10

After completing this chapter you should be able to

1 know and use the relationships $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$

2 solve simple trigonometrical equations of the form $\sin(\theta) = k$

3 solve more complex trigonometrical equations of the form $\sin(n\theta + \alpha) = k$

Trigonometrical identities and simple equations

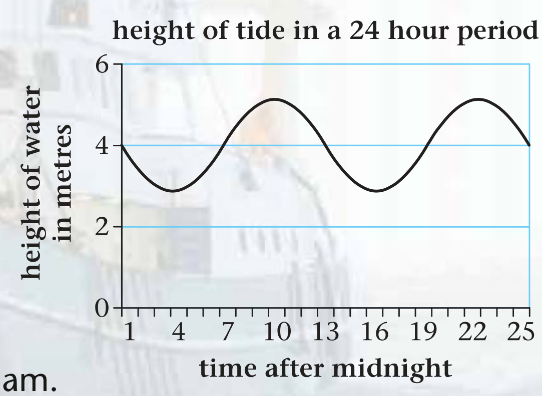
In Chapter 8 we described how many real life examples of trigonometrical functions occur in real life. Once you have mastered how to solve trigonometrical equations you can then start to solve real life problems. Try this one after you have completed the chapter.

The height of water h metres in a harbour is given by the formula

$$h = 4 - 2\sin(30t + 20)$$

where t is the time in hours after midnight.

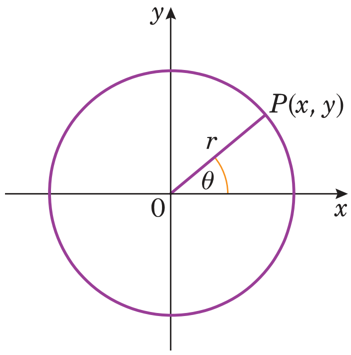
If the boat can only enter the harbour when there is more than 5 metres of water, show that it can first enter the harbour at 6:20 am.



10.1 You need to be able to use the relationships $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta \equiv 1$

You saw on page 123 that for all values of θ

$$\sin\theta = \frac{y}{r} \quad \cos\theta = \frac{x}{r} \quad \tan\theta = \frac{y}{x}$$



where (x, y) are the coordinates of the point P as it moves round the circumference of a circle with centre O and radius r, and OP makes an angle θ with the +ve x-axis.

Now, $\tan\theta = \dfrac{y}{x} = \dfrac{y}{r} \times \dfrac{r}{x} = \sin\theta \times \dfrac{1}{\cos\theta} = \dfrac{\sin\theta}{\cos\theta}$, so $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

■ **For all values of θ (except where $\cos\theta = 0$, i.e. for odd multiples of 90°, where $\tan\theta$ is not defined):**

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

You know from Chapter 4 that the equation of the circle with centre the origin and radius r is

$$x^2 + y^2 = r^2$$

Using the equations at the top of the page you can express the coordinates of P in terms of r and θ:

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

Since P lies on this circle:

$$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$$

So $\quad (\cos\theta)^2 + (\sin\theta)^2 \equiv 1$

■ **For all values of θ, $\cos^2\theta + \sin^2\theta \equiv 1$.**

Hint: Positive powers of trigonometric functions are written without brackets but with the power before the angle: for example $(\cos\theta)^n$ is written as $\cos^n\theta$, if n is positive.

Hint: This is sometimes known as Pythagoras' theorem in trigonometry.

Example 1

Simplify the following expressions:

a $\sin^2 3\theta + \cos^2 3\theta$

b $5 - 5\sin^2\theta$

c $\dfrac{\sin 2\theta}{\sqrt{1 - \sin^2 2\theta}}$

a $\sin^2 3\theta + \cos^2 3\theta = 1$

$\sin^2\theta + \cos^2\theta \equiv 1$, with θ replaced by 3θ.

b $5 - 5\sin^2\theta = 5(1 - \sin^2\theta)$

$= 5\cos^2\theta.$

Always look for factors.

$\sin^2\theta + \cos^2\theta \equiv 1$, so $1 - \sin^2\theta \equiv \cos^2\theta$.

c $\dfrac{\sin 2\theta}{\sqrt{1 - \sin^2 2\theta}} = \dfrac{\sin 2\theta}{\sqrt{\cos^2 2\theta}}$

$= \dfrac{\sin 2\theta}{\cos 2\theta}$

$= \tan 2\theta$

$\sin^2 2\theta + \cos^2 2\theta \equiv 1$, so $1 - \sin^2 2\theta \equiv \cos^2 2\theta$.

$\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$, so $\dfrac{\sin 2\theta}{\cos 2\theta} \equiv \tan 2\theta$.

Example 2

Show that $\dfrac{\cos^4\theta - \sin^4\theta}{\cos^2\theta} \equiv 1 - \tan^2\theta$

When you have to prove an identity like this you may quote the basic identities like '$\sin^2 + \cos^2 \equiv 1$'.

$$\text{LHS} = \frac{\cos^4\theta - \sin^4\theta}{\cos^2\theta}$$

Usually the best strategy is to start with the more complicated side (here the left-hand side, LHS) and try to produce the expression on the other side.

$$= \frac{(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)}{\cos^2\theta}$$

$$= \frac{(\cos^2\theta - \sin^2\theta)}{\cos^2\theta}$$

The numerator can be factorised as the 'difference of two squares'.

