2 The sine and cosine rule

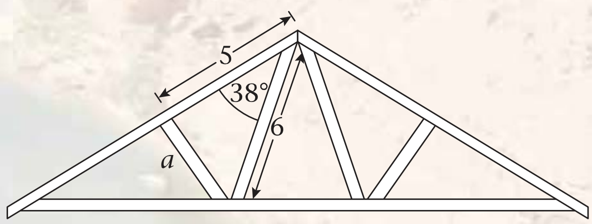
After completing this chapter you should be able to

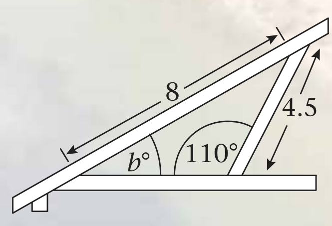
1. use the sine rule to find a missing side
2. use the sine rule to find a missing angle
3. use the cosine rule to find a missing side
4. use the cosine rule to find a missing angle
5. solve problems using combinations of the above and possibly Pythagoras' Theorem
6. find the area of a triangle using an appropriate formula.

Did you know?

...structural engineers use the sine and cosine rules to calculate the size of the trusses in bespoke pieces of engineering.

You can calculate a using the cosine rule and $b°$ using the sine rule.

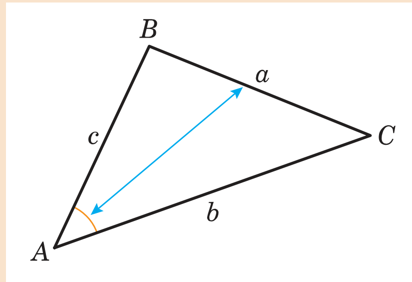




2.1 **The sine rule is:**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



Hint: Note that side *a* is opposite angle *A*.

You can use the sine rule to find an unknown length when you know two angles and one of the opposite sides.

- **When you are finding the length of a side use:**

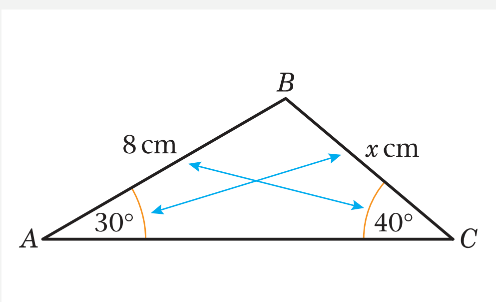
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

or $$\frac{a}{\sin A} = \frac{c}{\sin C}$$

or $$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Example 1

In $\triangle ABC$, $AB = 8$ cm, $\angle BAC = 30°$ and $\angle BCA = 40°$. Find BC.



Always draw a diagram and carefully add the data. Here $c = 8$ (cm), $C = 40°$, $A = 30°$, $a = x$ (cm)

In a triangle, the larger a side is, the larger the opposite angle is. Here, as $C > A$, then $c > a$, so you know that $8 > x$.

$$\frac{x}{\sin 30°} = \frac{8}{\sin 40°}$$

Using the sine rule, $\frac{a}{\sin A} = \frac{c}{\sin C}$.

$$\text{So} \quad x = \frac{8 \sin 30°}{\sin 40°}$$

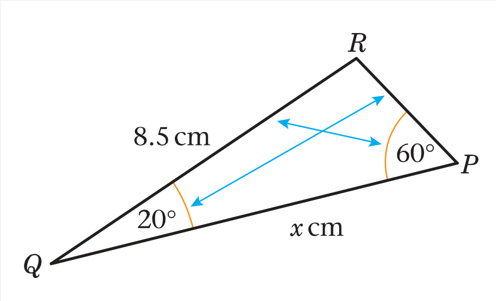
Multiply throughout by sin 30°.

$$= 6.22$$

Give answer to 3 significant figures.

Example 2

In $\triangle PQR$, $QR = 8.5$ cm, $\angle QPR = 60°$ and $\angle PQR = 20°$. Find PQ.



Here $p = 8.5$, $P = 60°$, $Q = 20°$, $r = x$.
To work out PQ you need $\angle R$.
$R = 180° - (60° + 20°) = 100°$. (Angles in a triangle)
As $100° > 60°$, you know that $x > 8.5$.

$$\frac{x}{\sin 100°} = \frac{8.5}{\sin 60°}$$

Using the sine rule, $\frac{r}{\sin R} = \frac{p}{\sin P}$.

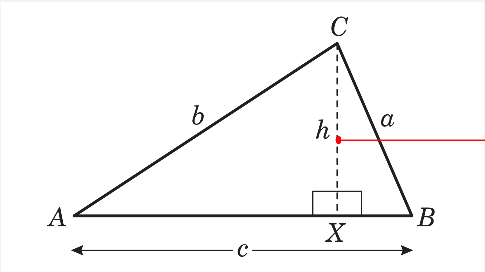
$$\text{So} \quad x = \frac{8.5 \sin 100°}{\sin 60°}$$

Multiply throughout by $\sin 100°$.

$$x = 9.67$$

Example 3

Prove the sine rule for a general triangle ABC.



In a general triangle ABC, draw the perpendicular from C to AB.

It meets AB at X.

The length of CX is h.

$$\sin B = \frac{h}{a} \Rightarrow h = a \sin B$$

Use the sine ratio in triangle CBX.

$$\text{and} \quad \sin A = \frac{h}{b} \Rightarrow h = b \sin A$$

Use the sine ratio in triangle CAX.

So $a \sin B = b \sin A$

$$\text{So} \quad \frac{a}{\sin A} = \frac{b}{\sin B}$$

Divide throughout by $\sin A \sin B$.

In a similar way, by drawing the perpendicular from B to the side AC, you can show that:

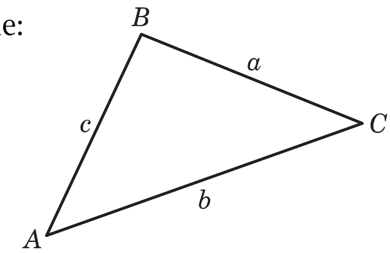
$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\text{So} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This is the sine rule and is true for all triangles.

Exercise 2A (Give answers to 3 significant figures.)

1 In each of parts **a** to **d**, given values refer to the general triangle:



a Given that $a = 8$ cm, $A = 30°$, $B = 72°$, find b.

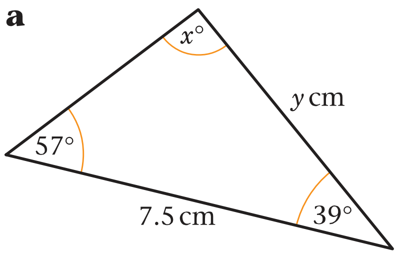
b Given that $a = 24$ cm, $A = 110°$, $C = 22°$, find c.

c Given that $b = 14.7$ cm, $A = 30°$, $C = 95°$, find a.

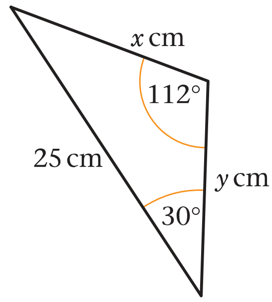
d Given that $c = 9.8$ cm, $B = 68.4°$, $C = 83.7°$, find a.

