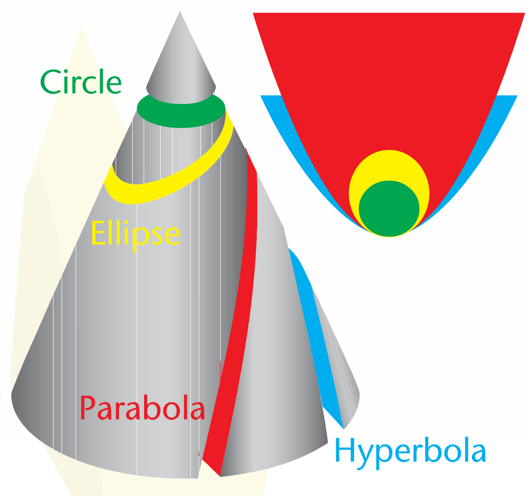
After completing this chapter you should be able to

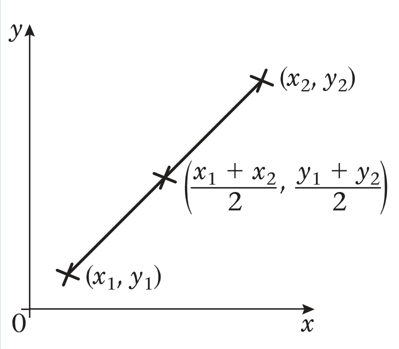
1 find the mid-point of a line
2 find the distance between a pair of points
3 know how to find the equation of a circle
4 use the properties of a circle to solve geometric problems.

4 Coordinate geometry in the (*x*, *y*) plane

A circle is a conic section – a curve formed by intersecting a cone with a plane. The ellipse, parabola and hyperbola are also conic sections; they were named by Apollonius of Perga who studied them at length in about 200 BC.



4.1 You can find the mid-point of the line joining (x_1, y_1) and (x_2, y_2) by using the formula $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.



Example 1

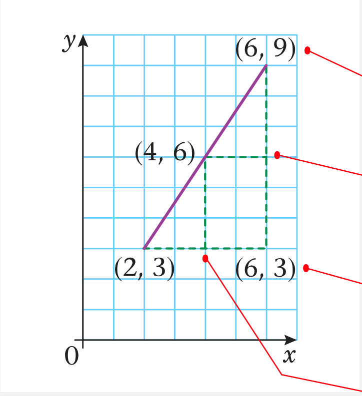
Find the mid-point of the line joining these pairs of points:

a (2, 3), (6, 9)

b $(2a, -4b)$, $(7a, 8b)$

c $(4, \sqrt{2})$, $(-4, 3\sqrt{2})$

a



The mid-point is $\left(\frac{2+6}{2}, \frac{3+9}{2}\right)$

$= \left(\frac{8}{2}, \frac{12}{2}\right)$

$= (4, 6)$

A diagram might help you work this out.

The y-coordinate is half way between 3 and 9, so use $\frac{y_1 + y_2}{2}$ with $y_1 = 3$ and $y_2 = 9$.

(6, 3) has the same x-coordinate as (6, 9) and the same y-coordinate as (2, 3).

The x-coordinate is half way between 2 and 6, so use $\frac{x_1 + x_2}{2}$ with $x_1 = 2$ and $x_2 = 6$.

Simplify.

b The mid-point is $\left(\frac{2a+7a}{2}, \frac{-4b+8b}{2}\right)$

$= \left(\frac{9a}{2}, \frac{4b}{2}\right)$

$= \left(\frac{9a}{2}, 2b\right)$

Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
Here $(x_1, y_1) = (2a, -4b)$ and $(x_2, y_2) = (7a, 8b)$.

Simplify.

c The mid-point is $\left(\frac{4+(-4)}{2}, \frac{\sqrt{2}+3\sqrt{2}}{2}\right)$

$= \left(\frac{4-4}{2}, \frac{4\sqrt{2}}{2}\right)$

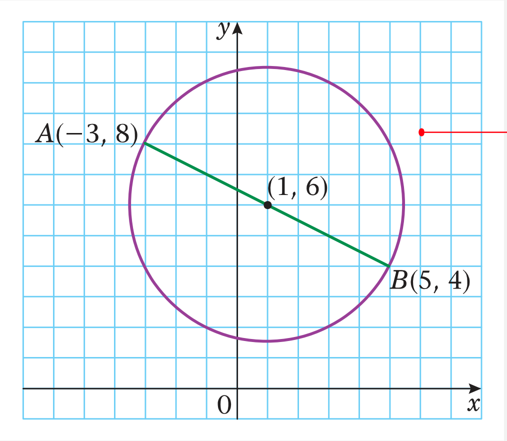
$= (0, 2\sqrt{2})$

Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
Here $(x_1, y_1) = (4, \sqrt{2})$ and $(x_2, y_2) = (-4, 3\sqrt{2})$.

Simplify.

Example 2

The line AB is a diameter of a circle, where A and B are $(-3, 8)$ and $(5, 4)$ respectively. Find the coordinates of the centre of the circle.



Draw a sketch.
Remember the centre of a circle is the mid-point of a diameter.

The centre of the circle is $\left(\frac{-3+5}{2}, \frac{8+4}{2}\right)$

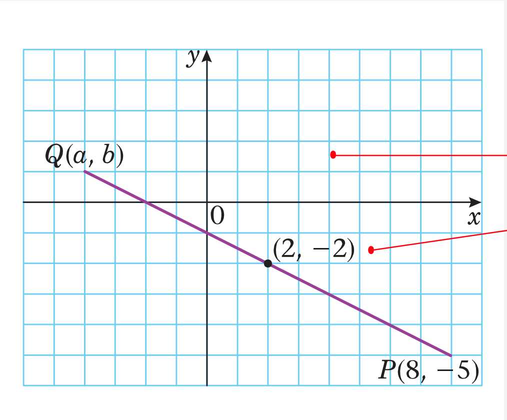
$= \left(\frac{2}{2}, \frac{12}{2}\right)$

$= (1, 6)$

Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
Here $(x_1, y_1) = (-3, 8)$ and $(x_2, y_2) = (5, 4)$.

Example 3

The line *PQ* is a diameter of the circle centre (2, −2). Given *P* is (8, −5), find the coordinates of *Q*.



Draw a sketch.

(2, −2) is the mid-point of (*a*, *b*) and (8, −5).

Let *Q* have coordinates (*a*, *b*).

$$\left(\frac{8+a}{2}, \frac{-5+b}{2}\right) = (2, -2)$$

Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Here $(x_1, y_1) = (8, -5)$ and $(x_2, y_2) = (a, b)$.

