 4 - 8 - 4$

$= 0$

So $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.

Use the factor theorem:
If $f(p) = 0$, then $x - p$ is a factor of $f(x)$.
Here $p = 2$, so $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.

Example 10

Factorise $2x^3 + x^2 - 18x - 9$.

$f(x) = 2x^3 + x^2 - 18x - 9$

Write the polynomial as a function.

$f(-1) = 2(-1)^3 + (-1)^2 - 18(-1) - 9 = 8$
$f(1) = 2(1)^3 + (1)^2 - 18(1) - 9 = -24$
$f(2) = 2(2)^3 + (2)^2 - 18(2) - 9 = -25$
$f(3) = 2(3)^3 + (3)^2 - 18(3) - 9 = 0$

Try values of x, e.g. $-1, 1, 2, 3, \ldots$ until you find $f(p) = 0$.

$f(p) = 0$.

So $(x - 3)$ is a factor of $2x^3 + x^2 - 18x - 9$.

Use the factor theorem:
If $f(p) = 0$, then $x - p$ is a factor of $f(x)$.
Here $p = 3$.

$$
\begin{array}{r}
2x^2 + 7x + 3 \\
x - 3\,\overline{)\,2x^3 + x^2 - 18x - 9} \\
\underline{2x^3 - 6x^2 \qquad\qquad} \\
7x^2 - 18x \qquad \\
\underline{7x^2 - 21x \qquad} \\
3x - 9 \\
\underline{3x - 9} \\
0
\end{array}
$$

Divide $2x^3 + x^2 - 18x - 9$ by $(x - 3)$.

You can check your division here:
$(x - 3)$ is a factor of $2x^3 + x^2 - 18x - 9$, so the remainder must $= 0$.

$2x^3 + x^2 - 18x - 9 = (x - 3)(2x^2 + 7x + 3)$
$= (x - 3)(2x + 1)(x + 3)$

$2x^2 + 7x + 3$ can also be factorised.

Example 11

Given that $(x + 1)$ is a factor of $4x^4 - 3x^2 + a$, find the value of a.

$f(x) = 4x^4 - 3x^2 + a$

Write the polynomial as a function.

$f(-1) = 0$

Use the factor theorem the other way around:
$x - p$ is a factor of $f(x)$, so $f(p) = 0$
Here $p = -1$.

$4(-1)^4 - 3(-1)^2 + a = 0$

$4 - 3 + a = 0$

$a = -1$

Substitute $x = -1$ and solve the equation for a.

Remember $(-1)^4 = 1$

Example 12

Show that if $(x - p)$ is a factor of $f(x)$ then $f(p) = 0$.

If $(x - p)$ is a factor of $f(x)$ then

$f(x) = (x - p) \times g(x)$

$g(x)$ is a polynomial expression.

So $f(p) = (p - p) \times g(p)$

Substitute $x = p$.

i.e. $f(p) = 0 \times g(p)$

$p - p = 0$

So $f(p) = 0$ as required.

Remember $0 \times$ *anything* $= 0$.

Exercise 1D

1 Use the factor theorem to show that:

a $(x - 1)$ is a factor of $4x^3 - 3x^2 - 1$

b $(x + 3)$ is a factor of $5x^4 - 45x^2 - 6x - 18$

c $(x - 4)$ is a factor of $-3x^3 + 13x^2 - 6x + 8$

2 Show that $(x - 1)$ is a factor of $x^3 + 6x^2 + 5x - 12$ and hence factorise the expression completely.

3 Show that $(x + 1)$ is a factor of $x^3 + 3x^2 - 33x - 35$ and hence factorise the expression completely.

4 Show that $(x - 5)$ is a factor of $x^3 - 7x^2 + 2x + 40$ and hence factorise the expression completely.

5 Show that $(x - 2)$ is a factor of $2x^3 + 3x^2 - 18x + 8$ and hence factorise the expression completely.

6 Each of these expressions has a factor $(x \pm p)$. Find a value of p and hence factorise the expression completely.

a $x^3 - 10x^2 + 19x + 30$ **b** $x^3 + x^2 - 4x - 4$ **c** $x^3 - 4x^2 - 11x + 30$