2 In each of the following triangles calculate the values of x and y.

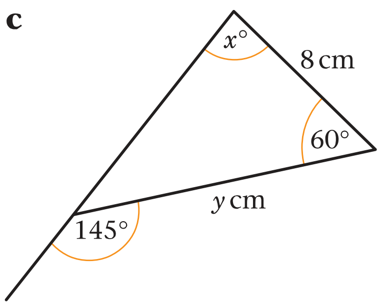
a



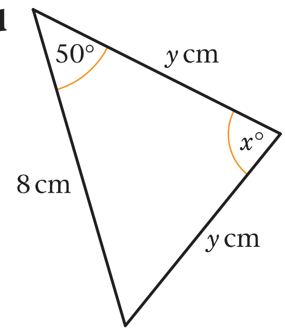
b



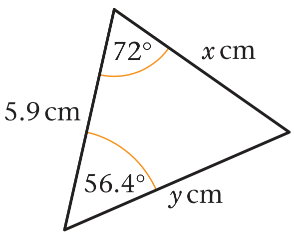
c



d



e



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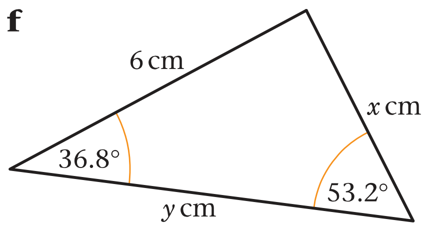
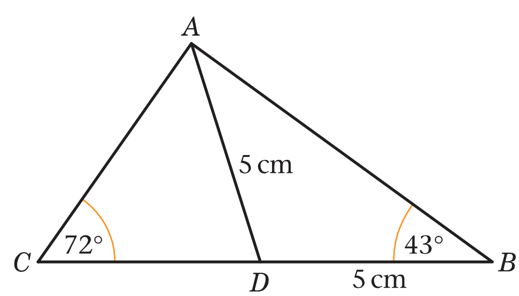


3 In $\triangle PQR$, $QR = \sqrt{3}$ cm, $\angle PQR = 45°$ and $\angle QPR = 60°$. Find **a** PR and **b** PQ.

4 Town B is 6 km, on a bearing of 020°, from town A. Town C is located on a bearing of 055° from town A and on a bearing of 120° from town B. Work out the distance of town C from **a** town A and **b** town B.

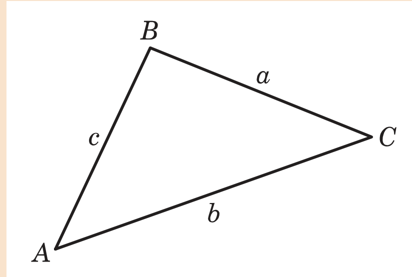
5 In the diagram $AD = DB = 5$ cm, $\angle ABC = 43°$ and $\angle ACB = 72°$.
Calculate **a** AB and **b** CD.



2.2 **You can use the sine rule to find an unknown angle in a triangle when you know two sides and one of their opposite angles.**

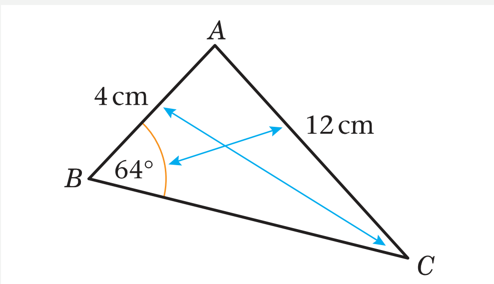
When you are finding an angle use:

$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin C}{c} \quad \text{or} \quad \frac{\sin B}{b} = \frac{\sin C}{c}$$



Example 4

In $\triangle ABC$, $AB = 4$ cm, $AC = 12$ cm and $\angle ABC = 64°$. Find $\angle ACB$.



$$\frac{\sin C}{4} = \frac{\sin 64°}{12}$$

Here $b = 12$ cm, $c = 4$ cm, $B = 64°$.

As you need to find angle C, use the sine rule $\frac{\sin C}{c} = \frac{\sin B}{b}$.

$$\text{So} \quad \sin C = \frac{4 \sin 64°}{12}$$

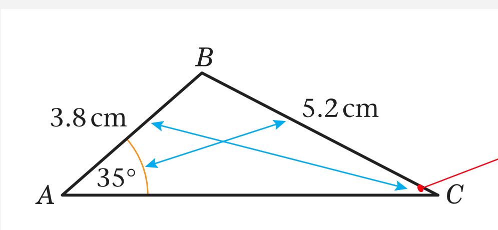
As $4 < 12$, you know that $C < 64°$.

$$C = 17.4°$$

$C = \sin^{-1}\left(\frac{4 \sin 64°}{12}\right)$.

Example 5

In $\triangle ABC$, $AB = 3.8$ cm, $BC = 5.2$ cm and $\angle BAC = 35°$. Find $\angle ABC$.



Here $a = 5.2$ cm, $c = 3.8$ cm and $A = 35°$. You first need to find angle C.

$$\frac{\sin C}{3.8} = \frac{\sin 35°}{5.2}$$

Use $\frac{\sin C}{c} = \frac{\sin A}{a}$.

$$\text{So} \quad \sin C = \frac{3.8 \sin 35°}{5.2}$$

$$C = 24.8°$$

You know that $C < 35°$.

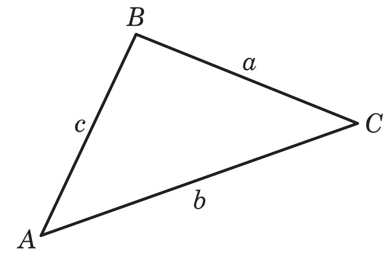
$$\text{So} \quad B = 120°$$

$B = 180° - (24.8° + 35°) = 120.2°$, which rounds to 120° to 3 significant figures.

Exercise 2B

(*Note:* Give answers to 3 significant figures, unless they are exact.)

1 In each of the following sets of data for a triangle ABC, find the value of x:



a $AB = 6$ cm, $BC = 9$ cm, $\angle BAC = 117°$, $\angle ACB = x°$.

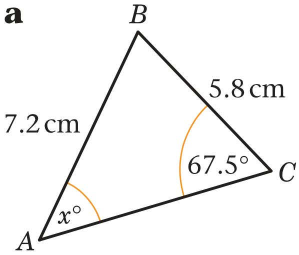
b $AC = 11$ cm, $BC = 10$ cm, $\angle ABC = 40°$, $\angle CAB = x°$.

c $AB = 6$ cm, $BC = 8$ cm, $\angle BAC = 60°$, $\angle ACB = x°$.

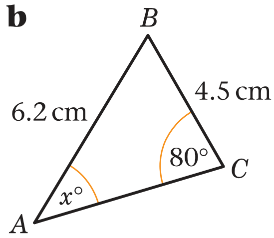
d $AB = 8.7$ cm, $AC = 10.8$ cm, $\angle ABC = 28°$, $\angle BAC = x°$.