So $\frac{8+a}{2} = 2$

$8 + a = 4$

$a = -4$

Compare the *x*-coordinates.
Rearrange the equation to find *a*.
Multiply each side by 2 to clear the fraction.
Subtract 8 from each side.

and $\frac{-5+b}{2} = -2$

$-5 + b = -4$

$b = 1$

Compare the *y*-coordinates.
Rearrange the equation to find *b*.
Multiply each side by 2 to clear the fraction.
Add 5 to each side.

So, *Q* is (−4, 1).

Exercise 4A

1 Find the mid-point of the line joining these pairs of points:

a (4, 2), (6, 8) **b** (0, 6), (12, 2) **c** (2, 2), (−4, 6)
d (−6, 4), (6, −4) **e** (−5, 3), (7, 5) **f** (7, −4), (−3, 6)
g (−5, −5), (−11, 8) **h** $(6a, 4b), (2a, -4b)$ **i** $(2p, -q), (4p, 5q)$
j $(-2s, -7t), (5s, t)$ **k** $(-4u, 0), (3u, -2v)$ **l** $(a + b, 2a - b), (3a - b, -b)$
m $(4\sqrt{2}, 1), (2\sqrt{2}, 7)$ **n** $(-\sqrt{3}, 3\sqrt{5}), (5\sqrt{3}, 2\sqrt{5})$
o $(\sqrt{2} - \sqrt{3}, 3\sqrt{2} + 4\sqrt{3}), (3\sqrt{2} + \sqrt{3}, -\sqrt{2} + 2\sqrt{3})$

2 The line *PQ* is a diameter of a circle, where *P* and *Q* are (−4, 6) and (7, 8) respectively. Find the coordinates of the centre of the circle.

3 The line RS is a diameter of a circle, where R and S are $\left(\frac{4a}{5}, -\frac{3b}{4}\right)$ and $\left(\frac{2a}{5}, \frac{5b}{4}\right)$ respectively. Find the coordinates of the centre of the circle.

4 The line AB is a diameter of a circle, where A and B are $(-3, -4)$ and $(6, 10)$ respectively. Show that the centre of the circle lies on the line $y = 2x$.

5 The line JK is a diameter of a circle, where J and K are $(\frac{3}{4}, \frac{4}{3})$ and $(-\frac{1}{2}, 2)$ respectively. Show that the centre of the circle lies on the line $y = 8x + \frac{2}{3}$.

6 The line AB is a diameter of a circle, where A and B are $(0, -2)$ and $(6, -5)$ respectively. Show that the centre of the circle lies on the line $x - 2y - 10 = 0$.

7 The line FG is a diameter of the circle centre $(6, 1)$. Given F is $(2, -3)$, find the coordinates of G.

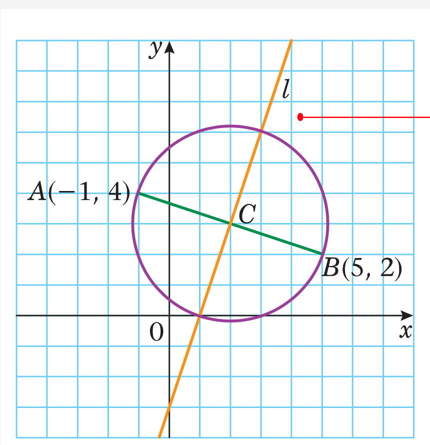
8 The line CD is a diameter of the circle centre $(-2a, 5a)$. Given D has coordinates $(3a, -7a)$, find the coordinates of C.

9 The points $M(3, p)$ and $N(q, 4)$ lie on the circle centre $(5, 6)$. The line MN is a diameter of the circle. Find the value of p and q.

10 The points $V(-4, 2a)$ and $W(3b, -4)$ lie on the circle centre $(b, 2a)$. The line VW is a diameter of the circle. Find the value of a and b.

Example 4

The line AB is a diameter of the circle centre C, where A and B are $(-1, 4)$ and $(5, 2)$ respectively. The line l passes through C and is perpendicular to AB. Find the equation of l.



Draw a sketch.

The centre of the circle is $\left(\frac{-1+5}{2}, \frac{4+2}{2}\right)$

$= \left(\frac{4}{2}, \frac{6}{2}\right)$

$= (2, 3)$

Use $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
Here $(x_1, y_1) = (-1, 4)$ and $(x_2, y_2) = (5, 2)$.

The gradient of the line AB is $\dfrac{2-4}{5-(-1)}$

$= \dfrac{-2}{6}$

$= -\dfrac{1}{3}$

So, the gradient of the line perpendicular to AB is 3.

The equation of the perpendicular line l is

$y - 3 = 3(x - 2)$

$y - 3 = 3x - 6$

So $y = 3x - 3$

Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Here $(x_1, y_1) = (-1, 4)$ and $(x_2, y_2) = (5, 2)$.

Simplify the fraction so divide by 2.

$\dfrac{-1}{3}$ is the same as $-\dfrac{1}{3}$.

Remember the product of the gradients of two perpendicular lines $= -1$, so $-\dfrac{1}{3} \times 3 = -1$.

The perpendicular line l passes through the point (2, 3) and has gradient 3, so use $y - y_1 = m(x - x_1)$ with $m = 3$ and $(x_1, y_1) = (2, 3)$.

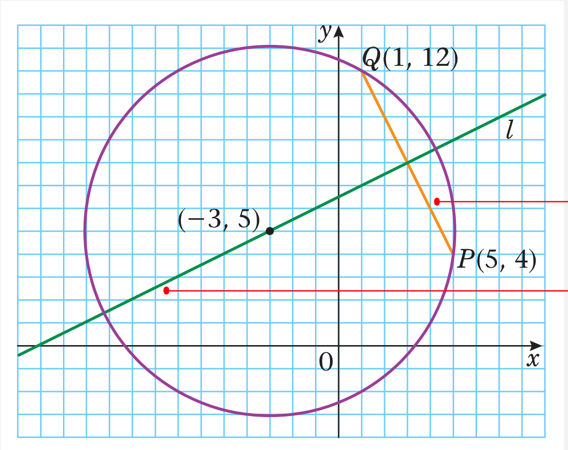
Rearrange the equation into the form $y = mx + c$.

Expand the brackets.

Add 3 to each side.

Example 5

The line PQ is a chord of the circle centre $(-3, 5)$, where P and Q are (5, 4) and (1, 12) respectively. The line l is perpendicular to PQ and bisects it. Show that l passes through the centre of the circle.



A chord is a line that joins any two points on the circumference of a circle.

The line l **bisects** PQ, so it passes through the mid-point of PQ.

First find the equation of l.

