After completing this chapter you should be able to

1 integrate simple functions within defined limits

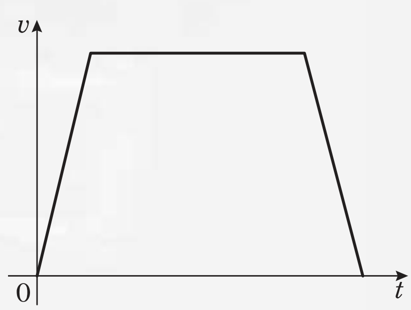
2 use integration to find areas under curves

3 use integration to find the area between a curve and a line

4 approximate the area under a curve by using the trapezium rule.

11 Integration

For anyone studying Mechanics the usefulness of this chapter should be well known. If you calculate the area under a velocity–time graph you find the total distance travelled.



Area under graph $= \int v \, \mathrm{d}t =$ distance travelled

11.1 You need to be able to integrate simple functions within defined limits. This is called definite integration.

You met indefinite integration in Book C1:

$$\int 3x^2\,\mathrm{d}x = \frac{3x^{2+1}}{3} + C$$

$$= x^3 + C$$

where C is an arbitrary constant.

You can also integrate a function between defined limits, e.g. $x = 1$ and $x = 2$.
You write this as

$$\int_1^2 3x^2\,\mathrm{d}x$$

Here is how you work out this definite integral:

$$\int_1^2 3x^2\,\mathrm{d}x = [x^3]_1^2$$

$$= (2^3) - (1^3)$$

$$= 8 - 1$$

$$= 7$$

Hint: The limits of the integral are from $x = 1$ to $x = 2$.

Hint: Notice the use of [ ] brackets. This is standard notation.

Hint: Notice the use of ( ). This is standard notation for this step.

Hint: Evaluate the integral at the upper limit.

Hint: Evaluate the integral at the lower limit.

There are three stages when you work out a definite integral:

The statement		*After integration* [square brackets]		*The evaluation* (round brackets)
$\int_a^b \ldots\,\mathrm{d}x$	$=$	$[\ldots]_a^b$	$=$	$(\ldots) - (\ldots)$

You should note the notation for evaluating definite integrals and aim to use it when answering questions.

■ **The definite integral is defined as**

$$\int_a^b \mathrm{f}'(x)\,\mathrm{d}x = [\mathrm{f}(x)]_a^b = \mathrm{f}(b) - \mathrm{f}(a)$$

provided f′ is the derived function of f throughout the interval (a, b).

Example 1

Evaluate the following

a $\int_1^4 (2x - 3x^{\frac{1}{2}} + 1)\,dx$ **b** $\int_{-1}^0 (x^{\frac{1}{3}} - 1)^2\,dx$.

a $\int_1^4 (2x - 3x^{\frac{1}{2}} + 1)\,dx$

$= \left[x^2 - \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + x\right]_1^4$

Remember: $\int_1^4 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_1^4$

$= [x^2 - 2x^{\frac{3}{2}} + x]_1^4$

Simplify the terms.

$= (4^2 - 2 \times 2^3 + 4) - (1 - 2 + 1)$

Evaluate expression at $x = 1$.

Evaluate expression at $x = 4$.
Note that $4^{\frac{3}{2}} = 2^3$.

$= 4 - 0$

$= 4.$

b $\int_{-1}^0 (x^{\frac{1}{3}} - 1)^2\,dx$

$= \int_{-1}^0 (x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1)\,dx$

First multiply out the bracket to put the expression in a form ready to be integrated. (See Book C1.)

$= \left[\frac{x^{\frac{5}{3}}}{\frac{5}{3}} - 2\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + x\right]_{-1}^0$

Remember: $\int_{-1}^0 x^{\frac{a}{b}}\,dx = \left[\frac{x^{\frac{a}{b}+1}}{(\frac{a}{b} + 1)}\right]_{-1}^0$

$= \left[\frac{3}{5}x^{\frac{5}{3}} - \frac{6}{4}x^{\frac{4}{3}} + x\right]_{-1}^0$

Simplify each term.

$= (0 + 0 + 0) - \left(-\frac{3}{5} - \frac{3}{2} - 1\right)$

Note that $(-1)^{\frac{4}{3}} = +1$

$= 3\frac{1}{10}$ or 3.1

Exercise 11A

1 Evaluate the following definite integrals:

a $\int_1^2 \left(\frac{2}{x^3} + 3x\right)dx$ **b** $\int_0^2 (2x^3 - 4x + 5)\,dx$

c $\int_4^9 \left(\sqrt{x} - \frac{6}{x^2}\right)dx$ **d** $\int_1^2 \left(6x - \frac{12}{x^4} + 3\right)dx$

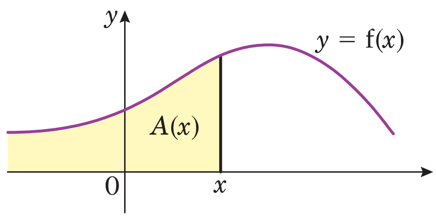
e $\int_1^8 (x^{-\frac{1}{3}} + 2x - 1)\,dx$

2 Evaluate the following definite integrals:

a $\int_1^3 \left(\frac{x^3 + 2x^2}{x}\right) dx$ **b** $\int_1^4 (\sqrt{x} - 3)^2\, dx$ **c** $\int_3^6 \left(x - \frac{3}{x}\right)^2 dx$

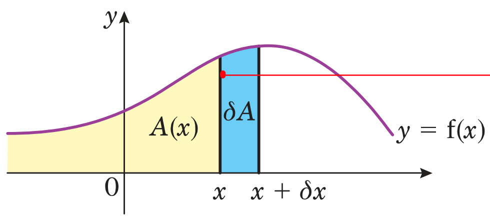
d $\int_0^1 x^2\left(\sqrt{x} + \frac{1}{x}\right) dx$ **e** $\int_1^4 \frac{2 + \sqrt{x}}{x^2}\, dx$

11.2 You need to be able to use definite integration to find areas under curves.



For any curve with equation $y = f(x)$, you can define the area under the curve and to the left of x as a function of x called $A(x)$. As x increases this area $A(x)$ also increases (since x moves further to the right).