2 In each of the diagrams shown below, work out the value of x:

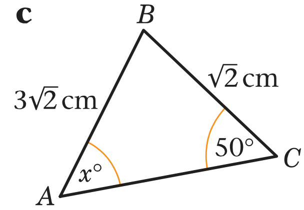
a



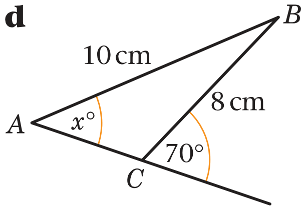
b



c



d



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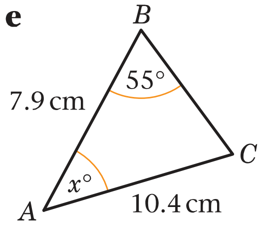

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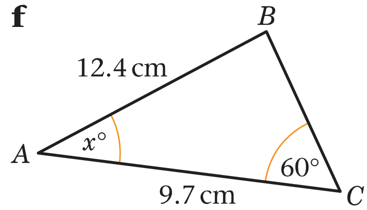
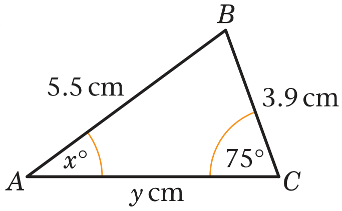


3 In $\triangle PQR$, $PQ = 15$ cm, $QR = 12$ cm and $\angle PRQ = 75°$. Find the two remaining angles.

4 In each of the following diagrams work out the values of x and y:

a



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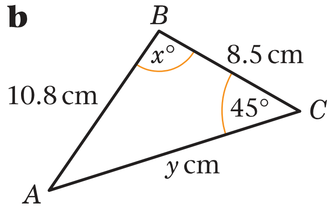

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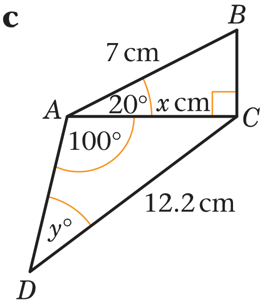

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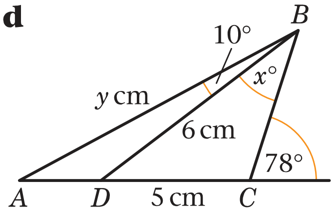

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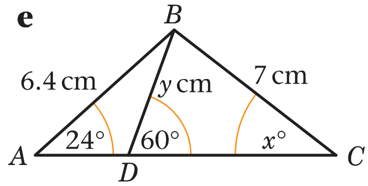

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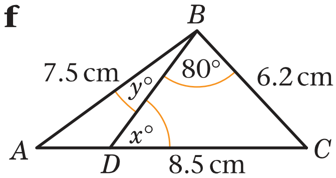


5 In $\triangle ABC$, $AB = x$ cm, $BC = (4 - x)$ cm, $\angle BAC = y°$ and $\angle BCA = 30°$.

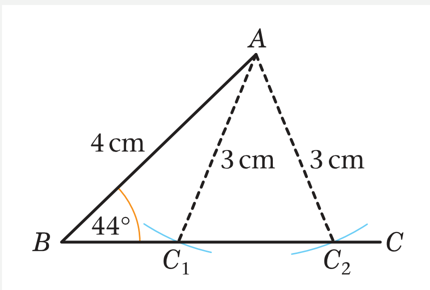
Given that $\sin y° = \dfrac{1}{\sqrt{2}}$, show that $x = 4(\sqrt{2} - 1)$.

2.3 You can sometimes find two solutions for a missing angle.

- **When the angle you are finding is larger than the given angle, there are two possible results. This is because you can draw two possible triangles with the data.**
- **In general, $\sin(180 - x)^\circ = \sin x^\circ$. For example $\sin 30^\circ = \sin 150^\circ$.**

Example 6

In $\triangle ABC$, $AB = 4$ cm, $AC = 3$ cm and $\angle ABC = 44^\circ$. Work out the two possible values of $\angle ACB$.



Here $\angle ACB > \angle ABC$, as $AB > AC$, and so there will be two possible results. The diagram shows why.
With $\angle ABC = 44^\circ$ and $AB = 4$ cm drawn, imagine putting a pair of compasses at A, then drawing an arc with centre A and radius 3 cm. This will intersect BC at C_1 and C_2 showing that there are two triangles ABC_1 and ABC_2 where $b = 3$ cm, $c = 4$ cm and $B = 44^\circ$.
(This would not happen if $AC > 4$ cm.)

$$\frac{\sin C}{4} = \frac{\sin 44^\circ}{3}$$

Use $\frac{\sin C}{c} = \frac{\sin B}{b}$, where $b = 3$, $c = 4$, $B = 44^\circ$.

$$\sin C = \frac{4 \sin 44^\circ}{3}$$

So $\quad C = 67.9^\circ$

This is the value your calculator will give to 3 s.f., which corresponds to $\triangle ABC_2$.

Or $\quad C = 112^\circ$ (3 s.f.)

As $\sin(180 - x)^\circ = \sin x^\circ$, $C = 180 - 67.9^\circ = 112.1^\circ$ is another possible answer. This corresponds to $\triangle ABC_1$.

Exercise 2C

(Give answers to 3 significant figures.)

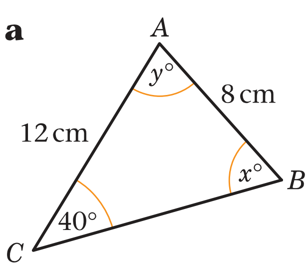
1 In $\triangle ABC$, $BC = 6$ cm, $AC = 4.5$ cm and $\angle ABC = 45^\circ$:

a Calculate the two possible values of $\angle BAC$.

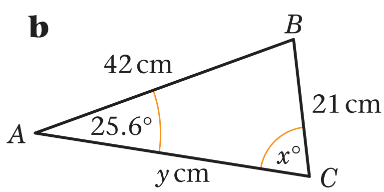
b Draw a diagram to illustrate your answers.

2 In each of the diagrams shown below, calculate the possible values of x and the corresponding values of y:

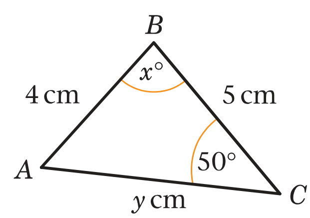
a



b



c



3 In each of the following cases $\triangle ABC$ has $\angle ABC = 30°$ and $AB = 10$ cm:

a Calculate the least possible length that AC could be.

b Given that $AC = 12$ cm, calculate $\angle ACB$.

c Given instead that $AC = 7$ cm, calculate the two possible values of $\angle ACB$.

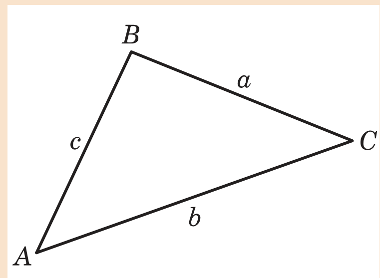
4 Triangle ABC is such that $AB = 4$ cm, $BC = 6$ cm and $\angle ACB = 36°$. Show that one of the possible values of $\angle ABC$ is 25.8° (to 3 s.f.). Using this value, calculate the length of AC.

5 Two triangles ABC are such that $AB = 4.5$ cm, $BC = 6.8$ cm and $\angle ACB = 30°$. Work out the value of the largest angle in each of the triangles.