The mid-point of PQ is $\left(\dfrac{5+1}{2}, \dfrac{4+12}{2}\right)$

$= (3, 8)$

Use $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

Here $(x_1, y_1) = (5, 4)$ and $(x_2, y_2) = (1, 12)$.

The gradient of the chord PQ is $\dfrac{12-4}{1-5}$

$= \dfrac{8}{-4}$

$= -2$

Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Here $(x_1, y_1) = (5, 4)$ and $(x_2, y_2) = (1, 12)$.

So, the gradient of the line perpendicular to PQ is $\frac{1}{2}$.

The equation of the perpendicular line is

$$y - 8 = \frac{1}{2}(x - 3)$$
$$y - 8 = \frac{1}{2}x - \frac{3}{2}$$
$$y = \frac{1}{2}x - \frac{3}{2} + 8$$
$$y = \frac{1}{2}x - \frac{3}{2} + \frac{16}{2}$$
$$y = \frac{1}{2}x + \frac{16 - 3}{2}$$
$$y = \frac{1}{2}x + \frac{13}{2}$$

Substitute $x = -3$.

$$y = \frac{1}{2}(-3) + \frac{13}{2}$$
$$= -\frac{3}{2} + \frac{13}{2}$$
$$= \frac{-3 + 13}{2}$$
$$= \frac{10}{2}$$
$$= 5$$

So l passes through $(-3, 5)$.

So l passes through the centre of the circle.

Use the product of the gradients of two perpendicular lines $= -1$, so $-2 \times \frac{1}{2} = -1$.

The perpendicular line passes through the point (3, 8) and has gradient $\frac{1}{2}$, so use $y - y_1 = m(x - x_1)$ with $m = \frac{1}{2}$ and $(x_1, y_1) = (3, 8)$.

Expand the brackets.

Rearrange the equation into the form $y = mx + c$ by adding 8 to each side.

Simplify.

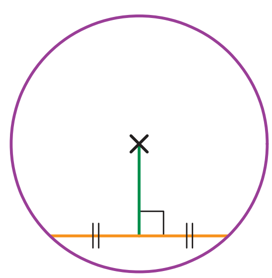
See if $(-3, 5)$ satisfies the equation of l, so substitute the x-coordinate into $y = \frac{1}{2}x + \frac{13}{2}$.

Simplify.

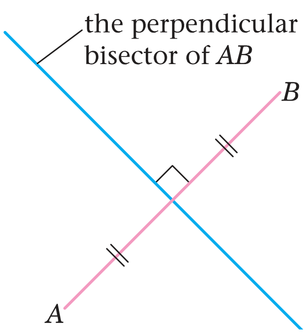
This is the y-coordinate, so $(-3, 5)$ is on the line.

The above example is a particular instance of this circle theorem:

■ **The perpendicular from the centre of a circle to a chord bisects the chord.**

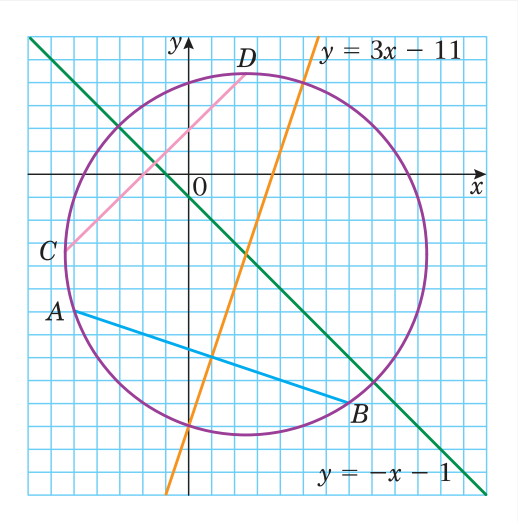


- A line that is perpendicular to a given line and bisects it is called the **perpendicular bisector.**



Example 6

The lines *AB* and *CD* are chords of a circle. The line $y = 3x - 11$ is the perpendicular bisector of *AB*. The line $y = -x - 1$ is the perpendicular bisector of *CD*. Find the coordinates of the centre of the circle.



Draw a sketch.

The perpendicular bisectors meet at the centre of the circle.

$y = 3x - 11$

$y = -x - 1$

Find where the perpendicular bisectors meet, so solve the equations simultaneously.

So $3x - 11 = -x - 1$ — Substitute $y = 3x - 11$ into $y = -x - 1$.

$4x - 11 = -1$ — Add x to each side.

$4x = 10$ — Add 11 to each side.

$x = \frac{10}{4}$ — Divide each side by 4.

$= \frac{5}{2}$ — Simplify the fraction so divide by 2.

Substitute $x = \frac{5}{2}$ — Substitute the x-coordinate into one of the equations to find the y-coordinate.

$y = 3\left(\frac{5}{2}\right) - 11$ — Here $y = 3x - 11$ and $x = \frac{5}{2}$. Simplify.

$= \frac{15}{2} - 11$

$= \frac{15}{2} - \frac{22}{2}$

$= \frac{15 - 22}{2}$

$= -\frac{7}{2}$

The centre of the circle is $\left(\frac{5}{2}, -\frac{7}{2}\right)$.

Exercise 4B

1 The line FG is a diameter of the circle centre C, where F and G are $(-2, 5)$ and $(2, 9)$ respectively. The line l passes through C and is perpendicular to FG.
Find the equation of l.

2 The line JK is a diameter of the circle centre P, where J and K are $(0, -3)$ and $(4, -5)$ respectively. The line l passes through P and is perpendicular to JK. Find the equation of l. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

3 The line AB is a diameter of the circle centre $(4, -2)$. The line l passes through B and is perpendicular to AB. Given that A is $(-2, 6)$,

a find the coordinates of B.

b Hence, find the equation of l.

4 The line PQ is a diameter of the circle centre $(-4, -2)$. The line l passes through P and is perpendicular to PQ. Given that Q is $(10, 4)$, find the equation of l.

5 The line RS is a chord of the circle centre $(5, -2)$, where R and S are $(2, 3)$ and $(10, 1)$ respectively. The line l is perpendicular to RS and bisects it. Show that l passes through the centre of the circle.

6 The line MN is a chord of the circle centre $(1, -\frac{1}{2})$, where M and N are $(-5, -5)$ and $(7, 4)$ respectively. The line l is perpendicular to MN and bisects it. Find the equation of l. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

7 The lines AB and CD are chords of a circle. The line $y = 2x + 8$ is the perpendicular bisector of AB. The line $y = -2x - 4$ is the perpendicular bisector of CD.
Find the coordinates of the centre of the circle.

