3

After completing this chapter you should be able to

1 know the shape of the graph of $y = a^x$

2 write an expression in logarithmic form

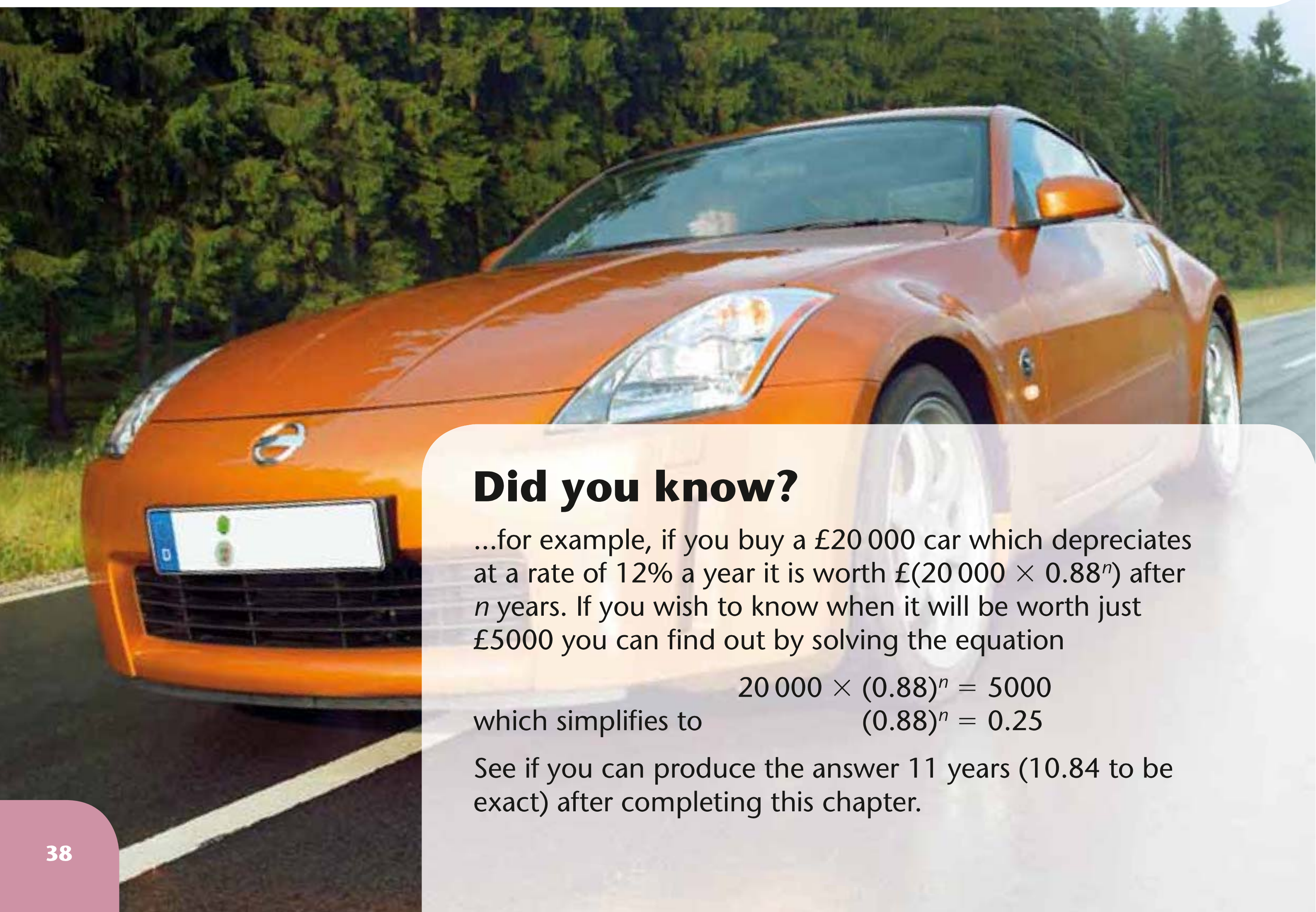
3 use the laws of logarithms

4 solve equations of the form $a^x = b$

5 change the base of a logarithm

You will use the above in C2 whilst studying geometric sequences and series. The ability to solve equations of the form $a^x = b$ has many practical uses.