If you look at a small increase in x, say δx, then the area increases by an amount $\delta A = A(x + \delta x) - A(x)$.



Hint: This vertical height will be y or $f(x)$.

This extra increase in the area δA is approximately rectangular and of magnitude $y\delta x$. (As we make δx smaller any error between the actual area and this will be negligible.)

So we have $\delta A \approx y\delta x$

or $\frac{\delta A}{\delta x} \approx y$

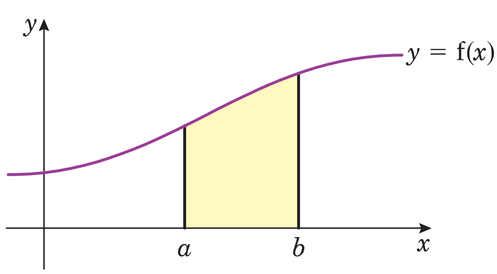
and if we take the limit $\lim_{\delta x \to 0}\left(\frac{\delta A}{\delta x}\right)$ then from Chapter 7 of Book C1 you will see that $\frac{dA}{dx} = y$.

Now if you know that $\frac{dA}{dx} = y$, then to find A you have to integrate, giving $A = \int y\, dx$.

■ **In particular if you wish to find the area between a curve, the x-axis and the lines $x = a$ and $x = b$ you have**

Area = $\int_a^b y\, dx$

where $y = f(x)$ is the equation of the curve.

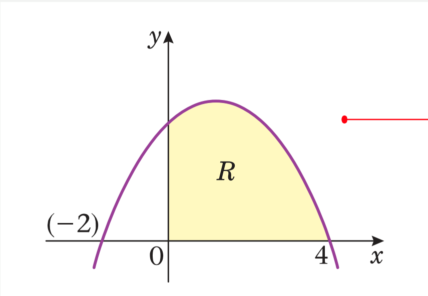


Example 2

Find the area of the region R bounded by the curve with equation $y = (4 - x)(x + 2)$ and the positive x- and y-axes.

When $x = 0, y = 8$

When $y = 0, x = 4$ or -2



A sketch of the curve will often help in this type of question. (See Chapter 4 of Book C1.)

The area of R is given by

$$A = \int_0^4 (4 - x)(x + 2)\, dx$$

So $$A = \int_0^4 (8 + 2x - x^2)\, dx$$

Multiply out the brackets.

$$A = \left[8x + x^2 - \frac{x^3}{3}\right]_0^4$$

Integrate.

$$A = \left(32 + 16 - \frac{64}{3}\right) - (0)$$

Use limits of 4 and 0.

So the area is $26\frac{2}{3}$

Example 3

The region R is enclosed by the curve with equation $y = x^2 + \frac{4}{x^2}$; $x > 0$, the x-axis and the lines $x = 1$ and $x = 3$. Find the area of R.

This curve is not one you would be expected to sketch but the limits of the integral are simply $x = 1$ and $x = 3$, so you can write down an expression for the area without referring to a sketch.

$$\text{Area} = \int_1^3 \left(x^2 + \frac{4}{x^2}\right) dx = \int_1^3 (x^2 + 4x^{-2})\, dx$$

Write the expression in a form suitable for integrating.

$$= \left[\frac{x^3}{3} - 4x^{-1}\right]_1^3$$

Now integrate.

$$= \left(9 - \frac{4}{3}\right) - \left(\frac{1}{3} - 4\right)$$

$$= 13 - \tfrac{5}{3} = 11\tfrac{1}{3}$$

You may be able to use a graphical calculator to check your answer, but you must show your working.

Exercise 11B

1 Find the area between the curve with equation $y = \mathrm{f}(x)$, the x-axis and the lines $x = a$ and $x = b$ in each of the following cases:

a $\mathrm{f}(x) = 3x^2 - 2x + 2;$ $\quad a = 0, b = 2$

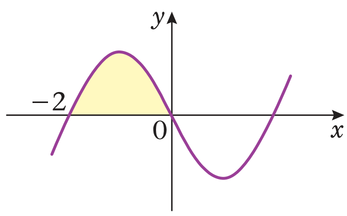
b $\mathrm{f}(x) = x^3 + 4x;$ $\quad a = 1, b = 2$

c $\mathrm{f}(x) = \sqrt{x} + 2x;$ $\quad a = 1, b = 4$

d $\mathrm{f}(x) = 7 + 2x - x^2;$ $\quad a = -1, b = 2$

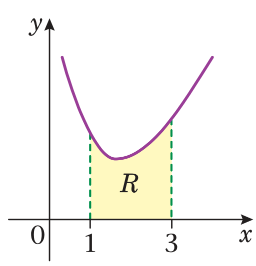
e $\mathrm{f}(x) = \dfrac{8}{x^3} + \sqrt{x};$ $\quad a = 1, b = 4$

2 The sketch shows part of the curve with equation $y = x(x^2 - 4)$.
Find the area of the shaded region.



3 The diagram shows a sketch of the curve with equation $y = 3x + \dfrac{6}{x^2} - 5, x > 0$.

The region R is bounded by the curve, the x-axis and the lines $x = 1$ and $x = 3$.
Find the area of R.



4 Find the area of the finite region between the curve with equation $y = (3 - x)(1 + x)$ and the x-axis.

5 Find the area of the finite region between the curve with equation $y = x(x - 4)^2$ and the x-axis.

6 Find the area of the finite region between the curve with equation $y = x^2(2 - x)$ and the x-axis.

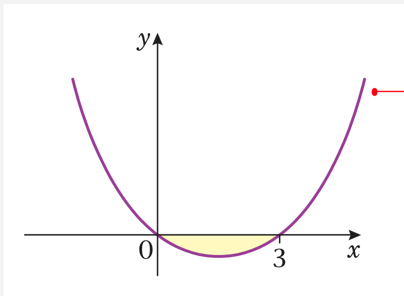
11.3 You need to be able to work out areas of curves under the x-axis.

In the examples so far the area that you were calculating was above the x-axis. If the area between a curve and the x-axis lies below the x-axis, then $\int y \, \mathrm{d}x$ will give a negative answer.