$\sin^2\theta + \cos^2\theta \equiv 1.$

$$= \frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$$

$$= 1 - \tan^2\theta = \text{RHS}$$

Divide through by $\cos^2\theta$ and note that $\dfrac{\sin^2\theta}{\cos^2\theta} = \left(\dfrac{\sin\theta}{\cos\theta}\right)^2 = \tan^2\theta.$

Example 3

Given that $\cos\theta = -\frac{3}{5}$ and that θ is reflex, find the value of: **a** $\sin\theta$ and **b** $\tan\theta$.

Method 1

a Since $\sin^2\theta + \cos^2\theta \equiv 1$,

$$\sin^2\theta = 1 - \left(-\frac{3}{5}\right)^2$$
$$= 1 - \frac{9}{25}$$
$$= \frac{16}{25}$$

So $\sin\theta = -\frac{4}{5}$

'θ is reflex' is critical information. Using your calculator to solve $\cos\theta = -\frac{3}{5}$, without consideration of the correct quadrant, would give wrong answers for both $\sin\theta$ and $\tan\theta$.

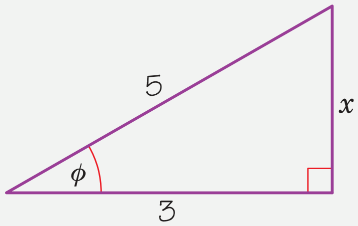
'θ is reflex' means θ is in 3rd or 4th quadrants, but as $\cos\theta$ is negative , θ must be in the 3rd quadrant. $\sin\theta = \pm\frac{4}{5}$ but in the third quadrant $\sin\theta$ is negative.

b $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$= \frac{\frac{-4}{5}}{\frac{-3}{5}}$$
$$= \frac{4}{3}$$

Method 2

a Use the right-angled triangle with the acute angle ϕ, where $\cos\phi = \frac{3}{5}$.



Using Pythagoras' theorem, $x = 4$

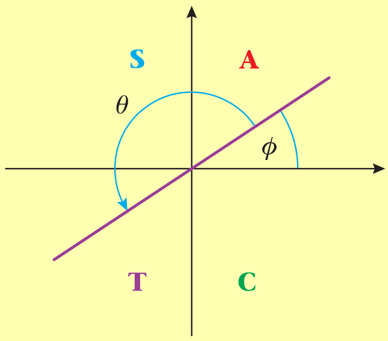
so $\sin\phi = \frac{4}{5}$

As $\sin\theta = -\sin\phi$, $\sin\theta = -\frac{4}{5}$

b Also from the triangle, $\tan\phi = \frac{4}{3}$

As $\tan\theta = +\tan\phi$, $\tan\theta = +\frac{4}{3}$

As θ is in the third quadrant you know from the work in Chapter 8, that $\theta = 180 + \phi$



so
$\sin\theta = -\sin\phi$ and $\tan\theta = +\tan\phi$

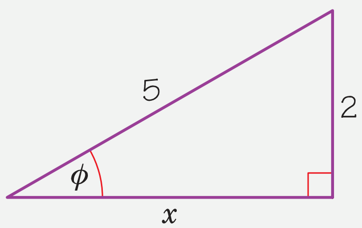
Hint: Given an angle θ of any size, you can always work with its related acute angle ϕ, and use a right-angled triangle work to find $\sin\phi$, $\cos\phi$ and $\tan\phi$. However, remember to consider the quadrant that θ is in when giving $\sin\theta$, $\cos\theta$ and $\tan\theta$.

Example 4

Given that $\sin \alpha = \frac{2}{5}$ and that α is obtuse, find the exact value of $\cos \alpha$.

Work with the acute angle ϕ, where $\sin \phi = \frac{2}{5}$.

Draw the right-angled triangle and work out the third side.



Using Pythagoras' theorem,

$$2^2 + x^2 = 5^2,$$

$$\text{so } x^2 = 21 \Rightarrow x = \sqrt{21}$$

So $\cos \phi = \frac{\sqrt{21}}{5}$

Therefore, $\cos \alpha = -\frac{\sqrt{21}}{5}$ as α is in the second quadrant.

'Exact' here means 'Do not use your calculator to find α'.

Alternatively, as in Method 1 of Example 3:
Using $\sin^2 \alpha + \cos^2 \alpha \equiv 1$,

$$\cos^2 \alpha = 1 - \frac{4}{25} = \frac{21}{25}$$

As α is obtuse, $\cos \alpha$ is $-$ve,

$$\text{so } \cos \alpha = -\frac{\sqrt{21}}{5}$$

Example 5

Given that $p = 3 \cos \theta$, and that $q = 2 \sin \theta$, show that $4p^2 + 9q^2 = 36$.

As $p = 3 \cos \theta$, and $q = 2 \sin \theta$,

$$\text{so } \cos \theta = \frac{p}{3} \text{ and } \sin \theta = \frac{q}{2}$$

Using $\sin^2 \theta + \cos^2 \theta \equiv 1$,

$$\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2 = 1$$

$$\text{so} \quad \frac{q^2}{4} + \frac{p^2}{9} = 1$$

$$\therefore \quad 9q^2 + 4p^2 = 36$$

You need to eliminate θ from the equations. As you can find $\sin \theta$ and $\cos \theta$ in terms of p and q, use the identity $\sin^2 \theta + \cos^2 \theta \equiv 1$.

Multiplying both sides by 36.