Exponential and logarithms



Did you know?

...for example, if you buy a £20 000 car which depreciates at a rate of 12% a year it is worth £$(20\,000 \times 0.88^n)$ after n years. If you wish to know when it will be worth just £5000 you can find out by solving the equation

$$20\,000 \times (0.88)^n = 5000$$

which simplifies to

$$(0.88)^n = 0.25$$

See if you can produce the answer 11 years (10.84 to be exact) after completing this chapter.

3.1 You need to be familiar with the function $y = a^x$ ($a > 0$) and to know the shape of its graph.

As an example, look at a table of values for $y = 2^x$:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Hint: In an expression such as 2^x, the x can be called a power, or an index, or an exponent. You will see in Section 3.2 that it can also be thought of as a logarithm.
A function that involves a variable power such as x is called an exponential function.

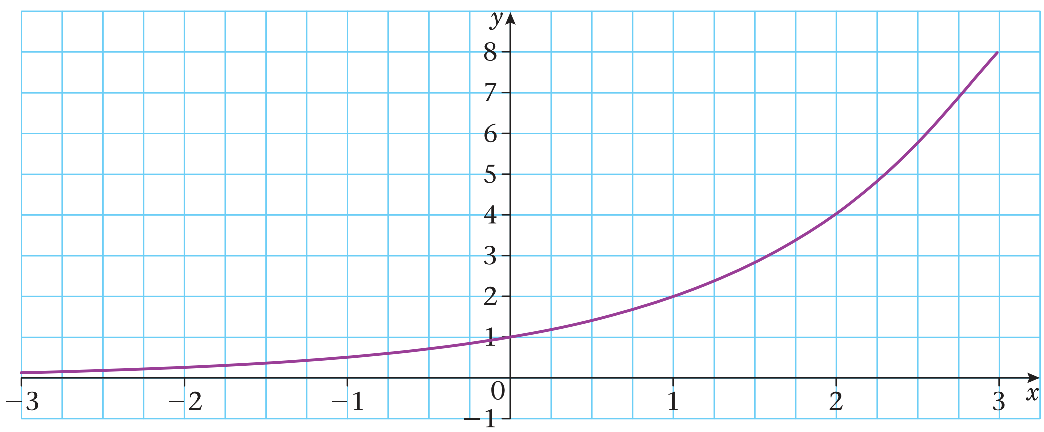
Note that

$2^0 = 1$ (in fact $a^0 = 1$ always if $a > 0$)

and $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$ (a negative index implies the 'reciprocal' of a positive index)

Hint: See Book C1, Chapter 1 for the rules of indices.

The graph of $y = 2^x$ looks like this:



Hint: The x-axis is an asymptote to the curve.

Other graphs of the type $y = a^x$ are of a similar shape, always passing through (0, 1).