2.4 **The cosine rule is:**

$$a^2 = b^2 + c^2 - 2bc\cos A \text{ or } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

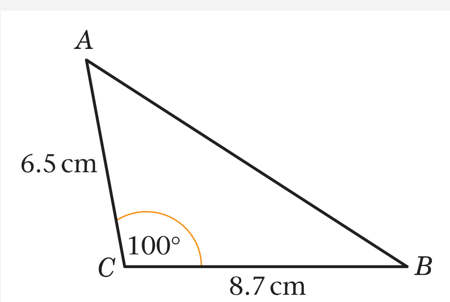
You can use the cosine rule to find an unknown side in a triangle when you know the lengths of two sides and the size of the angle between the sides.



- **In a question use $a^2 = b^2 + c^2 - 2bc\cos A$ to find a, given b, c and A.**
  or $b^2 = a^2 + c^2 - 2ac\cos B$ to find b, given a, c and B.
  or $c^2 = a^2 + b^2 - 2ab\cos C$ to find c, given a, b and C.

Example 7

Calculate the length of the side AB of the triangle ABC in which $AC = 6.5$ cm, $BC = 8.7$ cm and $\angle ACB = 100°$.



You have been given a, b and angle C, so use the cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$ to find c.

$c^2 = 8.7^2 + 6.5^2 - 2 \times 8.7 \times 6.5 \times \cos 100°$

Carefully set out this line of working before using your calculator.

$= 75.69 + 42.25 + 19.64$

This line may be omitted.

$= 137.58$

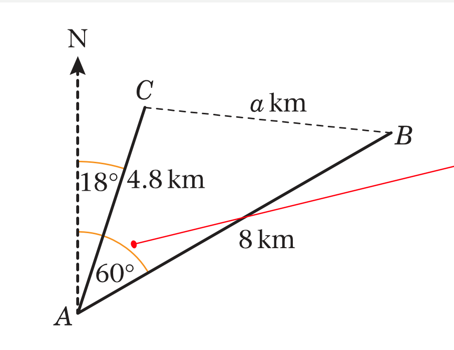
So $c = 11.729\ldots$

Find the square root.

So $AB = 11.7$ cm (3 s.f.)

Example 8

Coastguard station B is 8 km, on a bearing of 060°, from coastguard station A. A ship C is 4.8 km, on a bearing of 018°, away from A.
Calculate how far C is from B.



$a^2 = 4.8^2 + 8^2 - 2 \times 4.8 \times 8 \times \cos 42°$

$a = 5.47$

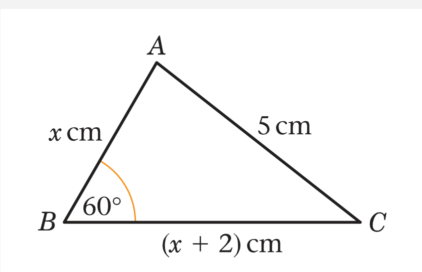
C is 5.47 km away from coastguard B.

Carefully transfer the given data to a diagram.
In $\triangle ABC$, $\angle CAB = 60° - 18° = 42°$.

You now have $b = 4.8$ km, $c = 8$ km and $A = 42°$.
Use the cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$.

Example 9

In $\triangle ABC$, $AB = x$ cm, $BC = (x + 2)$ cm, $AC = 5$ cm and $\angle ABC = 60°$.
Find the value of x.



$5^2 = (x + 2)^2 + x^2 - 2x(x + 2)\cos 60°$

So $25 = 2x^2 + 4x + 4 - x^2 - 2x$

So $x^2 + 2x - 21 = 0$

$x = \dfrac{-2 \pm \sqrt{88}}{2}$

$= 3.69$

The given data here are
$a = (x + 2)$, $c = x$, $b = 5$, $B = 60°$.

The sine rule cannot be used, but you can use $b^2 = a^2 + c^2 - 2ac \cos B$.

$(x + 2)^2 = x^2 + 4x + 4$; $\cos 60° = \frac{1}{2}$.

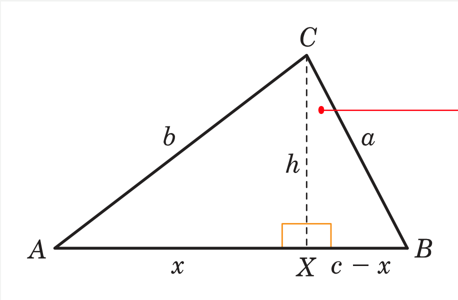
Rearrange to the form $ax^2 + bx + c = 0$.

Use the quadratic equation formula, where $b^2 - 4ac = 2^2 - 4(1)(-21) = 4 + 84 = 88$.

As $AB = x$ cm, x cannot be negative.

Example 10

Prove the cosine rule for a general triangle *ABC*.



The perpendicular from *C* to side *AB* is drawn and it meets *AB* at *X*.

The length of *CX* is *h*.

The length of *AX* is *x*, so $BX = c - x$.

$$h^2 + x^2 = b^2$$

Use Pythagoras' theorem in $\triangle CAX$.

and $h^2 + (c - x)^2 = a^2$

Use Pythagoras' theorem in $\triangle CBX$.

So $x^2 - (c - x)^2 = b^2 - a^2$

Subtract the two equations.

So $2cx - c^2 = b^2 - a^2$

$(c - x)^2 = c^2 - 2cx + x^2$.
So $x^2 - (c - x)^2 = x^2 - c^2 + 2cx - x^2$.

$$a^2 = b^2 + c^2 - 2cx \quad (1)$$

Rearrange.

but $x = b\cos A \quad (2)$

Use the cosine ratio $\cos A = \frac{x}{b}$ in $\triangle CAX$.

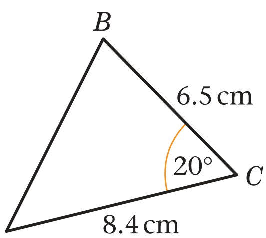
So $a^2 = b^2 + c^2 - 2bc\cos A$

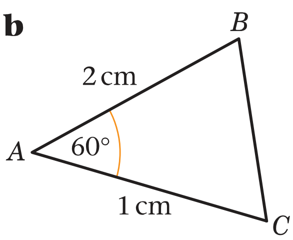
Combine (1) and (2). This is the cosine rule.

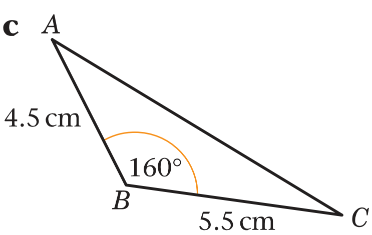
Exercise 2D

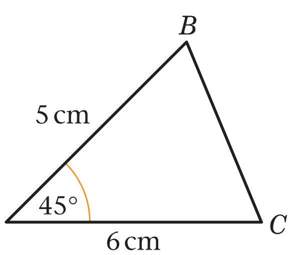
(*Note:* Give answers to 3 significant figures, where appropriate.)

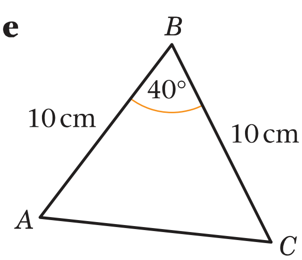
1 In each of the following triangles calculate the length of the third side:

a


b


c


d


e


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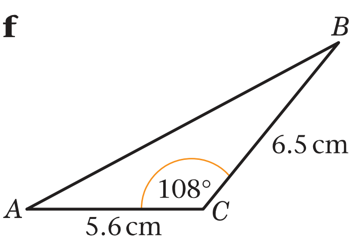


2 From a point *A* a boat sails due north for 7 km to *B*. The boat leaves *B* and moves on a bearing of 100° for 10 km until it reaches *C*. Calculate the distance of *C* from *A*.