8 The lines EF and GH are chords of a circle. The line $y = 3x - 24$ is the perpendicular bisector of EF. Given G and H are $(-2, 4)$ and $(4, 10)$ respectively, find the coordinates of the centre of the circle.

9 The points $P(3, 16)$, $Q(11, 12)$ and $R(-7, 6)$ lie on the circumference of a circle.

a Find the equation of the perpendicular bisector of

i PQ

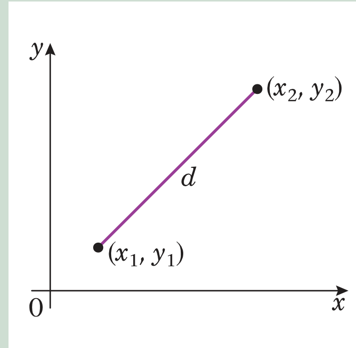
ii PR.

b Hence, find the coordinates of the centre of the circle.

10 The points $A(-3, 19)$, $B(9, 11)$ and $C(-15, 1)$ lie on the circumference of a circle.
Find the coordinates of the centre of the circle.

4.2 You can find the distance d between (x_1, y_1) and (x_2, y_2) by using the formula

$$d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}.$$

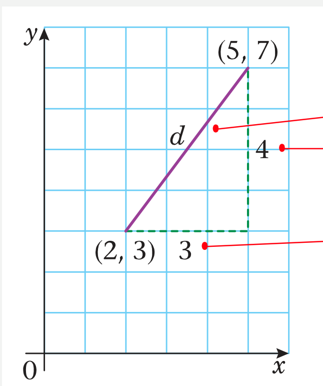


Example 7

Find the distance between these pairs of points:

a (2, 3), (5, 7) **b** $(4a, a)$, $(-3a, 2a)$ **c** $(2\sqrt{2}, -5\sqrt{2})$, $(4\sqrt{2}, \sqrt{2})$

a



Draw a sketch.

Let the distance between the points be d.

The difference in the y-coordinates is $7 - 3 = 4$.

The difference in the x-coordinates is $5 - 2 = 3$.

$d^2 = (5 - 2)^2 + (7 - 3)^2$

$d^2 = 3^2 + 4^2$

$d = \sqrt{(3^2 + 4^2)}$

$= \sqrt{25}$

$= 5$

Use Pythagoras' theorem:
$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

Take the square root of each side.

This is $d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$ with $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (5, 7)$.

b $d = \sqrt{[(-3a - 4a)^2 + (2a - a)^2]}$

$= \sqrt{[(-7a)^2 + a^2]}$

$= \sqrt{(49a^2 + a^2)}$

$= \sqrt{50a^2}$

$= \sqrt{25 \times 2 \times a^2}$

$= \sqrt{25} \times \sqrt{2} \times \sqrt{a^2}$

$= 5\sqrt{2}a$

Use $d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$. Here $(x_1, y_1) = (4a, a)$ and $(x_2, y_2) = (-3a, 2a)$.

$(-7a)^2 = -7a \times -7a$
$= 49a^2$

Simplify.

c $d = \sqrt{[(4\sqrt{2} - 2\sqrt{2})^2 + (\sqrt{2} - (-5\sqrt{2}))^2]}$

$4\sqrt{2} - 2\sqrt{2} = 2\sqrt{2}$

$\sqrt{2} - (-5\sqrt{2}) = \sqrt{2} + 5\sqrt{2}$

$= 6\sqrt{2}$

So $d = \sqrt{[(2\sqrt{2})^2 + (6\sqrt{2})^2]}$

$(2\sqrt{2})^2 = 2\sqrt{2} \times 2\sqrt{2}$

$= 2 \times 2 \times \sqrt{2} \times \sqrt{2}$

$= 4 \times 2$

$= 8$

$(6\sqrt{2})^2 = 6\sqrt{2} \times 6\sqrt{2}$

$= 6 \times 6 \times \sqrt{2} \times \sqrt{2}$

$= 36 \times 2$

$= 72$

So $d = \sqrt{(8 + 72)}$

$= \sqrt{80}$

$= \sqrt{(16 \times 5)}$

$= \sqrt{16} \times \sqrt{5}$

$= 4\sqrt{5}$

Use $d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$. Here $(x_1, y_1) = (2\sqrt{2}, -5\sqrt{2})$ and $(x_2, y_2) = (4\sqrt{2}, \sqrt{2})$.

Simplify the parts of the expression.

Simplify further.

Simplify the surd.

Example 8

The line PQ is a diameter of a circle, where P and Q are $(-1, 3)$ and $(6, -3)$ respectively. Find the radius of the circle.

The length of the diameter is:

$d = \sqrt{[(6 - (-1))^2 + (-3 - 3)^2]}$

$6 - (-1) = 6 + 1$

$= 7$

$-3 - 3 = -6$

So $d = \sqrt{[7^2 + (-6)^2]}$

$= \sqrt{(49 + 36)}$

$= \sqrt{85}$

So the length of the radius is $\frac{1}{2}\sqrt{85}$.

Use $d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$. Here $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (6, -3)$.

Simplify the parts of the expression.

Remember the radius is half the diameter.

Example 9

The line AB is a diameter of the circle, where A and B are $(-3, 21)$ and $(7, -3)$ respectively. The point $C(14, 4)$ lies on the circumference of the circle. Find the value of AB^2, AC^2 and BC^2. Hence, show that $\angle ACB = 90°$.

$AB^2 = (7-(-3))^2 + (-3-21)^2$
$= 10^2 + (-24)^2$
$= 676$

Use $d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$. Square each side so that $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. Here $(x_1, y_1) = (-3, 21)$ and $(x_2, y_2) = (7, -3)$.

$AC^2 = (14-(-3))^2 + (4-21)^2$
$= 17^2 + (-17)^2$
$= 578$

Here $(x_1, y_1) = (-3, 21)$ and $(x_2, y_2) = (14, 4)$.

$BC^2 = (14-7)^2 + (4-(-3))^2$
$= 7^2 + 7^2$
$= 98$

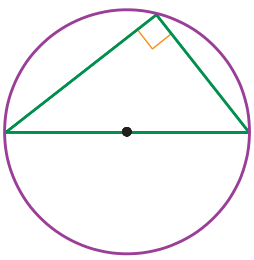
Here $(x_1, y_1) = (7, -3)$ and $(x_2, y_2) = (14, 4)$.

Now, $578 + 98 = 676$
So, $\angle ACB = 90°$.

Use Pythagoras' theorem to test if the triangle has a right angle. This is $AC^2 + BC^2 = AB^2$.