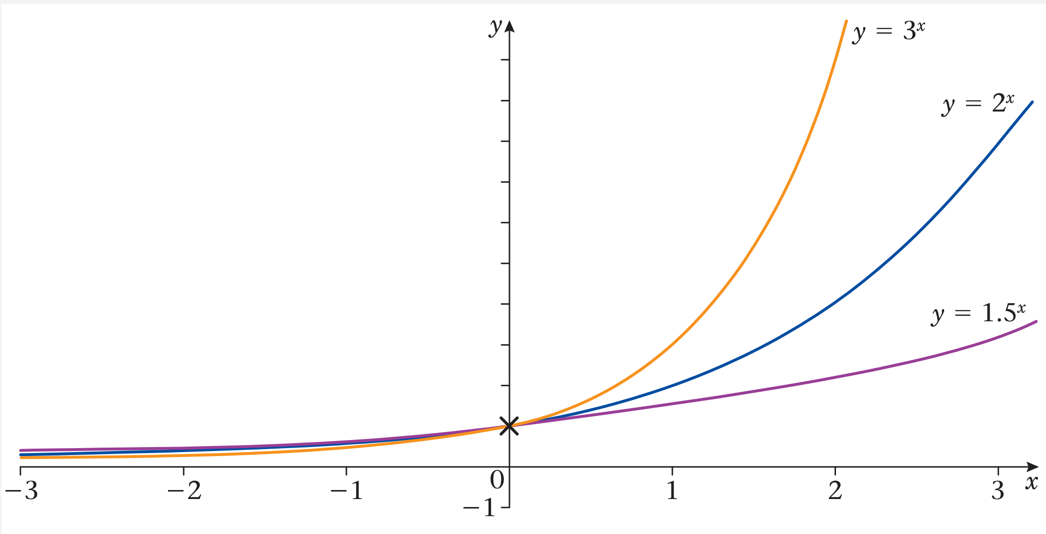
Example 1

a On the same axes sketch the graphs of $y = 3^x$, $y = 2^x$ and $y = 1.5^x$.

b On another set of axes sketch the graphs of $y = (\frac{1}{2})^x$ and $y = 2^x$.

a For all three graphs, $y = 1$ when $x = 0$.
When $x > 0$, $3^x > 2^x > 1.5^x$.
When $x < 0$, $3^x < 2^x < 1.5^x$.

$a^0 = 1$



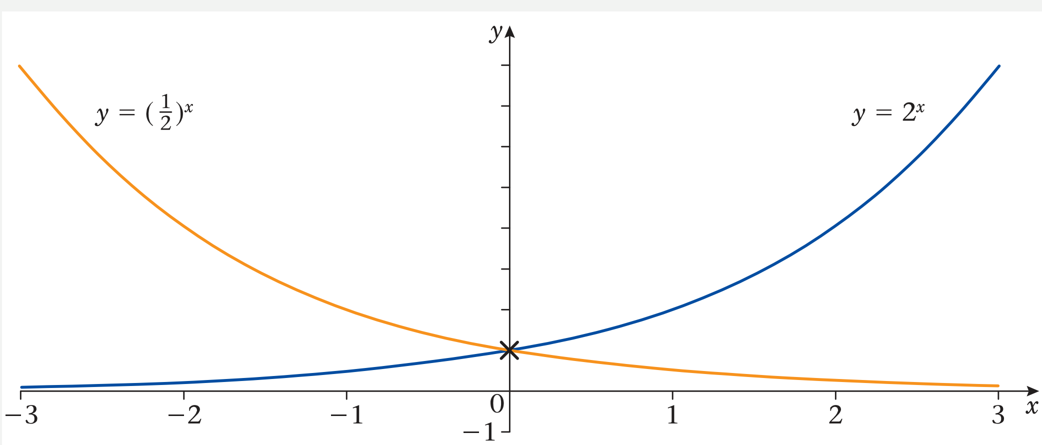
Work out the relative positions of the three graphs.

b $\frac{1}{2} = 2^{-1}$

So $y = \left(\frac{1}{2}\right)^x$ is the same as $y = (2^{-1})^x = 2^{-x}$.

$(a^m)^n = a^{mn}$

So the graph of $y = \left(\frac{1}{2}\right)^x$ is a reflection in the y-axis of the graph of $y = 2^x$.



Exercise 3A

1 **a** Draw an accurate graph of $y = (1.7)^x$, for $-4 \leqslant x \leqslant 4$.
b Use your graph to solve the equation $(1.7)^x = 4$.

2 **a** Draw an accurate graph of $y = (0.6)^x$, for $-4 \leqslant x \leqslant 4$.
b Use your graph to solve the equation $(0.6)^x = 2$

3 Sketch the graph of $y = 1^x$.

3.2 You need to know how to write an expression as a logarithm.

■ **$\log_a n = x$ means that $a^x = n$, where a is called the base of the logarithm.**

Example 2

Write as a logarithm $2^5 = 32$.

$2^5 = 32$

So $\log_2 32 = 5$

Here $a = 2$, $x = 5$, $n = 32$.

Base.

Logarithm.

In words, you would say '2 to the power 5 equals 32'.

In words you would say 'the logarithm of 32, to base 2, is 5'.

Example 3

Rewrite as a logarithm:
a $10^3 = 1000$
b $5^4 = 625$
c $2^{10} = 1024$

a $\log_{10} 1000 = 3$
b $\log_5 625 = 4$
c $\log_2 1024 = 10$

■ **$\log_a 1 = 0$ $(a > 0)$** — Because $a^0 = 1$.