Example 4

Find the area of the finite region bounded by the curve $y = x(x - 3)$ and the x-axis.

When $x = 0, y = 0$
When $y = 0, x = 0$ or 3



First sketch the curve.
It is U-shaped. It crosses the x-axis at 0 and 3.

$$\text{Area} = \int_0^3 x(x-3)\,dx$$

The limits on the integral will therefore be 0 and 3.

$$= \int_0^3 (x^2 - 3x)\,dx$$

Multiply out the bracket.

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3$$

Integrate as usual.

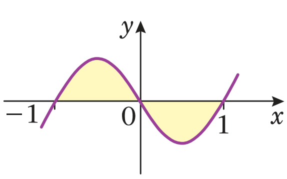
$$= \left(\frac{27}{3} - \frac{27}{2}\right) - (0)$$

$$= -\frac{27}{6} \text{ or } -\frac{9}{2} \text{ or } -4.5$$

So the area is 4.5

State the area as a positive quantity.

The following example shows that great care must be taken if you are trying to find an area which straddles the x-axis such as the shaded region below, bounded by the curve with equation $y = (x + 1)(x - 1)x = x^3 - x$.



Notice that:

$$\int_{-1}^{1} (x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= 0.$$

This is because:

$$\int_0^1 (x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1$$
$$= \left(\frac{1}{4} - \frac{1}{2}\right) - (0) = -\frac{1}{4}$$

and
$$\int_{-1}^0 (x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^0$$
$$= -\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4}$$

So the area of the shaded region is actually $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

For examples of this type you need to draw a sketch, unless one is given in the question.

Example 5

Sketch the curve with equation $y = x(x - 1)(x + 3)$ and find the area of the finite region bounded by the curve and the x-axis.

When $x = 0$, $y = 0$

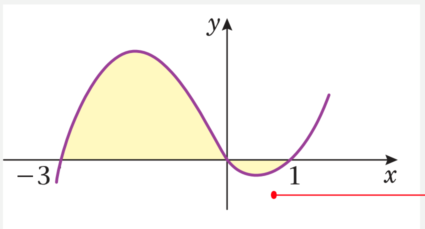
When $y = 0$, $x = 0$, 1 or -3

Find out where the curve intercepts the axes.

$x \to \infty$, $y \to \infty$

$x \to -\infty$, $y \to -\infty$

Find out what happens to y when x is large and positive or large and negative.



Since the area between $x = 0$ and 1 is below the axis the integral between these points will give a negative answer.

The area is given by $\int_{-3}^{0} y\,dx + -\int_0^1 y\,dx$

Now $\int y\,dx = \int (x^3 + 2x^2 - 3x)\,dx$

Multiply out the brackets.

$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]$$

So $\int_{-3}^{0} y\,dx = (0) - \left(\frac{81}{4} - \frac{2}{3} \times 27 - \frac{3}{2} \times 9\right)$

$= 11.25$

and $\int_0^1 y\,dx = \left(\frac{1}{4} + \frac{2}{3} - \frac{3}{2}\right) - (0)$

$= -\frac{7}{12}$

So the area required is $11.25 + \frac{7}{12} = 11\frac{5}{6}$

Exercise 11C

Sketch the following and find the area of the finite region or regions bounded by the curves and the x-axis:

1 $y = x(x + 2)$

2 $y = (x + 1)(x - 4)$

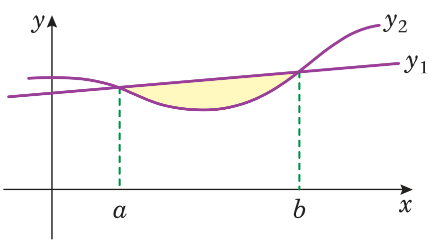
3 $y = (x + 3)x(x - 3)$

4 $y = x^2(x - 2)$

5 $y = x(x - 2)(x - 5)$

11.4 You need to be able to work out the area between a curve and a straight line.

Sometimes you may wish to find an area between a curve and a line. (The method also applies to finding the area between two curves, but this is not required in C2.)



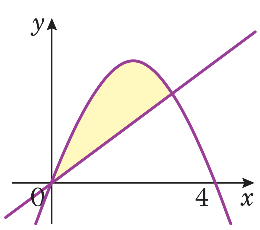
You find the area of the shaded region by calculating $\int_a^b (y_1 - y_2)\,dx$. This is because $\int_a^b y_1\,dx$ gives the area below the line (or curve) with equation y_1, and $\int_a^b y_2\,dx$ gives the area below y_2. So the shaded region is simply $\int_a^b y_1\,dx - \int_a^b y_2\,dx = \int_a^b (y_1 - y_2)\,dx$.

■ **The area between a line (equation y_1) and a curve (equation y_2) is given by**

$$\text{Area} = \int_a^b (y_1 - y_2)\,dx$$

Example 6

The diagram shows a sketch of part of the curve with equation $y = x(4 - x)$ and the line with equation $y = x$.



Find the area of the region bounded by the curve and the line.

Method 1

$$x = x(4 - x)$$
$$x = 4x - x^2$$
$$x^2 - 3x = 0$$
$$x(x - 3) = 0$$

So $x = 0$ or 3

So the line cuts the curve at $(0, 0)$ and $(3, 3)$

The area is given by $\int_0^3 [x(4 - x) - x]\, dx$

$$\text{Area} = \int_0^3 (3x - x^2)\, dx$$
$$= \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3$$
$$= \left(\frac{27}{2} - 9\right) - (0) = 4.5$$

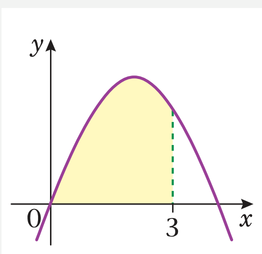
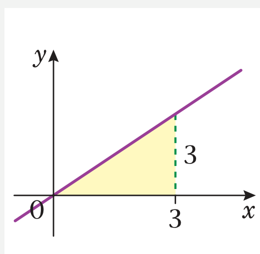
First find where the line and the curve cross.

Find y-coordinates by substituting back in one of the equations. The line $y = x$ is the simplest.

Use the formula with 'curve − line' since the curve is above the line.