Exercise 10A

1 Simplify each of the following expressions:

a $1 - \cos^2 \frac{1}{2}\theta$ **b** $5\sin^2 3\theta + 5\cos^2 3\theta$ **c** $\sin^2 A - 1$

d $\dfrac{\sin\theta}{\tan\theta}$ **e** $\dfrac{\sqrt{1-\cos^2 x}}{\cos x}$ **f** $\dfrac{\sqrt{1-\cos^2 3A}}{\sqrt{1-\sin^2 3A}}$

g $(1 + \sin x)^2 + (1 - \sin x)^2 + 2\cos^2 x$

h $\sin^4\theta + \sin^2\theta\cos^2\theta$

i $\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta$

2 Given that $2\sin\theta = 3\cos\theta$, find the value of $\tan\theta$.

3 Given that $\sin x\cos y = 3\cos x\sin y$, express $\tan x$ in terms of $\tan y$.

4 Express in terms of $\sin\theta$ only:

a $\cos^2\theta$ **b** $\tan^2\theta$ **c** $\cos\theta\tan\theta$

d $\dfrac{\cos\theta}{\tan\theta}$ **e** $(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$

5 Using the identities $\sin^2 A + \cos^2 A \equiv 1$ and/or $\tan A \equiv \dfrac{\sin A}{\cos A}$ $(\cos A \neq 0)$, prove that:

a $(\sin\theta + \cos\theta)^2 \equiv 1 + 2\sin\theta\cos\theta$ **b** $\dfrac{1}{\cos\theta} - \cos\theta \equiv \sin\theta\tan\theta$

c $\tan x + \dfrac{1}{\tan x} \equiv \dfrac{1}{\sin x\cos x}$ **d** $\cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$

e $(2\sin\theta - \cos\theta)^2 + (\sin\theta + 2\cos\theta)^2 \equiv 5$ **f** $2 - (\sin\theta - \cos\theta)^2 \equiv (\sin\theta + \cos\theta)^2$

g $\sin^2 x\cos^2 y - \cos^2 x\sin^2 y = \sin^2 x - \sin^2 y$

6 Find, without using your calculator, the values of:

a $\sin\theta$ and $\cos\theta$, given that $\tan\theta = \frac{5}{12}$ and θ is acute.

b $\sin\theta$ and $\tan\theta$, given that $\cos\theta = -\frac{3}{5}$ and θ is obtuse.

c $\cos\theta$ and $\tan\theta$, given that $\sin\theta = -\frac{7}{25}$ and $270° < \theta < 360°$.

7 Given that $\sin\theta = \frac{2}{3}$ and that θ is obtuse, find the exact value of: **a** $\cos\theta$, **b** $\tan\theta$.

8 Given that $\tan\theta = -\sqrt{3}$ and that θ is reflex, find the exact value of: **a** $\sin\theta$, **b** $\cos\theta$.

9 Given that $\cos\theta = \frac{3}{4}$ and that θ is reflex, find the exact value of: **a** $\sin\theta$, **b** $\tan\theta$.

10 In each of the following, eliminate θ to give an equation relating x and y:

a $x = \sin\theta$, $y = \cos\theta$

b $x = \sin\theta$, $y = 2\cos\theta$

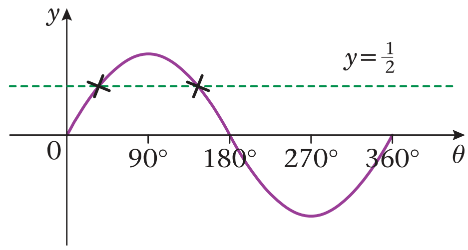
c $x = \sin\theta$, $y = \cos^2\theta$

d $x = \sin\theta$, $y = \tan\theta$

e $x = \sin\theta + \cos\theta$, $y = \cos\theta - \sin\theta$

10.2 You need to be able to solve simple trigonometrical equations of the form $\sin\theta = k$, $\cos\theta = k$, (where $-1 \leqslant k \leqslant 1$) and $\tan\theta = p$ ($p \in \mathbb{R}$)

You can show solutions to the equation $\sin\theta = \frac{1}{2}$ in the interval $0° \leqslant \theta \leqslant 360°$ by plotting the graphs of $y = \sin\theta$ and $y = \frac{1}{2}$ and seeing where they intersect.



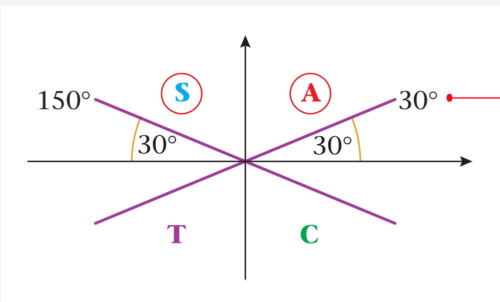
You can find the exact solutions by using what you already know about trigonometrical functions.

Example 6

Find the solutions of the equation $\sin\theta = \frac{1}{2}$ in the interval $0 \leqslant \theta \leqslant 360°$.

$\sin\theta = \frac{1}{2}$

So $\theta = 30°$ — Use $\sin^{-1}$ on your calculator to find one solution.



Putting 30° in the four positions shown gives the angles 30°, 150°, 210° and 330° but sine is only +ve in the 1st and 2nd quadrants.

So $x = 30°$

or $x = 180° - 30 = 150°$ — You can check this by putting sin 150° in your calculator.