■ **$\log_a a = 1$ $(a > 0)$** — Because $a^1 = a$.

Example 4

Find the value of

a $\log_3 81$ **b** $\log_4 0.25$ **c** $\log_{0.5} 4$ **d** $\log_a (a^5)$

a $\log_3 81 = 4$ — Because $3^4 = 81$.

b $\log_4 0.25 = -1$ — Because $4^{-1} = \frac{1}{4} = 0.25$.

c $\log_{0.5} 4 = -2$ — Because $0.5^{-2} = (\frac{1}{2})^{-2} = 2^2 = 4$.

d $\log_a (a^5) = 5$ — Because $a^5 = a^5$!

Exercise 3B

1 Rewrite as a logarithm:

a $4^4 = 256$ **b** $3^{-2} = \frac{1}{9}$ **c** $10^6 = 1\,000\,000$

d $11^1 = 11$ **e** $(0.2)^3 = 0.008$

2 Rewrite using a power:

a $\log_2 16 = 4$ **b** $\log_5 25 = 2$ **c** $\log_9 3 = \frac{1}{2}$

d $\log_5 0.2 = -1$ **e** $\log_{10} 100\,000 = 5$

3 Find the value of:

a $\log_2 8$ **b** $\log_5 25$

c $\log_{10} 10\,000\,000$ **d** $\log_{12} 12$

e $\log_3 729$ **f** $\log_{10} \sqrt{10}$

g $\log_4 (0.25)$ **h** $\log_{0.25} 16$

i $\log_a (a^{10})$ **j** $\log_{(\frac{2}{3})} (\frac{9}{4})$

4 Find the value of x for which:

a $\log_5 x = 4$ **b** $\log_x 81 = 2$

c $\log_7 x = 1$ **d** $\log_x (2x) = 2$

3.3 You need to be able to calculate logarithms to the base 10 using your calculator.

Example 5

Find the value of x for which $10^x = 500$.

$10^x = 500$

So $\log_{10} 500 = x$ — Since $10^2 = 100$ and $10^3 = 1000$, x must be somewhere between 2 and 3.

$x = \log_{10} 500$

$= 2.70$ (to 3 s.f.) — The log (or lg) button on your calculator gives values of logs to base 10.

Exercise 3C

Find from your calculator the value to 3 s.f. of:

1 $\log_{10} 20$

2 $\log_{10} 4$

3 $\log_{10} 7000$

4 $\log_{10} 0.786$

5 $\log_{10} 11$

6 $\log_{10} 35.3$

7 $\log_{10} 0.3$

8 $\log_{10} 999$

3.4 You need to know the laws of logarithms.

Suppose that	$\log_a x = b$ and $\log_a y = c$
Rewriting with powers:	$a^b = x$ and $a^c = y$
Multiplying:	$xy = a^b \times a^c = a^{b+c}$ (see Book C1, Chapter 1)
	$xy = a^{b+c}$
Rewriting as a logarithm:	$\log_a xy = b + c$

■ **$\log_a xy = \log_a x + \log_a y$ (the multiplication law)**

It can also be shown that:

■ **$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$ (the division law)**

Remember: $\frac{a^b}{a^c} = a^b \div a^c = a^{b-c}$

■ **$\log_a (x)^k = k \log_a x$ (the power law)**

Remember: $(a^b)^k = a^{bk}$

Note: You need to learn and remember the above three laws of logarithms.

Since $\frac{1}{x} = x^{-1}$, the power rule shows that $\log_a \left(\frac{1}{x}\right) = \log_a (x^{-1}) = -\log_a x$.

■ **$\log_a \left(\frac{1}{x}\right) = -\log_a x$**

Example 6

Write as a single logarithm:

a $\log_3 6 + \log_3 7$ **b** $\log_2 15 - \log_2 3$

c $2 \log_5 3 + 3 \log_5 2$ **d** $\log_{10} 3 - 4 \log_{10} (\frac{1}{2})$

a $\log_3 (6 \times 7)$

$= \log_3 42$ — Use the multiplication law.

b $\log_2 (15 \div 3)$

$= \log_2 5$ — Use the division law.

c $2\log_5 3 = \log_5 (3^2) = \log_5 9$
$3\log_5 2 = \log_5 (2^3) = \log_5 8$
$\log_5 9 + \log_5 8 = \log_5 72$

First apply the power law to both parts of the expression.
Then use the multiplication law.

d $4\log_{10}\left(\frac{1}{2}\right) = \log_{10}\left(\frac{1}{2}\right)^4 = \log_{10}\left(\frac{1}{16}\right)$
$\log_{10} 3 - \log_{10}\left(\frac{1}{16}\right) = \log_{10}\left(3 \div \frac{1}{16}\right)$
$= \log_{10} 48$

Use the power law first.
Then use the division law.