7 Factorise:

a $2x^3 + 5x^2 - 4x - 3$ **b** $2x^3 - 17x^2 + 38x - 15$

c $3x^3 + 8x^2 + 3x - 2$ **d** $6x^3 + 11x^2 - 3x - 2$

e $4x^3 - 12x^2 - 7x + 30$

8 Given that $(x - 1)$ is a factor of $5x^3 - 9x^2 + 2x + a$ find the value of a.

9 Given that $(x + 3)$ is a factor of $6x^3 - bx^2 + 18$ find the value of b.

10 Given that $(x - 1)$ and $(x + 1)$ are factors of $px^3 + qx^2 - 3x - 7$ find the value of p and q.

Hint for question 10: Solve simultaneous equations.

1.4 You can find the remainder when a polynomial is divided by $(ax - b)$ by using the remainder theorem:

If a polynomial f(x) is divided by $(ax - b)$ then the remainder is $f\left(\frac{b}{a}\right)$.

Example 13

Find the remainder when $x^3 - 20x + 3$ is divided by $(x - 4)$ using:

a algebraic division **b** the remainder theorem

a

$$\begin{array}{r} x^2 + 4x - 4 \\ x - 4 \overline{)x^3 + 0x^2 - 20x + 3} \\ \underline{x^3 - 4x^2} \\ 4x^2 - 20x \\ \underline{4x^2 - 16x} \\ -4x + 3 \\ \underline{-4x + 16} \\ -13 \end{array}$$

Divide $x^3 - 20x + 3$ by $(x - 4)$. Remember to use $0x^2$.

The remainder is -13.

b $f(x) = x^3 - 20x + 3$

$f(4) = (4)^3 - 20(4) + 3$
$= 64 - 80 + 3$
$= -13$

The remainder is -13.

Write the polynomial as a function.

Use the remainder theorem: If $f(x)$ is divided by $(ax - b)$, then the remainder is $f\left(\frac{b}{a}\right)$.

Compare $(x - 4)$ to $(ax - b)$, so $a = 1$, $b = 4$ and the remainder is $f\left(\frac{4}{1}\right)$, i.e. $f(4)$.

Substitute $x = 4$.

Example 14

When $8x^4 - 4x^3 + ax^2 - 1$ is divided by $(2x + 1)$ the remainder is 3. Find the value of a.

$f(x) = 8x^4 - 4x^3 + ax^2 - 1$

$f(-\frac{1}{2}) = 3$

$8(-\frac{1}{2})^4 - 4(-\frac{1}{2})^3 + a(-\frac{1}{2})^2 - 1 = 3$

$8(\frac{1}{16}) - 4(-\frac{1}{8}) + a(\frac{1}{4}) - 1 = 3$

$\frac{1}{2} + \frac{1}{2} + \frac{1}{4}a - 1 = 3$

$\frac{1}{4}a = 3$

$a = 12$

Use the remainder theorem: If $f(x)$ is divided by $(ax - b)$, then the remainder is $f\left(\frac{b}{a}\right)$.

Compare $(2x + 1)$ to $(ax - b)$, so $a = 2$, $b = -1$ and the remainder is $f(-\frac{1}{2})$.

Using the fact that the remainder is 3, substitute $x = -\frac{1}{2}$ and solve the equation for a.

$(-\frac{1}{2})^3 = -\frac{1}{2} \times -\frac{1}{2} \times -\frac{1}{2} = -\frac{1}{8}$

Exercise 1E

1 Find the remainder when:

a $4x^3 - 5x^2 + 7x + 1$ is divided by $(x - 2)$
b $2x^5 - 32x^3 + x - 10$ is divided by $(x - 4)$
c $-2x^3 + 6x^2 + 5x - 3$ is divided by $(x + 1)$
d $7x^3 + 6x^2 - 45x + 1$ is divided by $(x + 3)$
e $4x^4 - 4x^2 + 8x - 1$ is divided by $(2x - 1)$
f $243x^4 - 27x^3 - 3x + 7$ is divided by $(3x - 1)$
g $64x^3 + 32x^2 - 16x + 9$ is divided by $(4x + 1)$
h $81x^3 - 81x^2 + 9x + 6$ is divided by $(3x - 2)$
i $243x^6 - 780x^2 + 6$ is divided by $(3x + 4)$
j $125x^4 + 5x^3 - 9x$ is divided by $(5x + 3)$