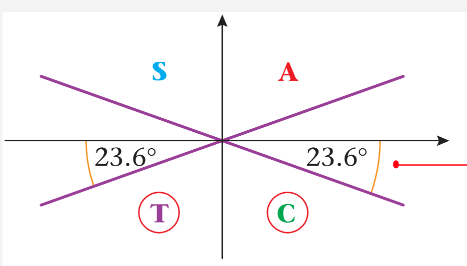
Example 7

Solve, in the interval $0 < x \leqslant 360°$, $5\sin x = -2$.

$5\sin x = -2$ — First rewrite in the form $\sin x = \ldots$

$\sin x = -0.4$

So one solution is $x = -23.6°$ (3 s.f.) — Note that this calculator solution is not in the given interval.



Sine is −ve so you need to look in the 3rd and 4th quadrants for your solutions.

$x = 203.6°$ (204° to 3 s.f.)

or $x = 336.4°$ (336° to 3 s.f.)

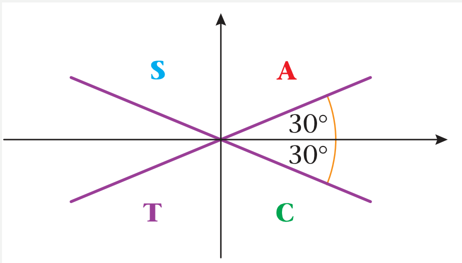
You can now read off the solutions in the given interval.
Note that in this case, if $\alpha = \sin^{-1}(-0.4)$ the solutions are $180 - \alpha$ and $360 + \alpha$.

- **A first solution of the equation $\sin x = k$ is your calculator value, $\alpha = \sin^{-1} k$. A second solution is $(180° - \alpha)$, or $(\pi - \alpha)$ if you are working in radians. Other solutions are found by adding or subtracting multiples of 360° or 2π radians.**

Example 8

Solve, in the interval $0 < x \leqslant 360°$, $\cos x = \dfrac{\sqrt{3}}{2}$.

A solution is $x = 30°$



$x = 30°$ or $330°$

The calculator solution of $\cos x = \dfrac{\sqrt{3}}{2}$, is $x = 30°$, a result you should know.

$\cos x$ is +ve so you need to look in the 1st and 4th quadrants.

Read off the solutions, in $0 < x \leqslant 360°$, from your diagram.

Note that these results are $\alpha°$ and $(360 - \alpha)°$ where $\alpha = \cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$.

- **A first solution of the equation $\cos x = k$ is your calculator value $\alpha = \cos^{-1} k$. A second solution is $(360° - \alpha)$, or $(2\pi - \alpha)$ if you are working in radians. Other solutions are found by adding or subtracting multiples of 360° or 2π radians.**

Example 9

Find the values of θ, in radians, in the interval $0 < \theta \leqslant 2\pi$, that satisfy the equation $\sin\theta = \sqrt{3}\cos\theta$.

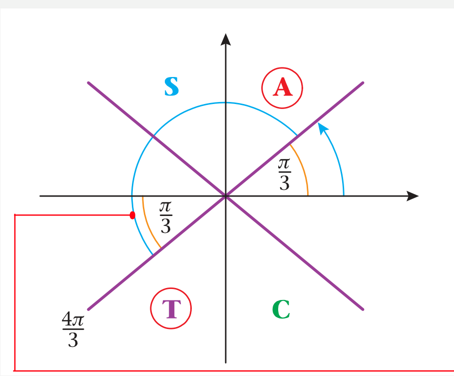
$\sin\theta = \sqrt{3}\cos\theta$

So $\tan\theta = \sqrt{3}$.

As the solutions of $\cos\theta = 0$ do not satisfy the equation you can divide both sides by $\cos\theta$.

So one solution is $\theta = \dfrac{\pi}{3}$

This is your calculator answer. (Use radian mode.)



Tangent is +ve in the 1st and 3rd quadrants, so insert the angle in the correct positions.

$\theta = \dfrac{\pi}{3}$ or $\dfrac{4\pi}{3}$

These results are α and $(\pi + \alpha)$ where $\alpha = \tan^{-1}\sqrt{3}$.

- **A first solution of the equation $\tan x = k$ is your calculator value $\alpha = \tan^{-1} k$. A second solution is $(180° + \alpha)$, or $(\pi + \alpha)$ if you are working in radians. Other solutions are found by adding or subtracting multiples of 360° or 2π radians.**

It is easier to find the solutions of $\sin\theta = -1$ or 0 or +1, $\cos\theta = -1$ or 0 or +1 or $\tan\theta = 0$ by considering the corresponding graphs of $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$ respectively.

Exercise 10B

1 Solve the following equations for θ, in the interval $0 < \theta \leqslant 360°$:

a $\sin\theta = -1$

b $\tan\theta = \sqrt{3}$

c $\cos\theta = \frac{1}{2}$

d $\sin\theta = \sin 15°$

e $\cos\theta = -\cos 40°$

f $\tan\theta = -1$

g $\cos\theta = 0$

h $\sin\theta = -0.766$

i $7\sin\theta = 5$

j $2\cos\theta = -\sqrt{2}$

k $\sqrt{3}\sin\theta = \cos\theta$

l $\sin\theta + \cos\theta = 0$

m $3\cos\theta = -2$

n $(\sin\theta - 1)(5\cos\theta + 3) = 0$

o $\tan\theta = \tan\theta\,(2 + 3\sin\theta)$

2 Solve the following equations for x, giving your answers to 3 significant figures where appropriate, in the intervals indicated:

a $\sin x° = -\dfrac{\sqrt{3}}{2}$, $-180 \leqslant x \leqslant 540$

b $2\sin x° = -0.3$, $-180 \leqslant x \leqslant 180$

c $\cos x° = -0.809$, $-180 \leqslant x \leqslant 180$

d $\cos x° = 0.84$, $-360 < x < 0$

e $\tan x° = -\dfrac{\sqrt{3}}{3}$, $0 \leqslant x \leqslant 720$

f $\tan x° = 2.90$, $80 \leqslant x \leqslant 440$

3 Solve, in the intervals indicated, the following equations for θ, where θ is measured in radians. Give your answer in terms of π or 2 decimal places.