Example 7

Write in terms of $\log_a x$, $\log_a y$ and $\log_a z$

a $\log_a (x^2yz^3)$ **b** $\log_a\left(\frac{x}{y^3}\right)$ **c** $\log_a\left(\frac{x\sqrt{y}}{z}\right)$ **d** $\log_a\left(\frac{x}{a^4}\right)$

a $\log_a (x^2yz^3)$
$= \log_a (x^2) + \log_a y + \log_a (z^3)$
$= 2\log_a x + \log_a y + 3\log_a z$

b $\log_a\left(\frac{x}{y^3}\right)$
$= \log_a x - \log_a (y^3)$
$= \log_a x - 3\log_a y$

c $\log_a\left(\frac{x\sqrt{y}}{z}\right)$
$= \log_a (x\sqrt{y}) - \log_a z$
$= \log_a x + \log_a \sqrt{y} - \log_a z$
$= \log_a x + \frac{1}{2}\log_a y - \log_a z$

Use the power law ($\sqrt{y} = y^{\frac{1}{2}}$).

d $\log_a\left(\frac{x}{a^4}\right)$
$= \log_a x - \log_a (a^4)$
$= \log_a x - 4\log_a a$
$= \log_a x - 4$

$\log_a a = 1$.

Exercise 3D

1 Write as a single logarithm:

a $\log_2 7 + \log_2 3$

b $\log_2 36 - \log_2 4$

c $3\log_5 2 + \log_5 10$

d $2\log_6 8 - 4\log_6 3$

e $\log_{10} 5 + \log_{10} 6 - \log_{10}(\frac{1}{4})$

2 Write as a single logarithm, then simplify your answer:

a $\log_2 40 - \log_2 5$

b $\log_6 4 + \log_6 9$

c $2\log_{12} 3 + 4\log_{12} 2$

d $\log_8 25 + \log_8 10 - 3\log_8 5$

e $2\log_{10} 20 - (\log_{10} 5 + \log_{10} 8)$

3 Write in terms of $\log_a x$, $\log_a y$ and $\log_a z$:

a $\log_a(x^3y^4z)$

b $\log_a\left(\frac{x^5}{y^2}\right)$

c $\log_a(a^2x^2)$

d $\log_a\left(\frac{x\sqrt{y}}{z}\right)$

e $\log_a \sqrt{ax}$

3.5 You need to be able to solve equations of the form $a^x = b$.

Example 8

Solve the equation $3^x = 20$, giving your answer to 3 significant figures.

$$3^x = 20$$

$$\log_{10}(3^x) = \log_{10} 20$$

Since there is no base 3 logarithm button on your calculator, any working must be done using base 10.

Take logs to base 10 on each side.

$$x\log_{10} 3 = \log_{10} 20$$

Use the power law.

$$x = \frac{\log_{10} 20}{\log_{10} 3}$$

Divide by $\log_{10} 3$.

$$x = \left(\frac{1.3010\ldots}{0.4771\ldots}\right)$$

$$= 2.73$$

Use your calculator (logs to base 10).

Example 9

Solve the equation $7^{x+1} = 3^{x+2}$, giving your answer to 4 decimal places.

$$7^{x+1} = 3^{x+2}$$

$$(x + 1)\log 7 = (x + 2)\log 3$$ — Use the power law.

$$x\log 7 + \log 7 = x\log 3 + 2\log 3$$ — Multiply out.

$$x\log 7 - x\log 3 = 2\log 3 - \log 7$$ — Collect x terms on left and 'number' terms on right.

$$x(\log 7 - \log 3) = 2\log 3 - \log 7$$ — Factorise.

$$x = \frac{2\log 3 - \log 7}{\log 7 - \log 3}$$ — Divide by (log 7 – log 3).

$$x = 0.2966$$

Example 10

Solve the equation $5^{2x} + 7(5^x) - 30 = 0$, giving your answer to 2 decimal places:

Let $y = 5^x$

$$y^2 + 7y - 30 = 0$$ — $5^{2x} = (5^x)^2 = y^2$

So $(y + 10)(y - 3) = 0$

So $y = -10$ or $y = 3$

If $y = -10$, $5^x = -10$ — No solution. 5^x cannot be negative.