3 The distance from the tee, T, to the flag, F, on a particular hole on a golf course is 494 yards. A golfer's tee shot travels 220 yards and lands at the point S, where $\angle STF = 22°$. Calculate how far the ball is from the flag.

4 In $\triangle ABC$, $AB = (x - 3)$ cm, $BC = (x + 3)$ cm, $AC = 8$ cm and $\angle BAC = 60°$. Use the cosine rule to find the value of x.

5 In $\triangle ABC$, $AB = x$ cm, $BC = (x - 4)$ cm, $AC = 10$ cm and $\angle BAC = 60°$. Calculate the value of x.

6 In $\triangle ABC$, $AB = (5 - x)$ cm, $BC = (4 + x)$ cm, $\angle ABC = 120°$ and $AC = y$ cm.

a Show that $y^2 = x^2 - x + 61$.

b Use the method of completing the square to find the minimum value of y^2, and give the value of x for which this occurs.

Hint for question 6b: Completing the square is in Book C1, Chapter 2.

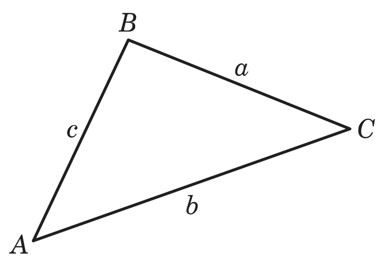
2.5 You can use the cosine rule to find an unknown angle in a triangle if you know the lengths of all three sides.

■ **You can find an unknown angle using a rearranged form of the cosine rule:**

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\text{or} \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\text{or} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



Example 11

Rearrange the equation $a^2 = b^2 + c^2 - 2bc\cos A$ in the form $\cos A = \ldots$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

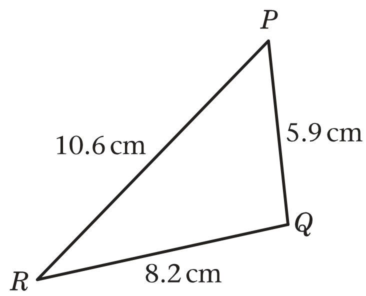
So $\quad 2bc\cos A = b^2 + c^2 - a^2$

So $\quad \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Divide throughout by $2bc$.

Example 12

In $\triangle PQR$, $PQ = 5.9$ cm, $QR = 8.2$ cm and $PR = 10.6$ cm.
Calculate the size of $\angle PQR$.



$$\cos Q = \frac{8.2^2 + 5.9^2 - 10.6^2}{2 \times 8.2 \times 5.9}$$

$$= -0.1065\ldots$$

$$Q = 96.1°$$

$$\angle PQR = 96.1°$$

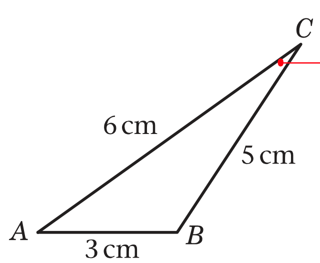
Here $p = 8.2$ cm, $r = 5.9$ cm, $q = 10.6$ cm, and you have to find angle Q.

Use the cosine rule $\cos Q = \dfrac{p^2 + r^2 - q^2}{2pr}$

$Q = \cos^{-1}(-0.1065\ldots)$

Example 13

Find the size of the smallest angle in a triangle whose sides have lengths 3 cm, 5 cm and 6 cm.



Label the triangle ABC.
The smallest angle is opposite the smallest side so angle C is required.

$$\cos C = \frac{6^2 + 5^2 - 3^2}{2 \times 6 \times 5}$$

$$C = 29.9°$$

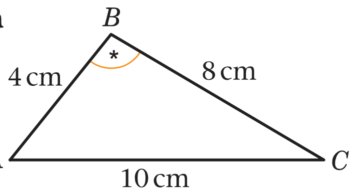
Use the cosine rule $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

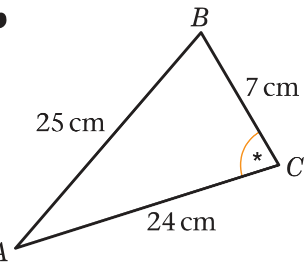
The size of the smallest angle is 29.9°.

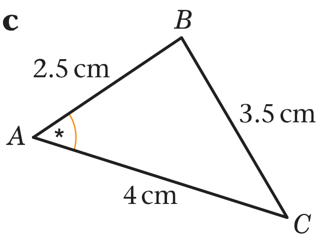
Exercise 2E

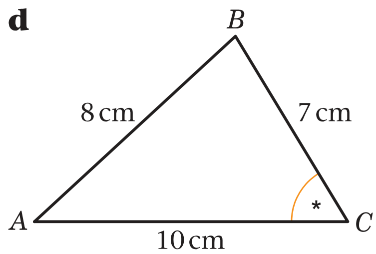
(Give answers to 3 significant figures.)

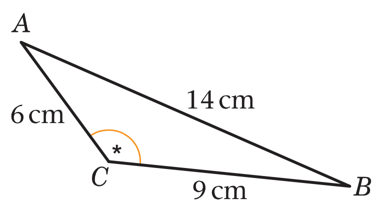
1 In the following triangles calculate the size of the angle marked *:

a


b


c


d


e


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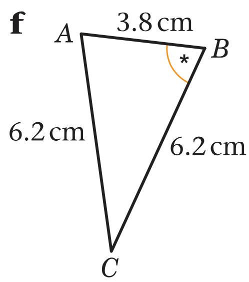


2 A helicopter flies on a bearing of 080° from A to B, where $AB = 50$ km.
It then flies for 60 km to a point C.
Given that C is 80 km from A, calculate the bearing of C from A.

3 In $\triangle ABC$, $AB = 5$ cm, $BC = 6$ cm and $AC = 10$ cm.
Calculate the value of the smallest angle.

4 In $\triangle ABC$, $AB = 9.3$ cm, $BC = 6.2$ cm and $AC = 12.7$ cm.
Calculate the value of the largest angle.

5 The lengths of the sides of a triangle are in the ratio $2 : 3 : 4$.
Calculate the value of the largest angle.

6 In $\triangle ABC$, $AB = x$ cm, $BC = 5$ cm and $AC = (10 - x)$ cm:

a Show that $\cos \angle ABC = \dfrac{4x - 15}{2x}$.

b Given that $\cos \angle ABC = -\frac{1}{7}$, work out the value of x.

2.6 You need to be able to use the sine rule, the cosine rule, the trigonometric ratios sin, cos and tan, and Pythagoras' theorem to solve problems.

In triangle work involving trigonometric calculations, the following strategy might help you.

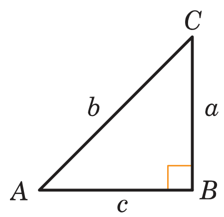
- When the triangle is right-angled or isosceles it is better to use sine, cosine, tangent or Pythagoras' theorem.