a $\sin\theta = 0$, $-2\pi < \theta \leqslant 2\pi$

b $\cos\theta = -\frac{1}{2}$, $-2\pi < \theta \leqslant \pi$

c $\sin\theta = \dfrac{1}{\sqrt{2}}$, $-2\pi < \theta \leqslant \pi$

d $\sin\theta = \tan\theta$, $0 < \theta \leqslant 2\pi$

e $2(1 + \tan\theta) = 1 - 5\tan\theta$, $-\pi < \theta \leqslant 2\pi$

f $2\cos\theta = 3\sin\theta$, $0 < \theta \leqslant 2\pi$

10.3 You need to be able to solve equations of the form $\sin(n\theta + \alpha) = k$, $\cos(n\theta + \alpha) = k$, and $\tan(n\theta + \alpha) = p$.

You can replace $(n\theta + \alpha)$ by X so that equation reduces to the type you have solved in Section 10.2. You must be careful to ensure that you give all the solutions in the given interval.

Example 10

Solve the equation $\cos 3\theta° = 0.766$, in the interval $0 \leqslant \theta \leqslant 360$.

Let $X = 3\theta$

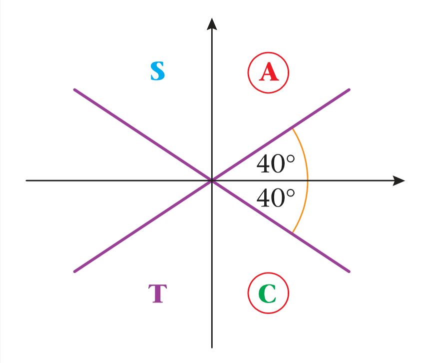
So $\cos X° = 0.766$

As $X = 3\theta$, then as $0 \leqslant \theta \leqslant 360$

So $3 \times 0 \leqslant X \leqslant 3 \times 360$

So the interval for X is $0 \leqslant X \leqslant 1080$

Replace 3θ by X and solve.



$X = 40.0, 320, 400, 680, 760, 1040$

The value of X from your calculator is 40.0. You need to list all values in the 1st and 4th quadrants for three complete revolutions.

i.e. $3\theta = 40.0, 320, 400, 680, 760, 1040$

Remember $X = 3\theta$.

So $\theta = 13.3, 107, 133, 227, 253, 347$

Divide by 3.

Always check that all of your solutions are in the given interval.

Example 11

Solve the equation $\sin(2\theta - 35)° = -1$, in the interval $-180 \leqslant \theta \leqslant 180$.

Let $X = 2\theta - 35$

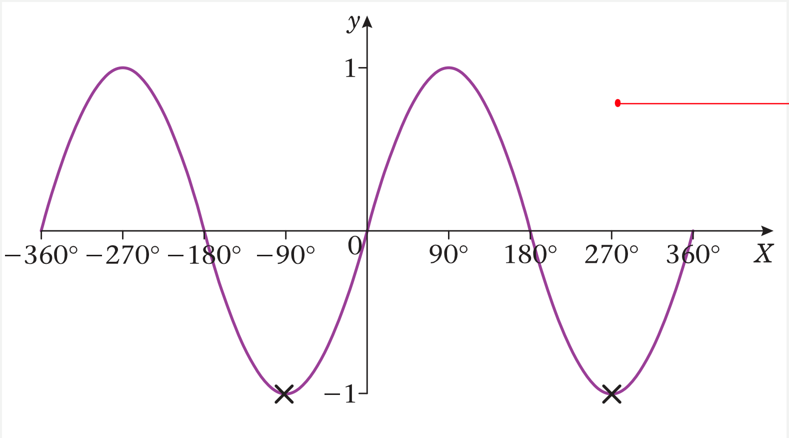
So $\sin X = -1$

Since $-180 \leqslant \theta \leqslant 180$

$-360 \leqslant 2\theta \leqslant 360$

$-395 \leqslant 2\theta - 35 \leqslant 325$

So the interval is $-395 \leqslant X \leqslant 325$



You can sketch a graph of sin X.

Refer to the graph for solutions of sin $X = -1$.