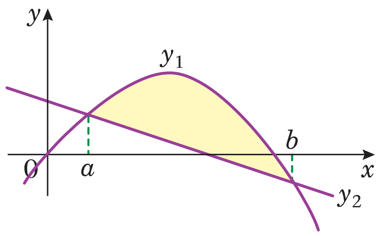
Simplify the expression to be integrated.

Method 2

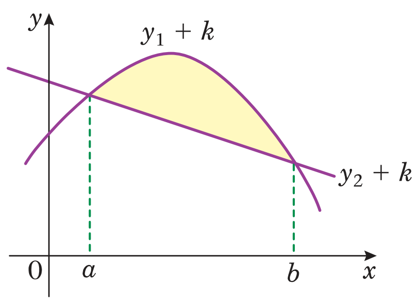
Area beneath curve	minus	Area of triangle
		
$\int_0^3 (4x - x^2)\, dx$	$-$	$\frac{1}{2} \times 3 \times 3$
$= \left[2x^2 - \frac{x^3}{3}\right]_0^3$	$-$	4.5
$= \left(18 - \frac{27}{3}\right) - (0)$	$-$	4.5
$= 9 - 4.5$		
$= 4.5$		

You should notice that you could have found this area by first finding the area beneath the curve between $x = 0$ and $x = 3$, and then subtracting the area of a triangle.

The $\int_a^b (y_1 - y_2)\,dx$ formula can be applied even if part of the region is below the x-axis. Consider the following:



If both the curve and the line are translated upwards by $+k$, where k is sufficiently large to ensure that the required area is totally above the x-axis, the diagram will look like this:



You should notice that since the translation is in the y-direction only, then the x-coordinates of the points of intersection are unchanged and so the limits of the integral will remain the same.

So the area in this case is given by $\int_a^b [y_1 + k - (y_2 + k)]\,dx$

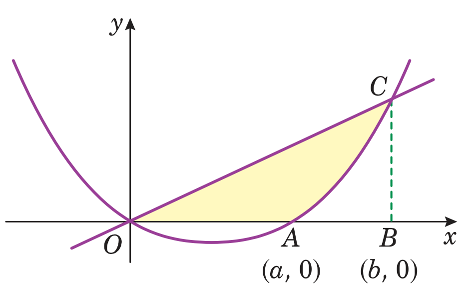
$$= \int_a^b (y_1 - y_2)\,dx$$

Notice that the value of k does not appear in the final formula so you can always use this approach for questions of this type.

Sometimes you will need to add or subtract an area found by integration to the area of a triangle, trapezium or other similar shape as the following example shows.

Example 7

The diagram shows a sketch of the curve with equation $y = x(x - 3)$ and the line with equation $y = 2x$.



Find the area of the shaded region OAC.

The required area is given by:

Area of triangle $OBC - \int_a^b x(x-3)\,dx$

The curve cuts the x-axis at $x = 3$

(and $x = 0$) so $a = 3$

The curve meets the line $y = 2x$ when

$2x = x(x-3)$ — Line = curve.

So $0 = x^2 - 5x$ — Simplify the equation.

$0 = x(x-5)$

$x = 0$ or 5, so $b = 5$

The point C is $(5, 10)$ — $y = 2 \times 5 = 10$, substituting $x = 5$ into the equation of the line.

Area of triangle $OBC = \frac{1}{2} \times 5 \times 10 = 25$

Area between curve, x-axis and the line $x = 5$ is

$$\int_3^5 x(x-3)\,dx = \int_3^5 (x^2 - 3x)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_3^5$$

$$= \left(\frac{125}{3} - \frac{75}{2}\right) - \left(\frac{27}{3} - \frac{27}{2}\right)$$

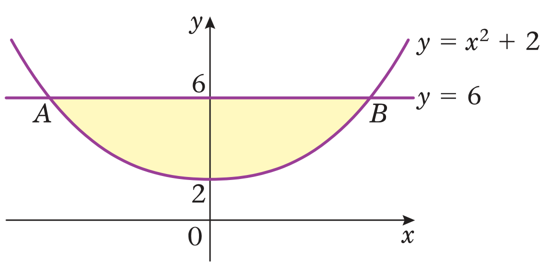
$$= \left(\frac{25}{6}\right) - \left(-\frac{27}{6}\right)$$

$$= \frac{52}{6} \text{ or } \frac{26}{3}$$

Shaded region is therefore $= 25 - \frac{26}{3} = \frac{49}{3}$ or $16\frac{1}{3}$

Exercise 11D

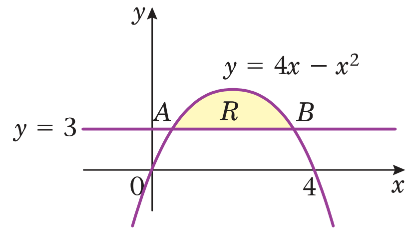
1 The diagram shows part of the curve with equation $y = x^2 + 2$ and the line with equation $y = 6$. The line cuts the curve at the points A and B.



a Find the coordinates of the points A and B.

b Find the area of the finite region bounded by AB and the curve.

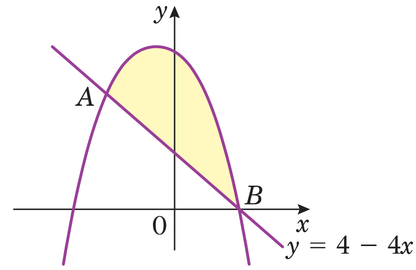
2 The diagram shows the finite region, R, bounded by the curve with equation $y = 4x - x^2$ and the line $y = 3$. The line cuts the curve at the points A and B.



a Find the coordinates of the points A and B.

b Find the area of R.

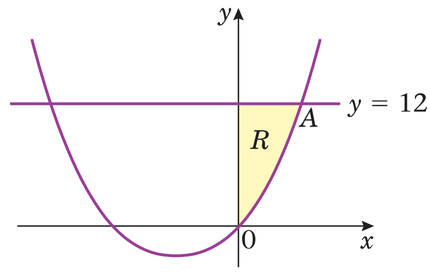
3 The diagram shows a sketch of part of the curve with equation $y = 9 - 3x - 5x^2 - x^3$ and the line with equation $y = 4 - 4x$. The line cuts the curve at the points A $(-1, 8)$ and B $(1, 0)$.