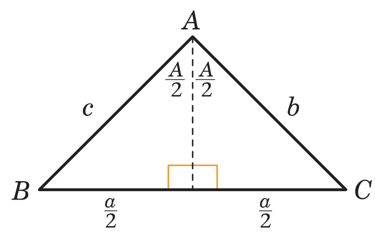
$\sin A = \frac{a}{b}$

$\cos A = \frac{c}{b}$

$\tan A = \frac{a}{c}$

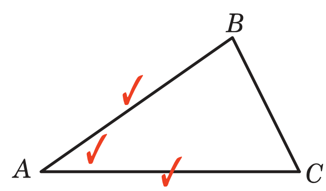
$a^2 + c^2 = b^2$

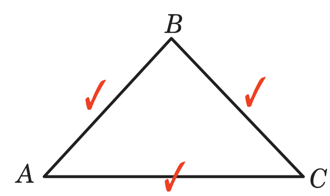




The line of symmetry produces two right-angled triangles.

- Use the **cosine rule** when you are given either **two sides and the angle between them** or **three sides**.

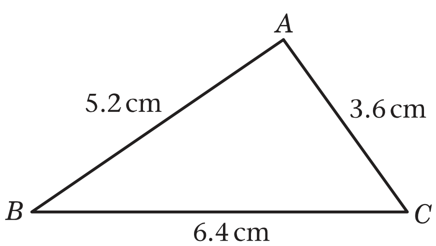




- For other combinations of given data, use the **sine rule**.
- When you have used the cosine rule once, it is generally better not to use it again, as the cosine rule involves more calculations and so may introduce more rounding errors.

Example 14

In $\triangle ABC$, $AB = 5.2$ cm, $BC = 6.4$ cm and $AC = 3.6$ cm.
Calculate the angles of the triangle.



$$\cos A = \frac{3.6^2 + 5.2^2 - 6.4^2}{2 \times 3.6 \times 5.2}$$

It is always better to work out the largest angle first, if you have a choice.

Here that is angle A, so use the cosine rule $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.

$$= -0.02564\ \ldots$$

Negative sign indicates an obtuse angle: $\angle BAC = 91.5°$

So $A = 91.5°$ (3 s.f.)

$$\sin C = \frac{5.2 \sin A}{6.4}$$

Store the calculator value for A.

To find a second angle it is better to use the sine rule rather than a second cosine rule.

Use $\frac{\sin C}{c} = \frac{\sin A}{a}$, and use the stored value for A.

So $C = 54.3°$

and $B = 34.2°$

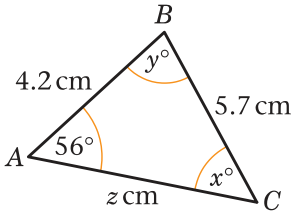
$\angle ACB = 54.3°$.
$\angle ABC = 180° - (91.5 + 54.3)°$.

Exercise 2F

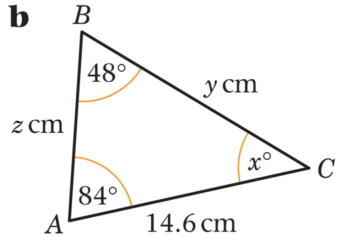
(*Note:* Try to use the neatest method, and give answers to 3 significant figures.)

1 In each triangle below find the values of x, y and z.

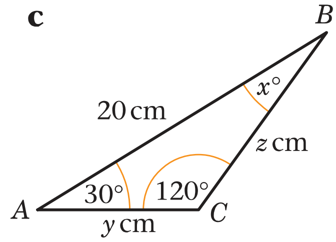
a



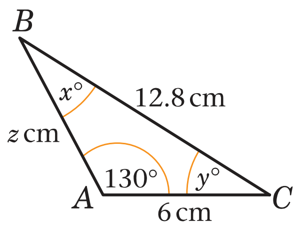
b



c



d



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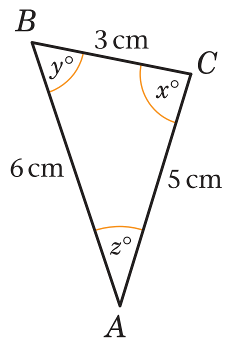
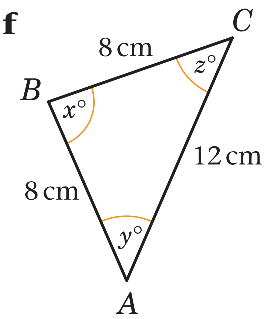
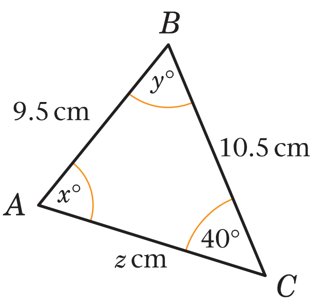


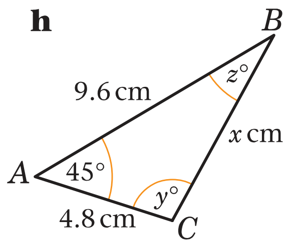
f



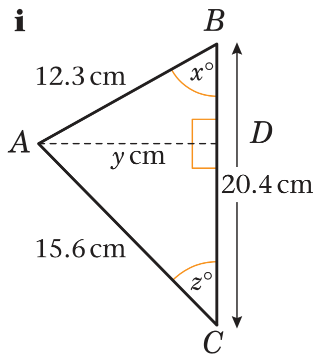
g



h

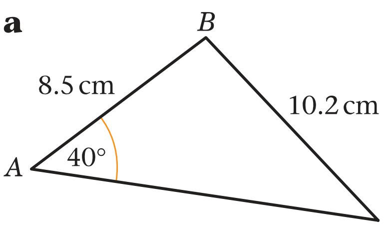


i



2 Calculate the size of the remaining angles and the length of the third side in the following triangles:

a



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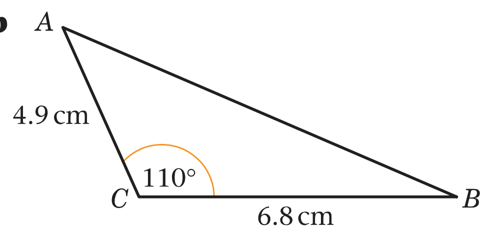


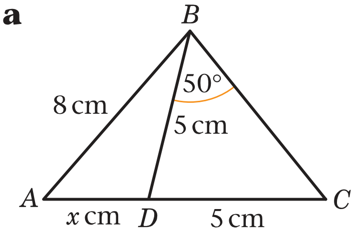
3 A hiker walks due north from A and after 8 km reaches B. She then walks a further 8 km on a bearing of 120° to C. Work out **a** the distance from A to C and **b** the bearing of C from A.

4 A helicopter flies on a bearing of 200° from A to B, where $AB = 70$ km. It then flies on a bearing of 150° from B to C, where C is due south of A. Work out the distance of C from A.

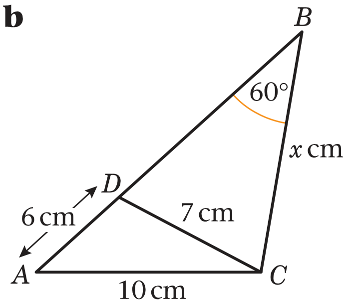
5 Two radar stations A and B are 16 km apart and A is due north of B. A ship is known to be on a bearing of 150° from A and 10 km from B. Show that this information gives two positions for the ship, and calculate the distance between these two positions.