The values of X are $-90, 270$

So $2\theta - 35 = -90, 270$

$2\theta = -55, 305$

$\theta = -27.5, 152.5$

Exercise 10C

1 Find the values of θ, in the interval $0 \leqslant \theta \leqslant 360°$, for which:

a $\sin 4\theta = 0$ **b** $\cos 3\theta = -1$ **c** $\tan 2\theta = 1$ **d** $\cos 2\theta = \frac{1}{2}$

e $\tan \frac{1}{2}\theta = -\frac{1}{\sqrt{3}}$ **f** $\sin(-\theta) = \frac{1}{\sqrt{2}}$ **g** $\tan(45° - \theta) = -1$

h $2\sin(\theta - 20°) = 1$ **i** $\tan(\theta + 75°) = \sqrt{3}$ **j** $\cos(50° + 2\theta) = -1$

2 Solve each of the following equations, in the interval given.
Give your answers to 3 significant figures where appropriate.

a $\sin(\theta - 10°) = -\frac{\sqrt{3}}{2}$, $0 < \theta \leqslant 360°$ **b** $\cos(70 - x)° = 0.6$, $-180 < x \leqslant 180$

c $\tan(3x + 25)° = -0.51$, $-90 < x \leqslant 180$ **d** $5\sin 4\theta + 1 = 0$, $-90° \leqslant \theta \leqslant 90°$

3 Solve the following equations for θ, in the intervals indicated. Give your answers in radians.

a $\sin\left(\theta - \frac{\pi}{6}\right) = -\frac{1}{\sqrt{2}}$, $-\pi < \theta \leqslant \pi$ **b** $\cos(2\theta + 0.2^c) = -0.2$, $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$

c $\tan\left(2\theta + \frac{\pi}{4}\right) = 1$, $0 \leqslant \theta \leqslant 2\pi$ **d** $\sin\left(\theta + \frac{\pi}{3}\right) = \tan\frac{\pi}{6}$, $0 \leqslant \theta \leqslant 2\pi$

10.4 **You need to be able to solve quadratic equations in sin θ or cos θ or tan θ. An equation like $\sin^2\theta + 2\sin\theta - 3 = 0$ can be solved in the same way as $x^2 + 2x - 3 = 0$, with sin θ replacing x.**

Example 12

Solve for θ, in the interval $0 \leqslant \theta \leqslant 360°$, the equations

a $2\cos^2\theta - \cos\theta - 1 = 0$

b $\sin^2(\theta - 30°) = \frac{1}{2}$

a $2\cos^2\theta - \cos\theta - 1 = 0$

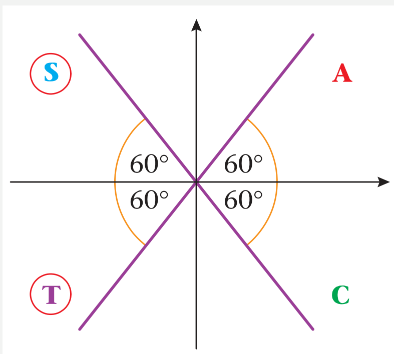
So $(2\cos\theta + 1)(\cos\theta - 1) = 0$

Compare with $2x^2 - x - 1 \equiv (2x + 1)(x - 1)$

So $\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$

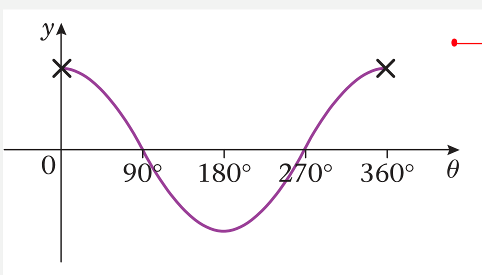
$\cos\theta = -\frac{1}{2}$ so $\theta = 120°$

Find one solution using your calculator.



120° makes an angle of 60° with the horizontal. But cosine is −ve in the 2nd and 3rd quadrants so $\theta = 120°$ or $\theta = 240°$.

$\theta = 120°$ or $\theta = 240°$



Sketch the graph of $y = \cos\theta$.

So $\theta = 0$ or $360°$

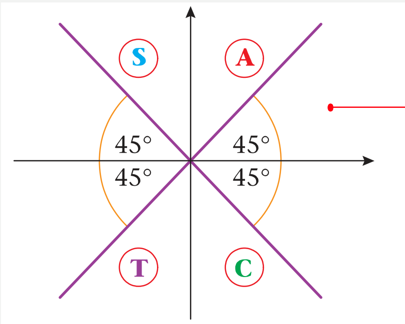
So the solutions are $\theta = 0, 120, 240, 360$

b $\sin^2(\theta - 30°) = \frac{1}{2}$

So $\sin(\theta - 30°) = \frac{1}{\sqrt{2}}$

or $\sin(\theta - 30°) = -\frac{1}{\sqrt{2}}$

So $\theta - 30° = 45°$ or $\theta - 30° = -45°$



So from $\sin(\theta - 30°) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta - 30° = 45°, 135°$

and from $\sin(\theta - 30°) = -\frac{1}{\sqrt{2}}$,

$\Rightarrow \quad \theta - 30° = 225°, 315°$

So the solutions are: $\theta = 75°, 165°, 255°, 345°$

> The solutions of $x^2 = k$ are $x = \pm\sqrt{k}$.

> Use your calculator to find one solution for each equation.

> Draw a diagram to find the quadrants where sine is +ve and the quadrants where sine is −ve.

In some equations you may need to use the identity $\sin^2 A + \cos^2 A \equiv 1$ before they are in a form to be solved by factorising (or use of 'the formula').

Example 13

Find the values of x, in the interval $-\pi \leqslant x \leqslant \pi$, satisfying the equation $2\cos^2 x + 9\sin x = 3\sin^2 x$.