This is a particular instance of this circle theorem:

■ **The angle in a semicircle is a right angle.**



Exercise 4C

1 Find the distance between these pairs of points:

a $(0, 1), (6, 9)$ **b** $(4, -6), (9, 6)$ **c** $(3, 1), (-1, 4)$
d $(3, 5), (4, 7)$ **e** $(2, 9), (4, 3)$ **f** $(0, -4), (5, 5)$
g $(-2, -7), (5, 1)$ **h** $(-4a, 0), (3a, -2a)$ **i** $(-b, 4b), (-4b, -2b)$
j $(2c, c), (6c, 4c)$ **k** $(-4d, d), (2d, -4d)$ **l** $(-e, -e), (-3e, -5e)$
m $(3\sqrt{2}, 6\sqrt{2}), (2\sqrt{2}, 4\sqrt{2})$ **n** $(-\sqrt{3}, 2\sqrt{3}), (3\sqrt{3}, 5\sqrt{3})$
o $(2\sqrt{3} - \sqrt{2}, \sqrt{5} + \sqrt{3}), (4\sqrt{3} - \sqrt{2}, 3\sqrt{5} + \sqrt{3})$

2 The point $(4, -3)$ lies on the circle centre $(-2, 5)$. Find the radius of the circle.

3 The point $(14, 9)$ is the centre of the circle radius 25. Show that $(-10, 2)$ lies on the circle.

4 The line MN is a diameter of a circle, where M and N are $(6, -4)$ and $(0, -2)$ respectively. Find the radius of the circle.

5 The line QR is a diameter of the circle centre C, where Q and R have coordinates $(11, 12)$ and $(-5, 0)$ respectively. The point P is $(13, 6)$.

a Find the coordinates of C. **b** Show that P lies on the circle.

6 The points $(-3, 19)$, $(-15, 1)$ and $(9, 1)$ are vertices of a triangle. Show that a circle centre $(-3, 6)$ can be drawn through the vertices of the triangle.

Hint for question 6: Show that the vertices are equidistant from the centre of the circle.

7 The line ST is a diameter of the circle c_1, where S and T are $(5, 3)$ and $(-3, 7)$ respectively. The line UV is a diameter of the circle c_2 centre $(4, 4)$. The point U is $(1, 8)$.

a Find the radius of **i** c_1 **ii** c_2.

b Find the distance between the centres of c_1 and c_2.

8 The points $U(-2, 8)$, $V(7, 7)$ and $W(-3, -1)$ lie on a circle.

a Show that $\triangle UVW$ has a right angle.

b Find the coordinates of the centre of the circle.

9 The points $A(2, 6)$, $B(5, 7)$ and $C(8, -2)$ lie on a circle.

a Show that $\triangle ABC$ has a right angle. **b** Find the area of the triangle.

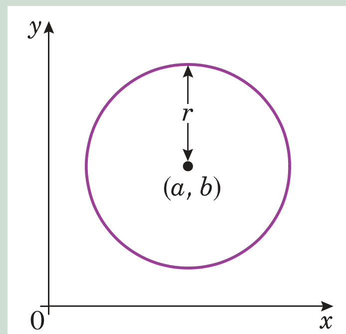
10 The points $A(-1, 9)$, $B(6, 10)$, $C(7, 3)$ and $D(0, 2)$ lie on a circle.

a Show that $ABCD$ is a square. **b** Find the area of $ABCD$.

c Find the centre of the circle.

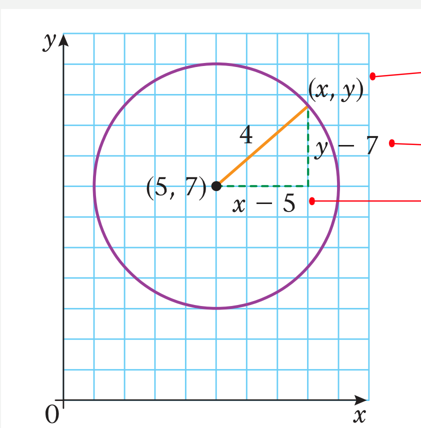
4.3

You can write the equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$, where (a, b) is the centre and r is the radius.



Example 10

Write down the equation of the circle with centre (5, 7) and radius 4.



Equation of circle is $(x - 5)^2 + (y - 7)^2 = 4^2$.

(x, y) is *any* point on the circumference of the circle. The distance between (x, y) and $(5, 7)$ is always 4.

The difference in the y-coordinates is $y - 7$.

The difference in the x-coordinates is $x - 5$.

To find the equation of the circle, use $d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$. Square each side so that $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. Here $(x_1, y_1) = (5, 7)$ and $(x_2, y_2) = (x, y)$. This is in the form $(x - a)^2 + (y - b)^2 = r^2$ with $(a, b) = (5, 7)$ and $r = 4$.

Example 11

Write down the coordinates of the centre and the radius of these circles:

a $(x + 3)^2 + (y - 1)^2 = 4^2$ **b** $(x - \frac{5}{2})^2 + (y - 3)^2 = 32$

a $(x + 3)^2 + (y - 1)^2 = 4^2$

$(x - (-3))^2 + (y - 1)^2 = 4^2$

So centre $= (-3, 1)$, radius $= 4$.

Write the equation in the form $(x - a)^2 + (y - b)^2 = r^2$, using $-(-3) = +3$.
So $a = -3$, $b = 1$ and $r = 4$.

b $\left(x - \frac{5}{2}\right)^2 + (y - 3)^2 = 32$

$\left(x - \frac{5}{2}\right)^2 + (y - 3)^2 = (\sqrt{32})^2$

$\sqrt{32} = \sqrt{(16 \times 2)}$

$= \sqrt{16} \times \sqrt{2}$

$= 4\sqrt{2}$

So centre $= \left(\frac{5}{2}, 3\right)$, radius $= 4\sqrt{2}$.

Write the equation in the form $(x - a)^2 + (y - b)^2 = r^2$.
So $a = \frac{5}{2}$, $b = 3$ and $r = \sqrt{32}$.
Simplify $\sqrt{32}$.

Example 12

Show that the circle $(x - 3)^2 + (y + 4)^2 = 20$ passes through $(5, -8)$.

$(x - 3)^2 + (y + 4)^2 = 20$

Substitute $(5, -8)$

$(5 - 3)^2 + (-8 + 4)^2 = 2^2 + (-4)^2$

$= 4 + 16$

$= 20$

So the circle passes through the point $(5, -8)$.

Substitute $x = 5$ and $y = -8$ into the equation of the circle.

$(5, -8)$ satisfies the equation of the circle.

Example 13

The line AB is a diameter of a circle, where A and B are (4, 7) and (−8, 3) respectively. Find the equation of the circle.