Find the area of the shaded region between AB and the curve.

4 Find the area of the finite region bounded by the curve with equation $y = (1 - x)(x + 3)$ and the line $y = x + 3$.

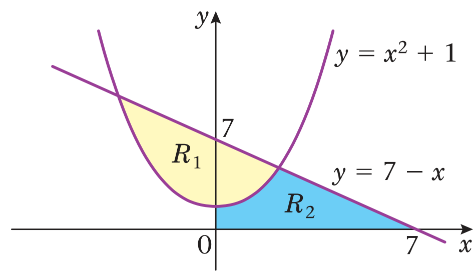
5 The diagram shows the finite region, R, bounded by the curve with equation $y = x(4 + x)$, the line with equation $y = 12$ and the y-axis.



a Find the coordinate of the point A where the line meets the curve.

b Find the area of R.

6 The diagram shows a sketch of part of the curve with equation $y = x^2 + 1$ and the line with equation $y = 7 - x$. The finite region R_1 is bounded by the line and the curve. The finite region R_2 is below the curve and the line and is bounded by the positive x- and y-axes as shown in the diagram.



a Find the area of R_1.

b Find the area of R_2.

7 The curve C has equation $y = x^{\frac{2}{3}} - \dfrac{2}{x^{\frac{1}{3}}} + 1$.

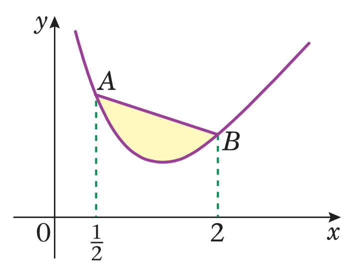
a Verify that C crosses the x-axis at the point $(1, 0)$.

b Show that the point A $(8, 4)$ also lies on C.

c The point B is $(4, 0)$. Find the equation of the line through AB.
The finite region R is bounded by C, AB and the positive x-axis.

d Find the area of R.

8 The diagram shows part of a sketch of the curve with equation $y = \dfrac{2}{x^2} + x$. The points A and B have x-coordinates $\frac{1}{2}$ and 2 respectively.

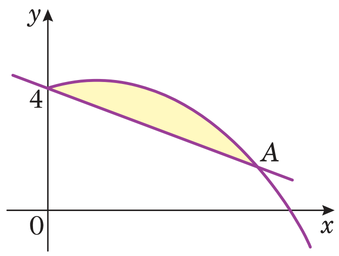


Find the area of the finite region between AB and the curve.

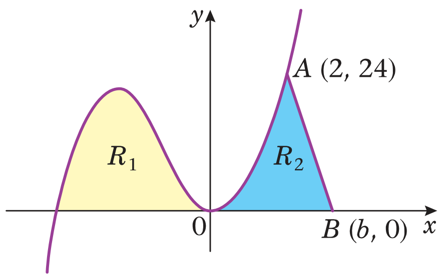
9 The diagram shows part of the curve with equation $y = 3\sqrt{x} - \sqrt{x^3} + 4$ and the line with equation $y = 4 - \frac{1}{2}x$.

a Verify that the line and the curve cross at the point A (4, 2).

b Find the area of the finite region bounded by the curve and the line.



10 The sketch shows part of the curve with equation $y = x^2(x + 4)$. The finite region R_1 is bounded by the curve and the negative x-axis. The finite region R_2 is bounded by the curve, the positive x-axis and AB, where A (2, 24) and B (b, 0).



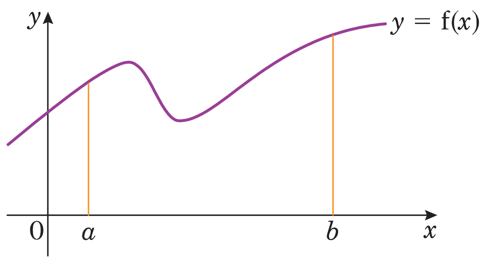
The area of R_1 = the area of R_2.

a Find the area of R_1.

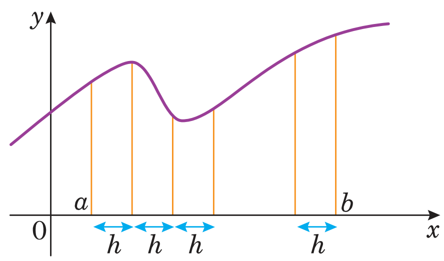
b Find the value of b.

11.5 Sometimes you may want to find the area beneath a curve but you may not be able to integrate the equation. You can find an approximation to the area using the trapezium rule.

Consider the curve $y = f(x)$:

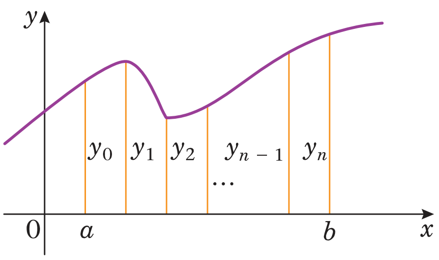


To find the area given by $\int_a^b y\,dx$, we divide the area up into n equal strips. Each strip will be of width h, so $h = \frac{b-a}{n}$.



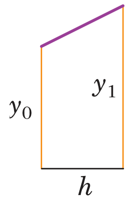
Next we calculate the value of y for each value of x that forms a boundary of one of the strips. So we find y for $x = a$, $x = a + h$, $x = a + 2h$, $x = a + 3h$ and so on up to $x = b$. We can label these values $y_0, y_1, y_2, y_3, \ldots, y_n$.

Hint: Notice that for n strips there will be $n + 1$ values of x and $n + 1$ values of y.



Finally we join adjacent points to form n trapeziums and approximate the original area by the sum of the areas of these n trapeziums.