$2\cos^2 x + 9\sin x = 3\sin^2 x$ can be written as

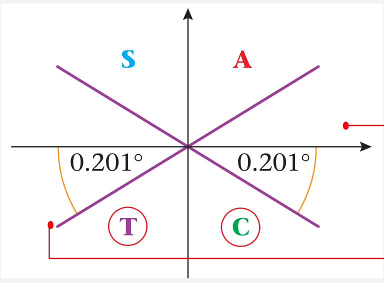
$2(1 - \sin^2 x) + 9\sin x = 3\sin^2 x$

which reduces to

$5\sin^2 x - 9\sin x - 2 = 0$

So $(5\sin x + 1)(\sin x - 2) = 0$

$\sin x = -\frac{1}{5}$



The solutions are $x = -2.94, -0.201$

> As $\sin^2 x + \cos^2 x \equiv 1$, you are able to rewrite $\cos^2 x$ as $(1 - \sin^2 x)$, and so form a quadratic equation in $\sin x$.

> The other equation, $\sin x = 2$, has no solutions.

> Your calculator value of x, in radian mode, is $x = -0.201$ (3 s.f.). Insert in the quadrant diagram.
> The smallest angle in the interval, in the 3rd quadrant, is $(-\pi + 0.201) = -2.94$; there are no values of x between 0 and π.

Exercise 10D

1 Solve for θ, in the interval $0 \leqslant \theta \leqslant 360°$, the following equations.
Give your answers to 3 significant figures where they are not exact.

a $4\cos^2\theta = 1$

b $2\sin^2\theta - 1 = 0$

c $3\sin^2\theta + \sin\theta = 0$

d $\tan^2\theta - 2\tan\theta - 10 = 0$

e $2\cos^2\theta - 5\cos\theta + 2 = 0$

f $\sin^2\theta - 2\sin\theta - 1 = 0$

g $\tan^2 2\theta = 3$

h $4\sin\theta = \tan\theta$

i $\sin\theta + 2\cos^2\theta + 1 = 0$

j $\tan^2(\theta - 45°) = 1$

k $3\sin^2\theta = \sin\theta\cos\theta$

l $4\cos\theta(\cos\theta - 1) = -5\cos\theta$

m $4(\sin^2\theta - \cos\theta) = 3 - 2\cos\theta$

n $2\sin^2\theta = 3(1 - \cos\theta)$

o $4\cos^2\theta - 5\sin\theta - 5 = 0$

p $\cos^2\frac{\theta}{2} = 1 + \sin\frac{\theta}{2}$

2 Solve for θ, in the interval $-180° \leqslant \theta \leqslant 180°$, the following equations.
Give your answers to 3 significant figures where they are not exact.

a $\sin^2 2\theta = 1$

b $\tan^2\theta = 2\tan\theta$

c $\cos\theta(\cos\theta - 2) = 1$

d $\sin^2(\theta + 10°) = 0.8$

e $\cos^2 3\theta - \cos 3\theta = 2$

f $5\sin^2\theta = 4\cos^2\theta$

g $\tan\theta = \cos\theta$

h $2\sin^2\theta + 3\cos\theta = 1$

3 Solve for x, in the interval $0 \leqslant x \leqslant 2\pi$, the following equations.
Give your answers to 3 significant figures unless they can be written in the form $\frac{a}{b}\pi$, where a and b are integers.

a $\tan^2\frac{1}{2}x = 1$

b $2\sin^2\left(x + \frac{\pi}{3}\right) = 1$

c $3\tan x = 2\tan^2 x$

d $\sin^2 x + 2\sin x\cos x = 0$

e $6\sin^2 x + \cos x - 4 = 0$

f $\cos^2 x - 6\sin x = 5$

g $2\sin^2 x = 3\sin x\cos x + 2\cos^2 x$

Mixed exercise 10E

1 Given that angle A is obtuse and $\cos A = -\sqrt{\frac{7}{11}}$, show that $\tan A = \frac{-2\sqrt{7}}{7}$.

2 Given that angle B is reflex and $\tan B = +\frac{\sqrt{21}}{2}$, find the exact value of: **a** $\sin B$, **b** $\cos B$.

3 **a** Sketch the graph of $y = \sin(x + 60)°$, in the interval $-360 \leqslant x \leqslant 360$, giving the coordinates of points of intersection with the axes.

b Calculate the values of the x-coordinates of the points in which the line $y = \frac{1}{2}$ intersects the curve.

4 Simplify the following expressions:

a $\cos^4\theta - \sin^4\theta$

b $\sin^2 3\theta - \sin^2 3\theta\cos^2 3\theta$

c $\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta$

5 **a** Given that $2(\sin x + 2\cos x) = \sin x + 5\cos x$, find the exact value of $\tan x$.

b Given that $\sin x \cos y + 3\cos x \sin y = 2\sin x \sin y - 4\cos x \cos y$, express $\tan y$ in terms of $\tan x$.

6 Show that, for all values of θ:

a $(1 + \sin\theta)^2 + \cos^2\theta = 2(1 + \sin\theta)$ **b** $\cos^4\theta + \sin^2\theta = \sin^4\theta + \cos^2\theta$

7 Without attempting to solving them, state how many solutions the following equations have in the interval $0 \leqslant \theta \leqslant 360°$. Give a brief reason for your answer.

a $2\sin\theta = 3$ **b** $\sin\theta = -\cos\theta$

c $2\sin\theta + 3\cos\theta + 6 = 0$ **d** $\tan\theta + \dfrac{1}{\tan\theta} = 0$

8 **a** Factorise $4xy - y^2 + 4x - y$.

b Solve the equation $4\sin\theta\cos\theta - \cos^2\theta + 4\sin\theta - \cos\theta = 0$, in the interval $0 \leqslant \theta \leqslant 360°$.