The length of AB is $\sqrt{[(4-(-8))^2+(7-3)^2]}$

$= \sqrt{12^2+4^2}$

$= \sqrt{144+16}$

$= \sqrt{160}$

$= \sqrt{16\times 10}$

$= \sqrt{16}\times\sqrt{10}$

$= 4\sqrt{10}$

So the radius is $2\sqrt{10}$.

The centre is $\left(\frac{4+(-8)}{2}, \frac{7+3}{2}\right) = (-2, 5)$.

The equation of the circle is

$(x+2)^2+(y-5)^2=(2\sqrt{10})^2$.

Use $d=\sqrt{[(x_2-x_1)^2+(y_2-y_1)^2]}$.
Here $(x_1, y_1) = (-8, 3)$ and $(x_2, y_2) = (4, 7)$.

Simplify.

Remember the radius = diameter ÷ 2.

Remember the centre of a circle is at the mid-point of a diameter.

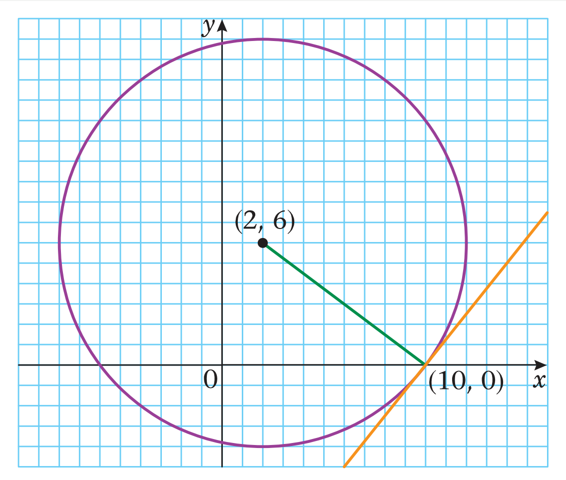
Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Here $(x_1, y_1) = (4, 7)$ and $(x_2, y_2) = (-8, 3)$.

Use $(x-a)^2+(y-b)^2=r^2$.
Here $(a, b) = (-2, 5)$ and $r = 2\sqrt{10}$.

Example 14

The line $4x - 3y - 40 = 0$ touches the circle $(x-2)^2+(y-6)^2=100$ at $P(10, 0)$. Show that the radius at P is perpendicular to the line.



$4x - 3y - 40 = 0$

$3y = 4x - 40$

$y = \frac{4}{3}x - \frac{40}{3}$

The gradient of the line is $\frac{4}{3}$.

First find the gradient of the line, so rearrange its equation into the form $y = mx + c$.

Add $3y$ to each side and turn the equation around.

Divide each term by 3.

Compare $y = \frac{4}{3}x - \frac{40}{3}$ to $y = mx + c$, so $m = \frac{4}{3}$.

$(x - 2)^2 + (y - 6)^2 = 100$

The centre is (2, 6).

To find the gradient of the radius at P, first find the centre of the circle from its equation.

Compare $(x - 2)^2 + (y - 6)^2 = 100$ to $(x - a)^2 + (y - b)^2 = r^2$, where (a, b) is the centre.

So the gradient of the radius at $P = \dfrac{6 - 0}{2 - 10}$

$= \dfrac{6}{-8}$

$= -\dfrac{3}{4}$

Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.
Here $(x_1, y_1) = (10, 0)$ and $(x_2, y_2) = (2, 6)$.

Simplify the fraction, so divide top and bottom by 2.

Now, $\dfrac{4}{3} \times -\dfrac{3}{4} = -1$.

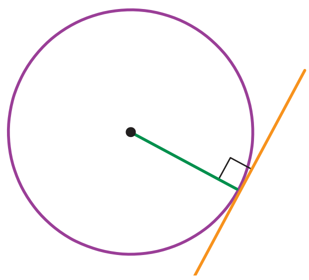
So, the radius at P is perpendicular to the line.

Test to see if the radius is perpendicular to the line.

Use the product of the gradients of two perpendicular lines $= -1$.

The above example is a particular instance of this circle theorem:

- **The angle between the tangent and a radius is 90°.**
- **A tangent meets a circle at one point only.**



Exercise 4D

1 Write down the equation of these circles:

a Centre (3, 2), radius 4

b Centre (−4, 5), radius 6

c Centre (5, −6), radius $2\sqrt{3}$

d Centre $(2a, 7a)$, radius $5a$

e Centre $(-2\sqrt{2}, -3\sqrt{2})$, radius 1

2 Write down the coordinates of the centre and the radius of these circles:

a $(x + 5)^2 + (y - 4)^2 = 9^2$

b $(x - 7)^2 + (y - 1)^2 = 16$

c $(x + 4)^2 + y^2 = 25$

d $(x + 4a)^2 + (y + a)^2 = 144a^2$

e $(x - 3\sqrt{5})^2 + (y + \sqrt{5})^2 = 27$

3 Find the centre and radius of these circles by first writing in the form $(x - a)^2 + (y - b)^2 = r^2$

a $x^2 + y^2 + 4x + 9y + 3 = 0$

b $x^2 + y^2 + 5x - 3y - 8 = 0$

c $2x^2 + 2y^2 + 8x + 15y - 1 = 0$

d $2x^2 + 2y^2 - 8x + 8y + 3 = 0$

4 In each case, show that the circle passes through the given point:

a $(x-2)^2+(y-5)^2=13$, $(4, 8)$

b $(x+7)^2+(y-2)^2=65$, $(0, -2)$

c $x^2+y^2=25^2$, $(7, -24)$

d $(x-2a)^2+(y+5a)^2=20a^2$, $(6a, -3a)$

e $(x-3\sqrt{5})^2+(y-\sqrt{5})^2=(2\sqrt{10})^2$, $(\sqrt{5}, -\sqrt{5})$

5 The point $(4, -2)$ lies on the circle centre $(8, 1)$. Find the equation of the circle.

6 The line PQ is the diameter of the circle, where P and Q are $(5, 6)$ and $(-2, 2)$ respectively. Find the equation of the circle.

7 The point $(1, -3)$ lies on the circle $(x-3)^2+(y+4)^2=r^2$. Find the value of r.

8 The line $y=2x+13$ touches the circle $x^2+(y-3)^2=20$ at $(-4, 5)$. Show that the radius at $(-4, 5)$ is perpendicular to the line.

9 The line $x+3y-11=0$ touches the circle $(x+1)^2+(y+6)^2=90$ at $(2, 3)$.

a Find the radius of the circle.

b Show that the radius at $(2, 3)$ is perpendicular to the line.