You may recall from GCSE maths that the area of a trapezium like this:



is given by $\frac{1}{2}(y_0 + y_1)h$. The required area under the curve is therefore given by:

$$\int_a^b y\,\mathrm{d}x \approx \tfrac{1}{2}h(y_0 + y_1) + \tfrac{1}{2}h(y_1 + y_2) + \ldots + \tfrac{1}{2}h(y_{n-1} + y_n)$$

Factorising gives:

$$\int_a^b y\,\mathrm{d}x \approx \tfrac{1}{2}h(y_0 + y_1 + y_1 + y_2 + y_2 \ldots + y_{n-1} + y_{n-1} + y_n)$$

or $$\int_a^b y\,\mathrm{d}x \approx \tfrac{1}{2}h[y_0 + 2(y_1 + y_2 \ldots + y_{n-1}) + y_n]$$

This formula is given in the Edexcel formula booklet but you will need to know how to use it.

■ **The trapezium rule:**

$$\int_a^b y\,\mathrm{d}x \approx \tfrac{1}{2}h[y_0 + 2(y_1 + y_2 \ldots + y_{n-1}) + y_n]$$

where $h = \dfrac{b-a}{n}$ and $y_i = \mathrm{f}(a + ih)$.

Hint: You do not need to remember how to develop this result.

Example 8

Use the trapezium rule with
a 4 strips **b** 8 strips
to estimate the area under the curve with equation $y = \sqrt{(2x + 3)}$ between the lines $x = 0$ and $x = 2$.

a Each strip will have width $\dfrac{2-0}{4} = 0.5$.

x	0	0.5	1	1.5	2
$y = \sqrt{(2x+3)}$	1.732	2	2.236	2.449	2.646

First work out the value of y at the boundaries of each of your strips.

It is sometimes helpful to put your working in a table.

So area $\approx \dfrac{1}{2} \times 0.5 \times [1.732 + 2(2 + 2.236 + 2.449) + 2.646]$

$= \dfrac{1}{2} \times 0.5 \times [17.748]$

$= 4.437$ or 4.44

b Each strip will be of width $\frac{2-0}{8} = 0.25$.

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
y	1.732	1.871	2	2.121	2.236	2.345	2.449	2.550	2.646

So area $\approx \frac{1}{2} \times 0.25 \times [1.732 + 2(1.871 + 2 + 2.121 + 2.236 + 2.345 + 2.449 + 2.550) + 2.646]$

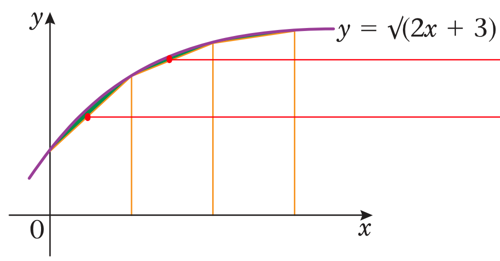
$= \frac{1}{2} \times 0.25 \times [35.522]$

$= 4.440\,25$ or 4.44 (2 d.p.)

Values of y.

The actual area in this case is 4.441 368 … and you can see (if you look at the calculations to 3 d.p.) in the above example that increasing the number of strips (or reducing their width) should improve the accuracy of the approximation.

A sketch of $y = \sqrt{(2x + 3)}$ looks like this:



Hint: The area is missed by the trapezium.

You can see that the trapezium rule will always underestimate the area since the curve bends 'outwards'.

Graphical calculators can be used to evaluate definite integrals. Calculators usually use a slightly different method from the trapezium rule to carry out these calculations and they will generally be more accurate. So, although the calculator can provide a useful check, you should remember that the trapezium rule is being used to *estimate* the value and you should not expect this estimate to be the same as the answer from a graphical calculator.

Exercise 11E

1 Copy and complete the table below and use the trapezium rule to estimate $\int_1^3 \frac{1}{x^2+1}\,dx$:

x	1	1.5	2	2.5	3
$y = \frac{1}{x^2+1}$	0.5	0.308		0.138	

2 Use the table below to estimate $\int_1^{2.5} \sqrt{(2x-1)}\,dx$ with the trapezium rule:

x	1	1.25	1.5	1.75	2	2.25	2.5
$y = \sqrt{(2x-1)}$	1	1.225	1.414	1.581	1.732	1.871	2

3 Copy and complete the table below and use it, together with the trapezium rule, to estimate $\int_0^2 \sqrt{(x^3+1)}\,dx$:

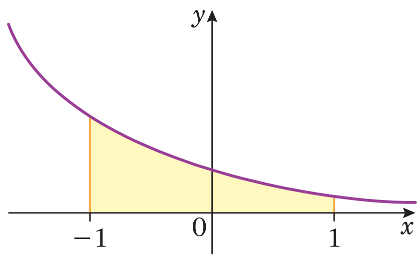
x	0	0.5	1	1.5	2
$y = \sqrt{(x^3+1)}$	1	1.061	1.414		

4 **a** Use the trapezium rule with 8 strips to estimate $\int_0^2 2^x\,dx$.

b With reference to a sketch of $y = 2^x$ explain whether your answer in part **a** is an underestimate or an overestimate of $\int_0^2 2^x\,dx$.

5 Use the trapezium rule with 6 strips to estimate $\int_0^3 \frac{1}{\sqrt{(x^2+1)}}\,dx$.

6 The diagram shows a sketch of part of the curve with equation $y = \frac{1}{x+2}$, $x > -2$.



a Copy and complete the table below and use the trapezium rule to estimate the area bounded by the curve, the x-axis and the lines $x = -1$ and $x = 1$.

x	−1	−0.6	−0.2	0.2	0.6	1
$y = \frac{1}{x+2}$	1	0.714			0.385	0.333

b State, with a reason, whether your answer in part **a** is an overestimate or an underestimate.

7 **a** Sketch the curve with equation $y = x^3 + 1$, for $-2 < x < 2$.

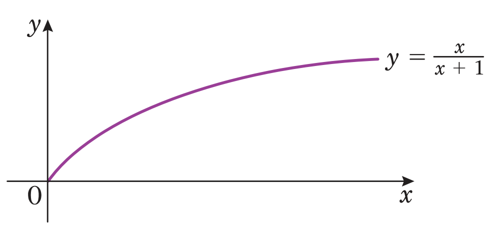
b Use the trapezium rule with 4 strips to estimate the value of $\int_{-1}^1 (x^3+1)\,dx$.

c Use integration to find the exact value of $\int_{-1}^1 (x^3+1)\,dx$.

d Comment on your answers to parts **b** and **c**.

