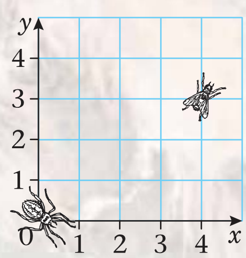
5 The binomial expansion

After completing this chapter you should be able to

1. use Pascal's Triangle to expand expressions of the form $(a + b)^n$
2. use combination and factorial notation to expand expressions of the form $(a + b)^n$
3. use the expansion of $(1 + x)^n$ to expand $(a + b)^n$.

You will revisit the above techniques in Core 4 when you will expand expressions when n is not a positive integer. In the meantime see if you can solve the problem below.

Sammy the spider has a rectangular shaped web. He sleeps and eats at the point (0, 0). How many different ways are there to catch a fly that lands at the point (4, 3) if he only can move ↑ or →



If you cannot solve this problem straight away write down how many different ways there are to get to all coordinate points. Complete Section 5.1 and see if you can see the link.

5.1 You can use Pascal's Triangle to quickly expand expressions such as $(x + 2y)^3$.

Consider the following:

$$(a+b)^1 = a + b$$

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 = (a+b)(a^2 + 2ab + b^2)\\ &= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)\\ &= a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

Similarly $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Setting these results out in order starting with $(a + b)^0$ we find that:

$$(a+b)^0 = 1$$

$$(a+b)^1 = 1a + 1b$$

$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

Hint: The terms all have the same index as the original expression. For example, look at the line for $(a + b)^3$. All of the terms have a total index of 3 (a^3, a^2b, ab^2 and b^3).

You should notice the following patterns:

- The coefficients form a pattern that is known as Pascal's Triangle.

```
                  1
              1       1
          1   +   2       1
      1       3   +   3       1
  1       4       6       4       1
1     5      10      10       5       1
1   6    15      20      15       6    1
```

Hint: To get from one line to the next you add adjacent pairs of numbers.

Example 1

Use Pascal's Triangle to find the expansions of:

a $(x + 2y)^3$

b $(2x - 5)^4$

a $(x + 2y)^3$

The coefficients are 1, 3, 3, 1 so:

$(x + 2y)^3 = 1x^3 + 3x^2(2y) + 3x(2y)^2 + 1(2y)^3$

$= x^3 + 6x^2y + 12xy^2 + 8y^3$

Index = 3 so look at the 4th line in Pascal's Triangle to find the coefficients.

Use the expansion of $(a + b)^3$. Remember $(2y)^2 = 4y^2$.

b $(2x - 5)^4$

The coefficients are 1, 4, 6, 4, 1 and terms are:

$(2x)^4, (2x)^3(-5)^1, (2x)^2(-5)^2, (2x)^1(-5)^3, (-5)^4$

So $(2x - 5)^4 = 1(2x)^4 + 4(2x)^3(-5)^1 + 6(2x)^2(-5)^2 + 4(2x)^1(-5)^3 + 1(-5)^4$

$= 16x^4 - 160x^3 + 600x^2 - 1000x + 625$

Index = 4 so look at the 5th line of Pascal's Triangle.

Use the expansion of $(a + b)^4$.

Careful with the negative numbers!

Example 2

The coefficient of x^2 in the expansion of $(2 - cx)^3$ is 294. Find the possible value(s) of the constant c.

The coefficients are 1, 3, 3, 1

The term in x^2 is $3 \times 2(-cx)^2 = 6c^2x^2$

So $6c^2 = 294$

$c^2 = 49$

$c = \pm 7$

Index = 3, so use the 4th line of Pascal's Triangle to find coefficients.

From the expansion of $(a + b)^3$ the x^2 term is $3ab^2$ where $a = 2$ and $b = -cx$.

Set up and solve an equation in c.

Exercise 5A

1 Write down the expansion of:

a $(x + y)^4$ **b** $(p + q)^5$ **c** $(a - b)^3$ **d** $(x + 4)^3$
e $(2x - 3)^4$ **f** $(a + 2)^5$ **g** $(3x - 4)^4$ **h** $(2x - 3y)^4$

2 Find the coefficient of x^3 in the expansion of:

a $(4 + x)^4$ **b** $(1 - x)^5$ **c** $(3 + 2x)^3$ **d** $(4 + 2x)^5$
e $(2 + x)^6$ **f** $(4 - \frac{1}{2}x)^4$ **g** $(x + 2)^5$ **h** $(3 - 2x)^4$