10 The point $P(1, -2)$ lies on the circle centre $(4, 6)$.

a Find the equation of the circle.

b Find the equation of the tangent to the circle at P.

11 The tangent to the circle $(x+4)^2+(y-1)^2=242$ at $(7, -10)$ meets the y-axis at S and the x-axis at T.

a Find the coordinates of S and T.

b Hence, find the area of $\triangle OST$, where O is the origin.

Example 15

Find where the circle $(x-5)^2+(y-4)^2=65$ meets the x-axis.

$(x-5)^2+(y-4)^2=65$

Substitute $y=0$ — The circle meets the x-axis when $y=0$, so substitute $y=0$ into the equation.

$(x-5)^2+(-4)^2=65$

$(x-5)^2+16=65$ — $-4\times-4=16$

$(x-5)^2=49$ — Subtract 16 from each side.

$x-5=\pm7$ — Take the square root of each side, so that $\sqrt{49}=\pm7$.

So $x-5=7$

$x=12$

and $x-5=-7$

$x=-2$ — Work out the values of x separately, adding 5 to each side in both cases.

So the circle meets the x-axis at $(-2, 0)$ and $(12, 0)$.

Example 16

Find where the line $y = x + 5$ meets the circle $x^2 + (y - 2)^2 = 29$.

$$x^2 + (y-2)^2 = 29$$
$$x^2 + (x+5-2)^2 = 29$$
$$x^2 + (x+3)^2 = 29$$
$$x^2 + x^2 + 6x + 9 = 29$$
$$2x^2 + 6x - 20 = 0$$
$$x^2 + 3x - 10 = 0$$
$$(x+5)(x-2) = 0$$

So $x = -5$ and $x = 2$.

Substitute $x = -5$

$$y = -5 + 5$$
$$= 0$$

Substitute $x = 2$

$$y = 2 + 5$$
$$= 7$$

So the line meets the circle at $(-5, 0)$ and $(2, 7)$.

Solve the equations simultaneously, so substitute $y = x + 5$ into the equation of the circle.

Simplify inside the brackets.

Expand the brackets.

Add the x^2 terms and subtract 29 from each side.

Divide each term by 2.

Factorise the quadratic:
$5 \times -2 = -10$
$5 + (-2) = +3$

Now find the y-coordinates, so substitute the values of x into the equation of the line.

Remember to write the answer as coordinates.

Example 17

Show that the line $y = x - 7$ does not meet the circle $(x + 2)^2 + y^2 = 33$.

$$(x+2)^2 + y^2 = 33$$
$$(x+2)^2 + (x-7)^2 = 33$$
$$x^2 + 4x + 4 + x^2 - 14x + 49 = 33$$
$$2x^2 - 10x + 20 = 0$$
$$x^2 - 5x + 10 = 0$$

Now $b^2 - 4ac = (-5)^2 - 4 \times 1 \times 10$
$= 25 - 40$
$= -15$

$b^2 - 4ac < 0$, so the line does not meet the circle.

Solve the equations simultaneously, so substitute $y = x - 7$ into the equation of the circle.

Expand the brackets.

Collect like terms and subtract 33 from each side.

Simplify the quadratic, so divide each term by 2.

Use the discriminant $b^2 - 4ac$ to test for roots of the quadratic equation.

Remember
If $b^2 - 4ac > 0$ there are two distinct roots.
If $b^2 - 4ac = 0$ there is a repeated root.
If $b^2 - 4ac < 0$ there are no real roots.

As $b^2 - 4ac < 0$, there is no solution to the quadratic equation. So, the line does not meet the circle.

Exercise 4E

1 Find where the circle $(x-1)^2+(y-3)^2=45$ meets the x-axis.

2 Find where the circle $(x-2)^2+(y+3)^2=29$ meets the y-axis.

3 The circle $(x-3)^2+(y+3)^2=34$ meets the x-axis at $(a, 0)$ and the y-axis at $(0, b)$. Find the possible values of a and b.

4 The line $y=x+4$ meets the circle $(x-3)^2+(y-5)^2=34$ at A and B. Find the coordinates of A and B.

5 Find where the line $x+y+5=0$ meets the circle $(x+3)^2+(y+5)^2=65$.

6 Show that the line $y=x-10$ does not meet the circle $(x-2)^2+y^2=25$.

7 Show that the line $x+y=11$ is a tangent to the circle $x^2+(y-3)^2=32$.

Hint for question 7: Show that the line meets the circle at one point only.

8 Show that the line $3x-4y+25=0$ is a tangent to the circle $x^2+y^2=25$.

9 The line $y=2x-2$ meets the circle $(x-2)^2+(y-2)^2=20$ at A and B.

a Find the coordinates of A and B.

b Show that AB is a diameter of the circle.

10 The line $x+y=a$ meets the circle $(x-p)^2+(y-6)^2=20$ at $(3, 10)$, where a and p are constants.

a Work out the value of a. **b** Work out the two possible values of p.

Mixed exercise 4F

1 The line $y=2x-8$ meets the coordinate axes at A and B. The line AB is a diameter of the circle. Find the equation of the circle.

2 The circle centre $(8, 10)$ meets the x-axis at $(4, 0)$ and $(a, 0)$.

a Find the radius of the circle. **b** Find the value of a.

3 The circle $(x-5)^2+y^2=36$ meets the x-axis at P and Q. Find the coordinates of P and Q.

4 The circle $(x+4)^2+(y-7)^2=121$ meets the y-axis at $(0, m)$ and $(0, n)$. Find the value of m and n.

5 The line $y=0$ is a tangent to the circle $(x-8)^2+(y-a)^2=16$. Find the value of a.

6 The point $A(-3, -7)$ lies on the circle centre $(5, 1)$. Find the equation of the tangent to the circle at A.

7 The circle $(x+3)^2+(y+8)^2=100$ meets the positive coordinate axes at $A(a, 0)$ and $B(0, b)$.

a Find the value of a and b.

b Find the equation of the line AB.

8 The circle $(x+2)^2+(y-5)^2=169$ meets the positive coordinate axes at $C(c, 0)$ and $D(0, d)$.

a Find the value of c and d. **b** Find the area of $\triangle OCD$, where O is the origin.

9 The circle, centre (p, q) radius 25, meets the x-axis at $(-7, 0)$ and $(7, 0)$, where $q > 0$.

a Find the value of p and q.

b Find the coordinates of the points where the circle meets the y-axis.

10 Show that $(0, 0)$ lies inside the circle $(x - 5)^2 + (y + 2)^2 = 30$.

11 The points $A(-4, 0)$, $B(4, 8)$ and $C(6, 0)$ lie on a circle. The lines AB and BC are chords of the circle. Find the coordinates of the centre of the circle.