8 Use the trapezium rule with 4 strips to estimate $\int_0^2 \sqrt{(3^x - 1)}\,dx$.

9 The sketch shows part of the curve with equation $y = \frac{x}{x+1}$, $x \geqslant 0$.



a Use the trapezium rule with 6 strips to estimate $\int_0^3 \frac{x}{x+1}\,dx$.

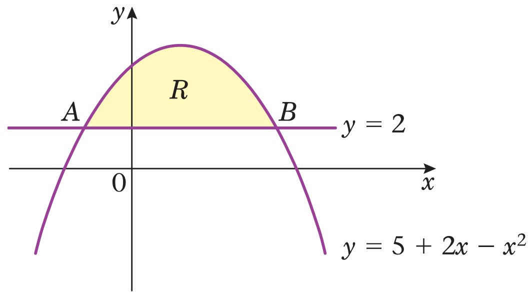
b With reference to the sketch state, with a reason, whether the answer in part **a** is an overestimate or an underestimate.

10 **a** Use the trapezium rule with n strips to estimate $\int_0^2 \sqrt{x}\,dx$ in the cases **i** $n = 4$ **ii** $n = 6$.

b Compare your answers from part **a** with the exact value of the integral and calculate the percentage error in each case.

Mixed exercise 11F

1 The diagram shows the curve with equation $y = 5 + 2x - x^2$ and the line with equation $y = 2$. The curve and the line intersect at the points A and B.

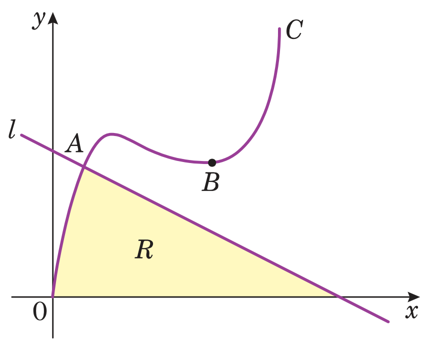


a Find the x-coordinates of A and B.

b The shaded region R is bounded by the curve and the line. Find the area of R.

E

2 The diagram shows part of the curve C with equation $y = x^3 - 9x^2 + px$, where p is a constant. The line l has equation $y + 2x = q$, where q is a constant. The point A is the intersection of C and l, and C has a minimum at the point B. The x-coordinates of A and B are 1 and 4 respectively.

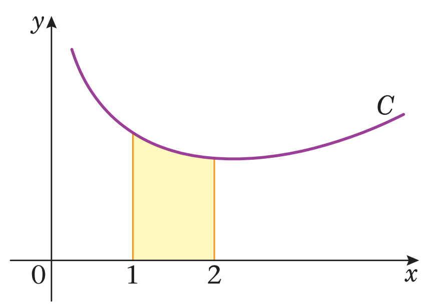


a Show that $p = 24$ and calculate the value of q.

b The shaded region R is bounded by C, l and the x-axis. Using calculus, showing all the steps in your working and using the values of p and q found in part **a**, find the area of R.

E

3 The diagram shows part of the curve C with equation $y = f(x)$, where $f(x) = 16x^{-\frac{1}{2}} + x^{\frac{3}{2}}$, $x > 0$.



a Use calculus to find the x-coordinate of the minimum point of C, giving your answer in the form $k\sqrt{3}$, where k is an exact fraction.

The shaded region shown in the diagram is bounded by C, the x-axis and the lines with equations $x = 1$ and $x = 2$.

b Using integration and showing all your working, find the area of the shaded region, giving your answer in the form $a + b\sqrt{2}$, where a and b are exact fractions.

E

4 **a** Find $\int (x^{\frac{1}{2}} - 4)(x^{-\frac{1}{2}} - 1)\,dx$.

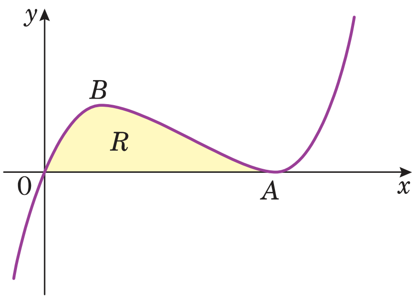
b Use your answer to part **a** to evaluate

$$\int_1^4 (x^{\frac{1}{2}} - 4)(x^{-\frac{1}{2}} - 1)\,dx.$$

giving your answer as an exact fraction.

E

5 The diagram shows part of the curve with equation $y = x^3 - 6x^2 + 9x$. The curve touches the x-axis at A and has a maximum turning point at B.



a Show that the equation of the curve may be written as $y = x(x - 3)^2$, and hence write down the coordinates of A.

b Find the coordinates of B.

c The shaded region R is bounded by the curve and the x-axis. Find the area of R.

E

6 Given that $y^{\frac{1}{2}} = x^{\frac{1}{3}} + 3$:

a Show that $y = x^{\frac{2}{3}} + Ax^{\frac{1}{3}} + B$, where A and B are constants to be found.

b Hence find $\int y\,dx$.

c Using your answer from part **b**, determine the exact value of $\int_1^8 y\,dx$.

E

7 Considering the function $y = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$, $x > 0$:

a Find $\frac{dy}{dx}$.

b Find $\int y\,dx$.

c Hence show that $\int_1^3 y\,dx = A + B\sqrt{3}$, where A and B are integers to be found.

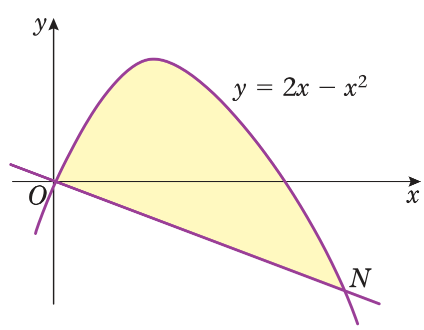
E

8 The diagram shows a sketch of the curve with equation $y = 2x - x^2$ and the line ON which is the normal to the curve at the origin O.

a Find an equation of ON.