If $y = 3$, $5^x = 3$

$$\log_{10}(5^x) = \log_{10} 3$$

$$x\log_{10} 5 = \log_{10} 3$$

$$x = \frac{\log_{10} 3}{\log_{10} 5}$$

$$x = 0.68 \text{ (2 d.p.)}$$

— Solve as in Example 8.

Exercise 3E

1 Solve, giving your answer to 3 significant figures:

a $2^x = 75$ **b** $3^x = 10$

c $5^x = 2$ **d** $4^{2x} = 100$

e $9^{x+5} = 50$ **f** $7^{2x-1} = 23$

g $3^{x-1} = 8^{x+1}$ **h** $2^{2x+3} = 3^{3x+2}$

i $8^{3-x} = 10^x$ **j** $3^{4-3x} = 4^{x+5}$

2 Solve, giving your answer to 3 significant figures:

a $2^{2x} - 6(2^x) + 5 = 0$ **b** $3^{2x} - 15(3^x) + 44 = 0$

c $5^{2x} - 6(5^x) - 7 = 0$ **d** $3^{2x} + 3^{x+1} - 10 = 0$

e $7^{2x} + 12 = 7^{x+1}$ **f** $2^{2x} + 3(2^x) - 4 = 0$

g $3^{2x+1} - 26(3^x) - 9 = 0$ **h** $4(3^{2x+1}) + 17(3^x) - 7 = 0$

Hint for question 2d: Note that $3^{x+1} = 3^x \times 3^1 = 3(3^x)$

3.6 To evaluate a logarithm using your calculator, you sometimes need to change the base of the logarithm.

Working in base a, suppose that: $\log_a x = m$

Writing this as a power: $a^m = x$

Taking logs to a different base b: $\log_b (a^m) = \log_b x$

Using the power law: $m \log_b a = \log_b x$

Writing m as $\log_a x$: $\log_b x = \log_a x \times \log_b a$

This can be written as:

■ $\log_a x = \dfrac{\log_b x}{\log_b a}$

Hint: This is the change of base rule for logarithms.

Using this rule, notice in particular that $\log_a b = \dfrac{\log_b b}{\log_b a}$, but $\log_b b = 1$, so:

■ $\log_a b = \dfrac{1}{\log_b a}$

Example 11

Find, to 3 significant figures, the value of $\log_8 11$:

$$\log_8 11 = \frac{\log_{10} 11}{\log_{10} 8} = 1.15$$

One method is to use the change of base rule to change to base 10.

$8^x = 11$ — Another method is to solve $8^x = 11$.

$\log_{10}(8^x) = \log_{10} 11$ — Take logs to base 10 of each side.

$x \log_{10} 8 = \log_{10} 11$ — Use the power law.

$x = \dfrac{\log_{10} 11}{\log_{10} 8}$ — Divide by $\log_{10} 8$.

$x = 1.15$

Example 12

Solve the equation $\log_5 x + 6\log_x 5 = 5$:

$$\log_5 x + \frac{6}{\log_5 x} = 5$$

Use change of base rule (special case).

Let $\log_5 x = y$

$$y + \frac{6}{y} = 5$$

$$y^2 + 6 = 5y$$

Multiply by y.

$$y^2 - 5y + 6 = 0$$

$$(y-3)(y-2) = 0$$

So $y = 3$ or $y = 2$

$\log_5 x = 3$ or $\log_5 x = 2$

$x = 5^3$ or $x = 5^2$

Write as powers.