2 When $2x^3 - 3x^2 - 2x + a$ is divided by $(x - 1)$ the remainder is -4. Find the value of a.

3 When $-3x^3 + 4x^2 + bx + 6$ is divided by $(x + 2)$ the remainder is 10. Find the value of b.

4 When $16x^3 - 32x^2 + cx - 8$ is divided by $(2x - 1)$ the remainder is 1. Find the value of c.

5 Show that $(x - 3)$ is a factor of $x^6 - 36x^3 + 243$.

6 Show that $(2x - 1)$ is a factor of $2x^3 + 17x^2 + 31x - 20$.

7 $f(x) = x^2 + 3x + q$. Given $f(2) = 3$, find $f(-2)$.

Hint for question 7: First find q.

8 $g(x) = x^3 + ax^2 + 3x + 6$. Given $g(-1) = 2$, find the remainder when $g(x)$ is divided by $(3x - 2)$.

9 The expression $2x^3 - x^2 + ax + b$ gives a remainder 14 when divided by $(x - 2)$ and a remainder -86 when divided by $(x + 3)$. Find the value of a and b.

10 The expression $3x^3 + 2x^2 - px + q$ is divisible by $(x - 1)$ but leaves a remainder of 10 when divided by $(x + 1)$.
Find the value of a and b.

Hint for question 10: Solve simultaneous equations.

Mixed exercise 1F

1 Simplify these fractions as far as possible:

a $\dfrac{3x^4 - 21x}{3x}$ **b** $\dfrac{x^2 - 2x - 24}{x^2 - 7x + 6}$ **c** $\dfrac{2x^2 + 7x - 4}{2x^2 + 9x + 4}$

2 Divide $3x^3 + 12x^2 + 5x + 20$ by $(x + 4)$.

3 Simplify $\dfrac{2x^3 + 3x + 5}{x + 1}$.

4 Show that $(x - 3)$ is a factor of $2x^3 - 2x^2 - 17x + 15$. Hence express $2x^3 - 2x^2 - 17x + 15$ in the form $(x - 3)(Ax^2 + Bx + C)$, where the values A, B and C are to be found.

5 Show that $(x - 2)$ is a factor of $x^3 + 4x^2 - 3x - 18$. Hence express $x^3 + 4x^2 - 3x - 18$ in the form $(x - 2)(px + q)^2$, where the values p and q are to be found.

6 Factorise completely $2x^3 + 3x^2 - 18x + 8$.

7 Find the value of k if $(x - 2)$ is a factor of $x^3 - 3x^2 + kx - 10$.

8 Find the remainder when $16x^5 - 20x^4 + 8$ is divided by $(2x - 1)$.

9 $f(x) = 2x^2 + px + q$. Given that $f(-3) = 0$, and $f(4) = 21$:

a find the value of p and q

b factorise $f(x)$

10 $h(x) = x^3 + 4x^2 + rx + s$. Given $h(-1) = 0$, and $h(2) = 30$:

a find the value of r and s

b find the remainder when $h(x)$ is divided by $(3x - 1)$

11 $g(x) = 2x^3 + 9x^2 - 6x - 5$.

a Factorise $g(x)$

b Solve $g(x) = 0$

12 The remainder obtained when $x^3 - 5x^2 + px + 6$ is divided by $(x + 2)$ is equal to the remainder obtained when the same expression is divided by $(x - 3)$.
Find the value of p.

13 The remainder obtained when $x^3 + dx^2 - 5x + 6$ is divided by $(x - 1)$ is twice the remainder obtained when the same expression is divided by $(x + 1)$.
Find the value of d.

14 **a** Show that $(x - 2)$ is a factor of $f(x) = x^3 + x^2 - 5x - 2$.

b Hence, or otherwise, find the exact solutions of the equation $f(x) = 0$. *E*

15 Given that -1 is a root of the equation $2x^3 - 5x^2 - 4x + 3$, find the two positive roots. *E*

Summary of key points

1 If $f(x)$ is a polynomial and $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

2 If $f(x)$ is a polynomial and $f\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of $f(x)$.

3 If a polynomial $f(x)$ is divided by $(ax - b)$ then the remainder is $f\left(\frac{b}{a}\right)$.