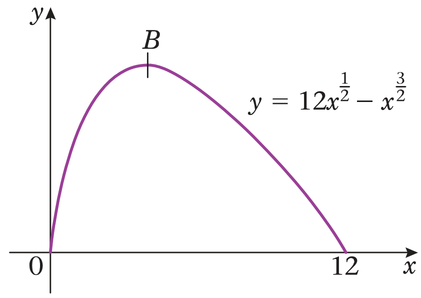
b Show that the x-coordinate of the point N is $2\frac{1}{2}$ and determine its y-coordinate.

c The shaded region shown is bounded by the curve and the line ON. Without using a calculator, determine the area of the shaded region.



E

9 The diagram shows a sketch of the curve with equation $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$ for $0 \leqslant x \leqslant 12$.



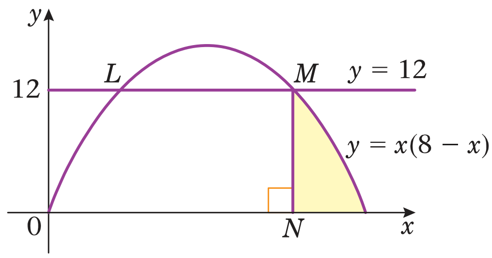
a Show that $\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$.

b At the point B on the curve the tangent to the curve is parallel to the x-axis. Find the coordinates of the point B.

c Find, to 3 significant figures, the area of the finite region bounded by the curve and the x-axis.

E

10 The diagram shows the curve C with equation $y = x(8 - x)$ and the line with equation $y = 12$ which meet at the points L and M.

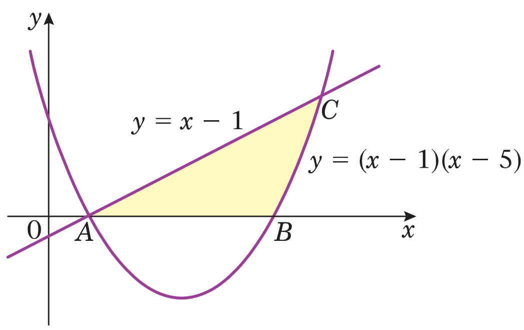


a Determine the coordinates of the point M.

b Given that N is the foot of the perpendicular from M on to the x-axis, calculate the area of the shaded region which is bounded by NM, the curve C and the x-axis.

E

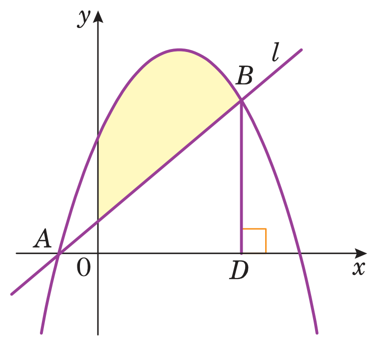
11 The diagram shows the line $y = x - 1$ meeting the curve with equation $y = (x - 1)(x - 5)$ at A and C. The curve meets the x-axis at A and B.



a Write down the coordinates of A and B and find the coordinates of C.

b Find the area of the shaded region bounded by the line, the curve and the x-axis.

12 A and B are two points which lie on the curve C, with equation $y = -x^2 + 5x + 6$. The diagram shows C and the line l passing through A and B.



a Calculate the gradient of C at the point where $x = 2$.

The line l passes through the point with coordinates $(2, 3)$ and is parallel to the tangent to C at the point where $x = 2$.

b Find an equation of l.

c Find the coordinates of A and B.

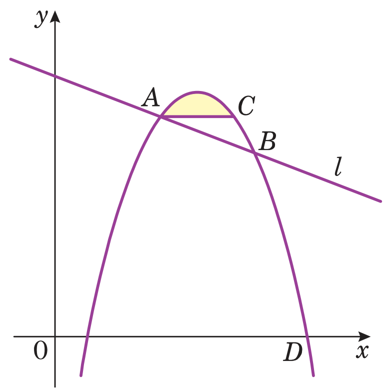
The point D is the foot of the perpendicular from B on to the x-axis.

d Find the area of the region bounded by C, the x-axis, the y-axis and BD.

e Hence find the area of the shaded region.

E

13 The diagram shows part of the curve with equation $y = p + 10x - x^2$, where p is a constant, and part of the line l with equation $y = qx + 25$, where q is a constant. The line l cuts the curve at the points A and B. The x-coordinates of A and B are 4 and 8 respectively. The line through A parallel to the x-axis intersects the curve again at the point C.



a Show that $p = -7$ and calculate the value of q.

b Calculate the coordinates of C.

c The shaded region in the diagram is bounded by the curve and the line AC. Using algebraic integration and showing all your working, calculate the area of the shaded region.

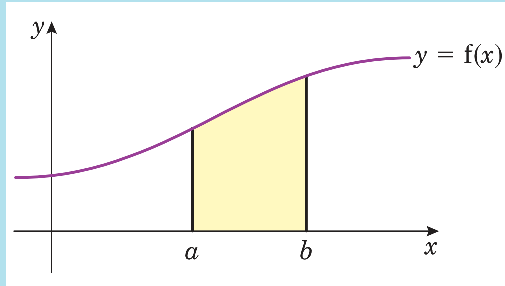
E

Summary of key points

1 The definite integral $\int_a^b f'(x)\,dx = f(b) - f(a)$.

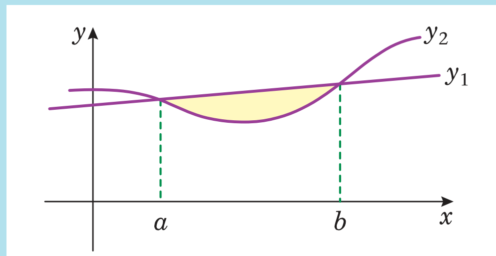
2 The area beneath the curve with equation $y = f(x)$ and between the lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x)\,dx$$



3 The area between a line (equation y_1) and a curve (equation y_2) is given by

$$\text{Area} = \int_a^b (y_1 - y_2)\,dx$$



4 **Trapezium rule (in the formula booklet):**

$$\int_a^b y\,dx \approx \tfrac{1}{2}h[y_0 + 2(y_1 + y_2 \ldots + y_{n-1}) + y_n]$$

where $h = \dfrac{b-a}{n}$ and $y_i = f(a + ih)$.