9 **a** Express $4\cos 3\theta° - \sin(90 - 3\theta)°$ as a single trigonometric function.

b Hence solve $4\cos 3\theta° - \sin(90 - 3\theta)° = 2$ in the interval $0 \leqslant \theta \leqslant 360$. Give your answers to 3 significant figures.

10 Find, in radians to two decimal places, the value of x in the interval $0 \leqslant x \leqslant 2\pi$, for which $3\sin^2 x + \sin x - 2 = 0$. *E*

11 Given that $2\sin 2\theta = \cos 2\theta$:

a show that $\tan 2\theta = 0.5$.

b Hence find the value of θ, to one decimal place, in the interval $0 \leqslant \theta < 360°$ for which $2\sin 2\theta° = \cos 2\theta°$. *E*

12 Find all the values of θ in the interval $0 \leqslant \theta < 360$ for which:

a $\cos(\theta + 75)° = 0.5$,

b $\sin 2\theta° = 0.7$, giving your answers to one decimal place. *E*

13 **a** Find the coordinates of the point where the graph of $y = 2\sin(2x + \frac{5}{6}\pi)$ crosses the y-axis.

b Find the values of x, where $0 \leqslant x \leqslant 2\pi$, for which $y = \sqrt{2}$. *E*

14 Find, giving your answers in terms of π, all values of θ in the interval $0 < \theta < 2\pi$, for which:

a $\tan\left(\theta + \dfrac{\pi}{3}\right) = 1$ **b** $\sin 2\theta = -\dfrac{\sqrt{3}}{2}$ *E*

15 Find the values of x in the interval $0 < x < 270°$ which satisfy the equation

$$\frac{\cos 2x + 0.5}{1 - \cos 2x} = 2$$

16 Find, to the nearest integer, the values of x in the interval $0 \leqslant x < 180°$ for which $3\sin^2 3x - 7\cos 3x - 5 = 0$. *E*

17 Find, in degrees, the values of θ in the interval $0 \leqslant \theta < 360°$ for which

$2\cos^2\theta - \cos\theta - 1 = \sin^2\theta$

Give your answers to 1 decimal place, where appropriate. *E*

18 Consider the function f(x) defined by

$f(x) \equiv 3 + 2\sin(2x + k)°,\ 0 < x < 360$

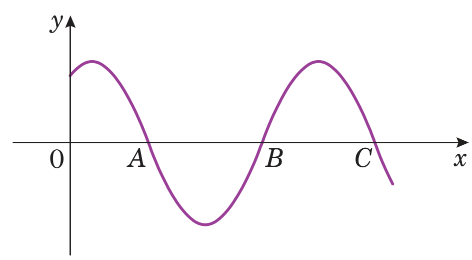
where k is a constant and $0 < k < 360$. The curve with equation $y = f(x)$ passes through the point with coordinates $(15, 3 + \sqrt{3})$.

a Show that $k = 30$ is a possible value for k and find the other possible value of k.

b Given that $k = 30$, solve the equation $f(x) = 1$. *E*

19 **a** Determine the solutions of the equation $\cos(2x - 30)° = 0$ for which $0 \leqslant x \leqslant 360$.

b The diagram shows part of the curve with equation $y = \cos(px - q)°$, where p and q are positive constants and $q < 180$. The curve cuts the x-axis at points A, B and C, as shown.

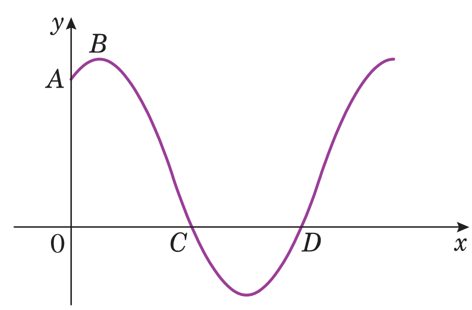


Given that the coordinates of A and B are (100, 0) and (220, 0) respectively:

i write down the coordinates of C,

ii find the value of p and the value of q. *E*

20 The diagram shows part of the curve with equation $y = f(x)$, where $f(x) = 1 + 2\sin(px° + q°)$, p and q being positive constants and $q \leqslant 90$. The curve cuts the y-axis at the point A and the x-axis at the points C and D. The point B is a maximum point on the curve.



Given that the coordinates of A and C are (0, 2) and (45, 0) respectively:

a calculate the value of q,

b show that $p = 4$,

c find the coordinates of B and D. *E*

Summary of key points

1 $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (providing $\cos\theta \neq 0$, when $\tan\theta$ is not defined)

2 $\sin^2\theta + \cos^2\theta = 1$

3 A first solution of the equation $\sin x = k$ is your calculator value, $\alpha = \sin^{-1} k$. A second solution is $(180° - \alpha)$, or $(\pi - \alpha)$ if you are working in radians. Other solutions are found by adding or subtracting multiples of 360° or 2π radians.

4 A first solution of the equation $\cos x = k$ is your calculator value of $\alpha = \cos^{-1} k$. A second solution is $(360° - \alpha)$, or $(2\pi - \alpha)$ if you are working in radians. Other solutions are found by adding or subtracting multiples of 360° or 2π radians.

5 A first solution of the equation $\tan x = k$ is your calculator value $\alpha = \tan^{-1} k$. A second solution is $(180° + \alpha)$, or $(\pi + \alpha)$ if you are working in radians. Other solutions are found by adding or subtracting multiples of 360° or 2π radians.