3 Fully expand the expression $(1 + 3x)(1 + 2x)^3$.

4 Expand $(2 + y)^3$. Hence or otherwise, write down the expansion of $(2 + x - x^2)^3$ in ascending powers of x.

5 Find the coefficient of the term in x^3 in the expansion of $(2 + 3x)^3(5 - x)^3$.

6 The coefficient of x^2 in the expansion of $(2 + ax)^3$ is 54. Find the possible values of the constant a.

7 The coefficient of x^2 in the expansion of $(2 - x)(3 + bx)^3$ is 45. Find possible values of the constant b.

8 Find the term independent of x in the expansion of $\left(x^2 - \dfrac{1}{2x}\right)^3$.

5.2 You can use combinations and factorial notation to help you expand binomial expressions. For larger indices, it is quicker than using Pascal's Triangle.

Suppose that three people A, B and C are running a race. There are six different outcomes for their finishing positions.
The number can be calculated as:

A, B, C
A, C, B
B, A, C
B, C, A
C, A, B
C, B, A

$3 \times 2 \times 1$

There are three runners in the race: A, B or C.

After the winner has crossed the line there are 2 choices for second place.

After the first and second places have been awarded, there is only 1 place left for the third place.

We can represent $3 \times 2 \times 1$ using what is termed factorial notation.
$3!$, pronounced '3 factorial' $= 3 \times 2 \times 1$.

■ **$n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \ldots \times 3 \times 2 \times 1$**

Note: By definition, $0! = 1$

Suppose you wish to choose any two letters from A, B and C, where order does not matter. There are three different outcomes. We can represent this by ${}^3C_2$ or $\binom{3}{2} = \dfrac{3!}{2!1!}$.

- **The number of ways of choosing r items from a group of n items is written ^nC_r or $\binom{n}{r}$ and is calculated by $\dfrac{n!}{(n-r)!r!}$**

  e.g. $^3C_2 = \dfrac{3!}{(3-2)!2!} = \dfrac{6}{1 \times 2} = 3$

Exercise 5B

1 Find the values of the following:

a $4!$ **b** $6!$ **c** $\dfrac{8!}{6!}$ **d** $\dfrac{10!}{9!}$

e $^4C_2$ **f** $^8C_6$ **g** $^5C_2$ **h** $^6C_3$

i $^{10}C_9$ **j** $^6C_2$ **k** $^8C_5$ **l** $^nC_3$

2 Calculate:

a $^4C_0$ **b** $\binom{4}{1}$ **c** $^4C_2$ **d** $\binom{4}{3}$ **e** $\binom{4}{4}$

Now look at line 5 of Pascal's Triangle. Can you find any connection?

3 Write using combination notation:

a Line 3 of Pascal's Triangle.

b Line 5 of Pascal's Triangle.

4 Why is $^6C_2$ equal to $\binom{6}{4}$?

a Answer using ideas on choosing from a group.

b Answer by calculating both quantities.

5.3 You can use $\binom{n}{r}$ to work out the coefficients in the binomial expansion.

- **The binomial expansion is**

$$(a+b)^n = \underbrace{(a+b)(a+b)\ \ldots\ (a+b)}_{n \text{ times}}$$

$$= {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + {}^nC_3a^{n-3}b^3 + \ldots + {}^nC_nb^n$$

$$\text{or } \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b + \ldots + \binom{n}{n}b^n$$

- **Similarly,**

$$(a+bx)^n = {}^nC_0a^n + {}^nC_1a^{n-1}bx + {}^nC_2a^{n-2}b^2x^2 + {}^nC_3a^{n-3}b^3x^3 + \ldots {}^nC_nb^nx^n$$

$$\text{or } \binom{n}{0}a^n + \binom{n}{1}a^{n-1}bx + \binom{n}{2}a^{n-2}b^2x^2 + \binom{n}{3}a^{n-3}b^3x^3 + \ldots + \binom{n}{n}b^nx^n$$

Hint: You do not need to memorise **both** these forms of the binomial expansion. You should be able to work out this form from the expansion of $(a+b)^n$.

Example 3

Use the binomial theorem to find the expansion of $(3 - 2x)^5$:

$$(3-2x)^5 = 3^5 + \binom{5}{1}3^4(-2x) + \binom{5}{2}3^3(-2x)^2 + \binom{5}{3}3^2(-2x)^3 + \binom{5}{4}3^1(-2x)^4 + (-2x)^5$$

$$= 243 - 810x + 1080x^2 - 720x^3 + 240x^4 - 32x^5$$

There will be 6 terms.
The terms have a total index of 5.
Use $(a + bx)^n$ where $a = 3$, $b = -2x$ and $n = 5$.
There are $\binom{5}{2}$ ways of choosing 2 '$-2x$' terms from 5 brackets.