12 The points $R(-4, 3)$, $S(7, 4)$ and $T(8, -7)$ lie on a circle.

a Show that $\triangle RST$ has a right angle. **b** Find the equation of the circle.

13 The points $A(-7, 7)$, $B(1, 9)$, $C(3, 1)$ and $D(-7, 1)$ lie on a circle. The lines AB and CD are chords of the circle.

a Find the equation of the perpendicular bisector of **i** AB **ii** CD.

b Find the coordinates of the centre of the circle.

14 The centres of the circles $(x - 8)^2 + (y - 8)^2 = 117$ and $(x + 1)^2 + (y - 3)^2 = 106$ are P and Q respectively.

a Show that P lies on $(x + 1)^2 + (y - 3)^2 = 106$.

b Find the length of PQ.

15 The line $y = -3x + 12$ meets the coordinate axes at A and B.

a Find the coordinates of A and B.

b Find the coordinates of the mid-point of AB.

c Find the equation of the circle that passes through A, B and O, where O is the origin.

16 The points $A(-5, 5)$, $B(1, 5)$, $C(3, 3)$ and $D(3, -3)$ lie on a circle. Find the equation of the circle.

17 The line AB is a chord of a circle centre $(2, -1)$, where A and B are $(3, 7)$ and $(-5, 3)$ respectively. AC is a diameter of the circle. Find the area of $\triangle ABC$.

18 The points $A(-1, 0)$, $B(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $C\,(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ are the vertices of a triangle.

a Show that the circle $x^2 + y^2 = 1$ passes through the vertices of the triangle.

b Show that $\triangle ABC$ is equilateral.

19 The points $P(2, 2)$, $Q(2 + \sqrt{3}, 5)$ and $R(2 - \sqrt{3}, 5)$ lie on the circle $(x - 2)^2 + (y - 4)^2 = r^2$.

a Find the value of r. **b** Show that $\triangle PQR$ is equilateral.

20 The points $A(-3, -2)$, $B(-6, 0)$ and $C(p, q)$ lie on a circle centre $(-\frac{5}{2}, 2)$. The line BC is a diameter of the circle.

a Find the value of p and q.

b Find the gradient of **i** AB **ii** AC.

c Show that AB is perpendicular to AC.

21 Find the centre and radius of the circle with equation $x^2 + y^2 - 6x - 2y - 6 = 0$.

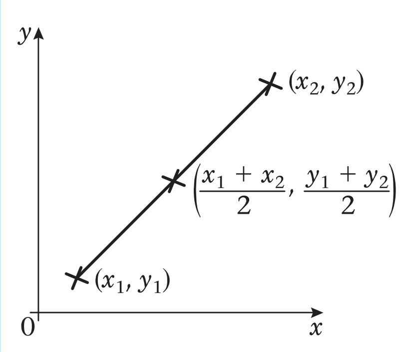
22 Find the centre and radius of the circle with equation $x^2 + y^2 - 10x + 16y - 11 = 0$.

23 Find the centre and radius of the circle with equation $x^2 + y^2 + 8x - 6y + 18 = 0$.

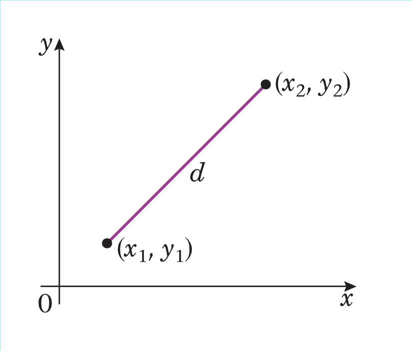
24 Find the centre and radius of the circle with equation $x^2 + y^2 - x - 3y - 2 = 0$.

Summary of key points

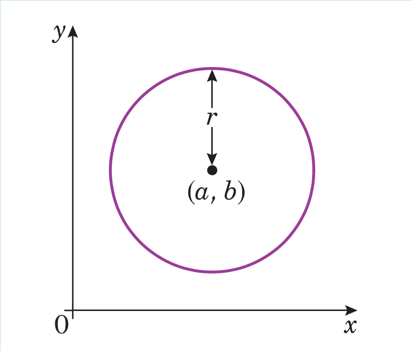
1 The mid-point of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.



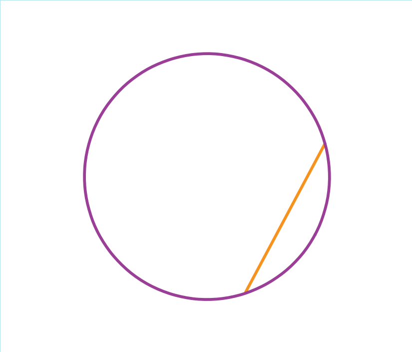
2 The distance d between (x_1, y_1) and (x_2, y_2) is $d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$.



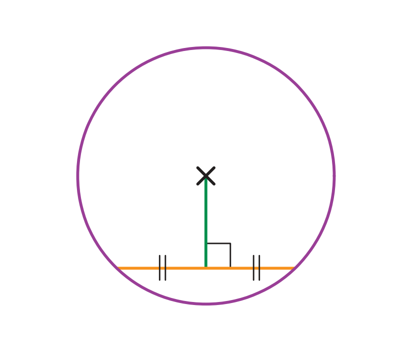
3 The equation of the circle centre (a, b) radius r is $(x - a)^2 + (y - b)^2 = r^2$.



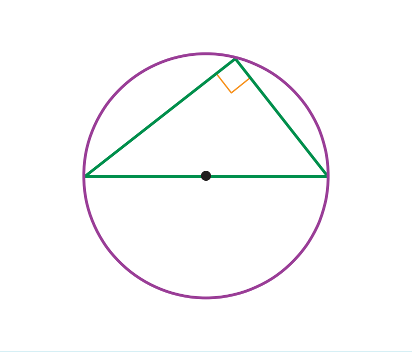
4 A chord is a line that joins two points on the circumference of a circle.



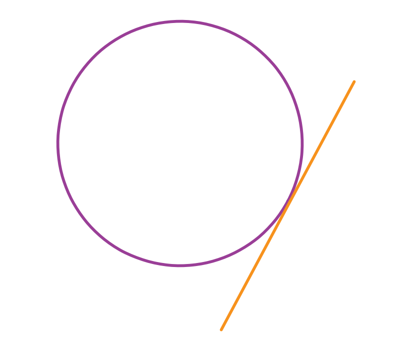
5 The perpendicular from the centre of a circle to a chord bisects the chord.



6 The angle in a semicircle is a right angle.



7 A tangent is a line that meets a circle at one point only.



8 The angle between a tangent and a radius is 90°.