6 Find x in each of the following diagrams:

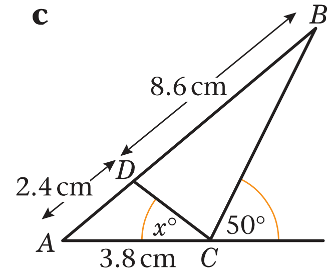
a



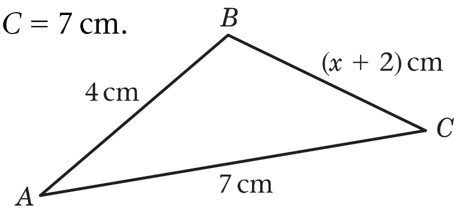
b



c



7 In $\triangle ABC$, shown right, $AB = 4$ cm, $BC = (x + 2)$ cm and $AC = 7$ cm.



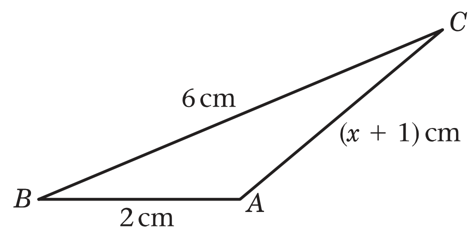
a Explain how you know that $1 < x < 9$.

b Work out the value of x for the cases when

i $\angle ABC = 60°$ and

ii $\angle ABC = 45°$, giving your answers to 3 significant figures.

8 In the triangle shown right, $\cos \angle ABC = \frac{5}{8}$.
Calculate the value of x.



9 In $\triangle ABC$, $AB = \sqrt{2}$ cm, $BC = \sqrt{3}$ cm and $\angle BAC = 60°$. Show that $\angle ACB = 45°$ and find AC.

10 In $\triangle ABC$, $AB = (2 - x)$ cm, $BC = (x + 1)$ cm and $\angle ABC = 120°$:

a Show that $AC^2 = x^2 - x + 7$.

b Find the value of x for which AC has a minimum value.

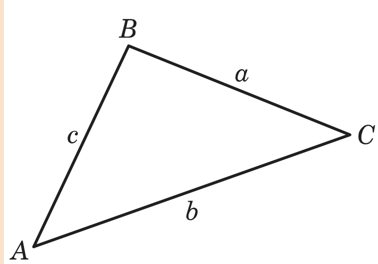
11 Triangle ABC is such that $BC = 5\sqrt{2}$ cm, $\angle ABC = 30°$ and $\angle BAC = \theta$, where $\sin \theta = \dfrac{\sqrt{5}}{8}$.

Work out the length of AC, giving your answer in the form $a\sqrt{b}$, where a and b are integers.

12 The perimeter of $\triangle ABC = 15$ cm. Given that $AB = 7$ cm and $\angle BAC = 60°$, find the lengths of AC and BC.

2.7 You can calculate the area of a triangle using the formula:

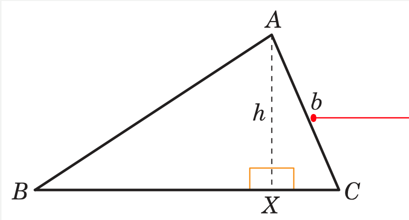
Area of a triangle = $\frac{1}{2}ab \sin C$ or $\frac{1}{2}ac \sin B$ or $\frac{1}{2}bc \sin A$.



You can use this formula when you know the lengths of two sides and the size of the angle between them.

Example 15

Show that the area of this triangle is $\frac{1}{2}ab\sin C$.



Area of $\triangle ABC = \frac{1}{2}ah$ (1)

But $h = b\sin C$ (2)

So Area $= \frac{1}{2}ab\sin C$

The perpendicular from A to BC is drawn and it meets BC at X. The length of $AX = h$.

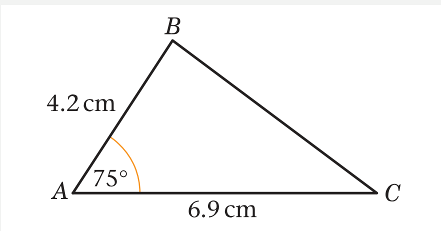
Area of triangle $= \frac{1}{2} \times$ base $\times$ height.

Use the sine ratio $\sin C = \frac{h}{b}$ in $\triangle AXC$.

Substitute (2) into (1).

Example 16

Work out the area of the triangle shown below.



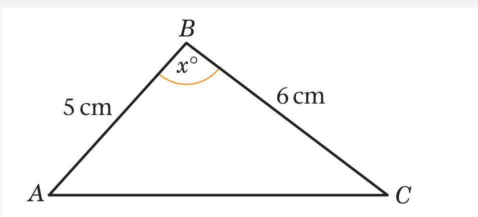
Area of $\triangle ABC = \frac{1}{2} \times 6.9 \times 4.2 \times \sin 75°\ \text{cm}^2$

$= 14.0\ \text{cm}^2$ (3 s.f.)

Here $b = 6.9$ cm, $c = 4.2$ cm and angle $A = 75°$, so use:
Area $= \frac{1}{2}bc\sin A$.

Example 17

In $\triangle ABC$, $AB = 5$ cm, $BC = 6$ cm and $\angle ABC = x°$. Given that the area of $\triangle ABC$ is $12\ \text{cm}^2$ and that AC is the longest side, find the value of x.



Area $\triangle ABC = \frac{1}{2} \times 5 \times 6 \times \sin x°\ \text{cm}^2$

So $12 = \frac{1}{2} \times 5 \times 6 \times \sin x°$

So $\sin x° = 0.8$

$x = 126.9$

$= 127$ (3 s.f.)

Here $a = 6$ cm, $c = 5$ cm and angle $B = x°$, so use:
Area $= \frac{1}{2}ac\sin B$.

Area of $\triangle ABC$ is $12\ \text{cm}^2$.

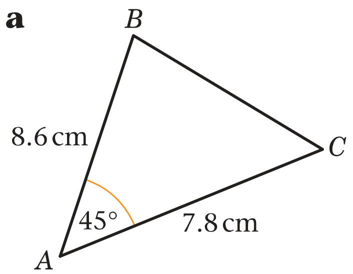
$\sin x° = \frac{12}{15}$.

There are two values of x for which $\sin x° = 0.8$, 53.1 and 126.9, but here you know B is the largest angle because AC is the largest side.