$x = 125$ or $x = 25$

Exercise 3F

1 Find, to 3 decimal places:

a $\log_7 120$ **b** $\log_3 45$ **c** $\log_2 19$

d $\log_{11} 3$ **e** $\log_6 4$

2 Solve, giving your answer to 3 significant figures:

a $8^x = 14$ **b** $9^x = 99$ **c** $12^x = 6$

3 Solve, giving your answer to 3 significant figures:

a $\log_2 x = 8 + 9\log_x 2$

b $\log_4 x + 2\log_x 4 + 3 = 0$

c $\log_2 x + \log_4 x = 2$

Mixed exercise 3G

1 Find the possible values of x for which $2^{2x+1} = 3(2^x) - 1$.

2 **a** Express $\log_a(p^2q)$ in terms of $\log_a p$ and $\log_a q$.

b Given that $\log_a(pq) = 5$ and $\log_a(p^2q) = 9$, find the values of $\log_a p$ and $\log_a q$.

3 Given that $p = \log_q 16$, express in terms of p,

a $\log_q 2$,

b $\log_q(8q)$.

4 **a** Given that $\log_3 x = 2$, determine the value of x.

b Calculate the value of y for which $2\log_3 y - \log_3 (y + 4) = 2$.

c Calculate the values of z for which $\log_3 z = 4\log_z 3$. *E*

5 **a** Using the substitution $u = 2^x$, show that the equation $4^x - 2^{(x+1)} - 15 = 0$ can be written in the form $u^2 - 2u - 15 = 0$.

b Hence solve the equation $4^x - 2^{(x+1)} - 15 = 0$, giving your answer to 2 decimal places. *E*

6 Solve, giving your answers as exact fractions, the simultaneous equations:

$$8^y = 4^{2x+3}$$
$$\log_2 y = \log_2 x + 4.$$ *E*

7 Find the values of x for which $\log_3 x - 2\log_x 3 = 1$. *E*

8 Solve the equation

$$\log_3 (2 - 3x) = \log_9 (6x^2 - 19x + 2).$$ *E*

9 If $xy = 64$ and $\log_x y + \log_y x = \frac{5}{2}$, find x and y. *E*

10 Prove that if $a^x = b^y = (ab)^{xy}$, then $x + y = 1$. *E*

11 **a** Show that $\log_4 3 = \log_2 \sqrt{3}$.

b Hence or otherwise solve the simultaneous equations:

$$2\log_2 y = \log_4 3 + \log_2 x,$$
$$3^y = 9^x,$$

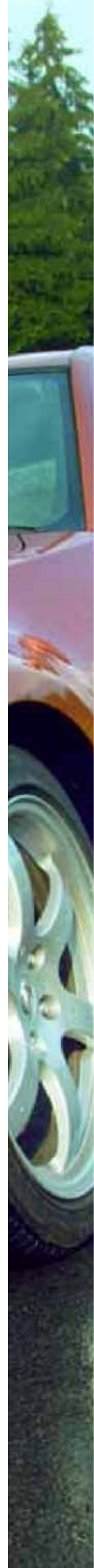
given that x and y are positive. *E*

12 **a** Given that $3 + 2\log_2 x = \log_2 y$, show that $y = 8x^2$.

b Hence, or otherwise, find the roots α and β, where $\alpha < \beta$, of the equation $3 + 2\log_2 x = \log_2 (14x - 3)$.

c Show that $\log_2 \alpha = -2$.

d Calculate $\log_2 \beta$, giving your answer to 3 significant figures. *E*



Summary of key points

1 A function $y = a^x$, or $f(x) = a^x$, where a is a constant, is called an exponential function.

2 $\log_a n = x$ means that $a^x = n$, where a is called the base of the logarithm.

3 $\log_a 1 = 0$
$\log_a a = 1$

4 $\log_{10} x$ is sometimes written as $\log x$.

5 The laws of logarithms are

$\log_a xy = \log_a x + \log_a y$ (the multiplication law)

$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$ (the division law)

$\log_a (x)^k = k \log_a x$ (the power law)

6 From the power law,

$$\log_a \left(\frac{1}{x}\right) = -\log_a x$$

7 You can solve an equation such as $a^x = b$ by first taking logarithms (to base 10) of each side.

8 The change of base rule for logarithms can be written as $\log_a x = \dfrac{\log_b x}{\log_b a}$

9 From the change of base rule, $\log_a b = \dfrac{1}{\log_b a}$