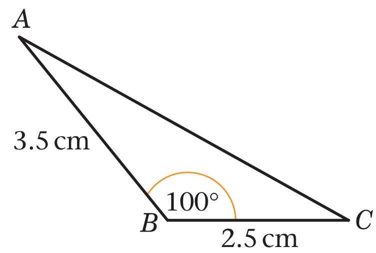
Exercise 2G

1 Calculate the area of the following triangles:

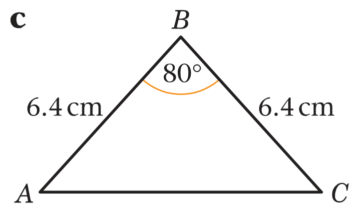
a



b

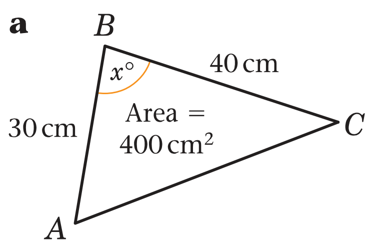


c

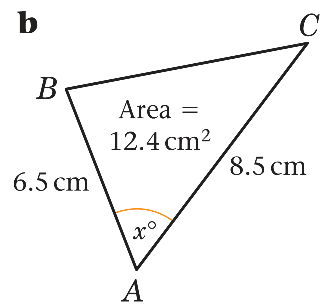


2 Work out the possible values of x in the following triangles:

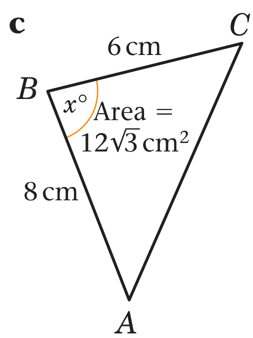
a



b

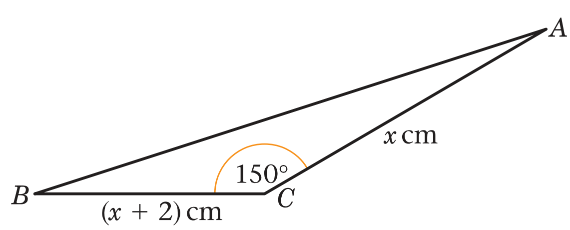


c



3 A fenced triangular plot of ground has area 1200 m^2. The fences along the two smaller sides are 60 m and 80 m respectively and the angle between them is $\theta°$. Show that $\theta = 150$, and work out the total length of fencing.

4 In triangle ABC, shown right, $BC = (x + 2)$ cm, $AC = x$ cm and $\angle BCA = 150°$. Given that the area of the triangle is 5 cm^2, work out the value of x, giving your answer to 3 significant figures.



5 In $\triangle PQR$, $PQ = (x + 2)$ cm, $PR = (5 - x)$ cm and $\angle QPR = 30°$. The area of the triangle is A cm^2:

a Show that $A = \frac{1}{4}(10 + 3x - x^2)$.

b Use the method of completing the square, or otherwise, to find the maximum value of A, and give the corresponding value of x.

6 In $\triangle ABC$, $AB = x$ cm, $AC = (5 + x)$ cm and $\angle BAC = 150°$. Given that the area of the triangle is $3\frac{3}{4}$ cm^2:

a Show that x satisfies the equation $x^2 + 5x - 15 = 0$.

b Calculate the value of x, giving your answer to 3 significant figures.

Mixed exercise 2H

(Give non-exact answers to 3 significant figures.)

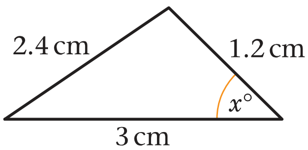
1 The area of a triangle is 10 cm². The angle between two of the sides, of length 6 cm and 8 cm respectively, is obtuse. Work out:

a The size of this angle.

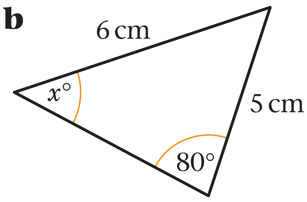
b The length of the third side.

2 In each triangle below, find the value of x and the area of the triangle:

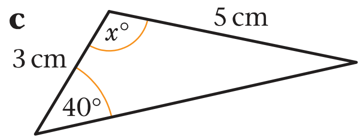
a



b



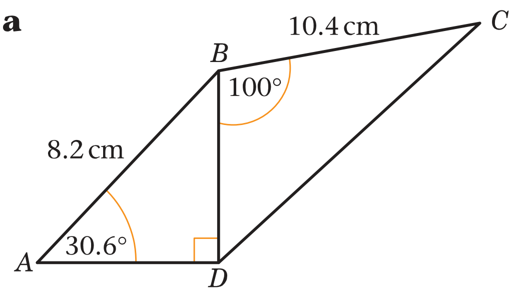
c



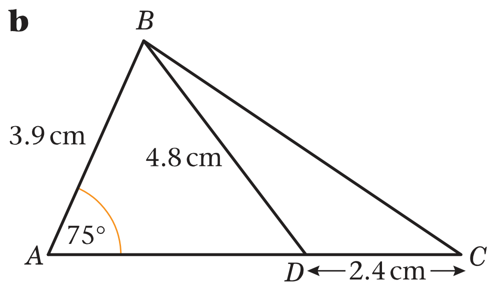
3 The sides of a triangle are 3 cm, 5 cm and 7 cm respectively. Show that the largest angle is 120°, and find the area of the triangle.

4 In each of the figures below calculate the total area:

a



b



5 In $\triangle ABC$, $AB = 10$ cm, $BC = a\sqrt{3}$ cm, $AC = 5\sqrt{13}$ cm and $\angle ABC = 150°$. Calculate:

a The value of a.

b The exact area of $\triangle ABC$.

6 In a triangle, the largest side has length 2 cm and one of the other sides has length $\sqrt{2}$ cm. Given that the area of the triangle is 1 cm², show that the triangle is right-angled and isosceles.

7 The three points A, B and C, with coordinates $A(0, 1)$, $B(3, 4)$ and $C(1, 3)$ respectively, are joined to form a triangle:

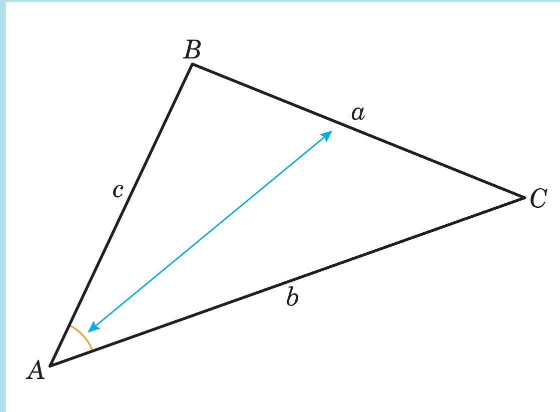
a Show that $\cos \angle ACB = -\frac{4}{5}$.

b Calculate the area of $\triangle ABC$.

8 The longest side of a triangle has length $(2x - 1)$ cm. The other sides have lengths $(x - 1)$ cm and $(x + 1)$ cm. Given that the largest angle is 120°, work out:

a the value of x and **b** the area of the triangle.

Summary of key points



1 The sine rule is

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

2 You can use the sine rule to find an unknown side in a triangle if you know two angles and the length of one of their opposite sides.

3 You can use the sine rule to find an unknown angle in a triangle if you know the lengths of two sides and one of their opposite angles.

4 The cosine rule is

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{or} \quad b^2 = a^2 + c^2 - 2ac\cos B \quad \text{or} \quad c^2 = a^2 + b^2 - 2ab\cos C$$

5 You can use the cosine rule to find an unknown side in a triangle if you know the lengths of two sides and the angle between them.

6 You can use the cosine rule to find an unknown angle if you know the lengths of all three sides.

7 You can find an unknown angle using a rearranged form of the cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{or} \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad \text{or} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

8 You can find the area of a triangle using the formula

area $= \frac{1}{2}ab\sin C$

if you know the length of two sides (a and b) and the value of the angle C between them.