Example 4

Find the first four terms in ascending powers of x of $\left(1 - \frac{x}{4}\right)^{10}$ and, by using a suitable substitution, use your result to find an approximate value to $(0.975)^{10}$. Use your calculator to find the degree of accuracy of your approximation.

$$\left(1 - \frac{x}{4}\right)^{10}$$

Terms are 1^{10}, $1^9\left(-\frac{x}{4}\right)^1$, $1^8\left(-\frac{x}{4}\right)^2$, and $1^7\left(-\frac{x}{4}\right)^3$

All terms have total index = 10.

Coefficients are ${}^{10}C_0$ ${}^{10}C_1$ ${}^{10}C_2$ ${}^{10}C_3$

You are selecting 2 '$-\frac{x}{4}$'s from 10 brackets.

Combining, we get the first four terms to equal:

$${}^{10}C_0 1^{10} + {}^{10}C_1(1)^9\left(-\frac{x}{4}\right)^1 + {}^{10}C_2(1)^8\left(-\frac{x}{4}\right)^2 + {}^{10}C_3(1)^7\left(-\frac{x}{4}\right)^3 + \ldots$$

$$= 1 - 2.5x + 2.8125x^2 - 1.875x^3 \ldots$$

We want $\left(1 - \frac{x}{4}\right) = 0.975$

$$\frac{x}{4} = 0.025$$

$$x = 0.1$$

Calculate the value of x.

Substitute $x = 0.1$ into the expansion for $\left(1 - \frac{x}{4}\right)^{10}$:

$$0.975^{10} = 1 - 0.25 + 0.028\,125 - 0.001\,875$$
$$= 0.77625$$

Substitute $x = 0.1$ into your expansion.

Using a calculator $(0.975)^{10} = 0.776\,329\,62$

So approximation is correct to 4 decimal places.

Exercise 5C

1 Write down the expansion of the following:

a $(2x + y)^4$ **b** $(p - q)^5$ **c** $(1 + 2x)^4$ **d** $(3 + x)^4$
e $(1 - \frac{1}{2}x)^4$ **f** $(4 - x)^4$ **g** $(2x + 3y)^5$ **h** $(x + 2)^6$

2 Find the term in x^3 of the following expansions:

a $(3 + x)^5$ **b** $(2x + y)^5$ **c** $(1 - x)^6$ **d** $(3 + 2x)^5$
e $(1 + x)^{10}$ **f** $(3 - 2x)^6$ **g** $(1 + x)^{20}$ **h** $(4 - 3x)^7$

3 Use the binomial theorem to find the first four terms in the expansion of:

a $(1 + x)^{10}$ **b** $(1 - 2x)^5$ **c** $(1 + 3x)^6$ **d** $(2 - x)^8$
e $(2 - \frac{1}{2}x)^{10}$ **f** $(3 - x)^7$ **g** $(x + 2y)^8$ **h** $(2x - 3y)^9$

4 The coefficient of x^2 in the expansion of $(2 + ax)^6$ is 60.
Find possible values of the constant a.

5 The coefficient of x^3 in the expansion of $(3 + bx)^5$ is -720.
Find the value of the constant b.

6 The coefficient of x^3 in the expansion of $(2 + x)(3 - ax)^4$ is 30.
Find the values of the constant a.

7 Write down the first four terms in the expansion of $\left(1 - \frac{x}{10}\right)^6$.

By substituting an appropriate value for x, find an approximate value to $(0.99)^6$. Use your calculator to find the degree of accuracy of your approximation.

8 Write down the first four terms in the expansion of $\left(2 + \frac{x}{5}\right)^{10}$.

By substituting an appropriate value for x, find an approximate value to $(2.1)^{10}$. Use your calculator to find the degree of accuracy of your approximation.

5.4 You need to be able to expand $(1 + x)^n$ and $(a + bx)^n$ using the binomial expansion.

■ $$(1 + x)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}x^1 + \binom{n}{2}1^{n-2}x^2 + \binom{n}{3}1^{n-3}x^3 + \binom{n}{4}1^{n-4}x^4 + \ldots + \binom{n}{r}1^{n-r}x^r$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots$$

Example 5

Find the first four terms in the binomial expansion of **a** $(1 + 2x)^5$ and **b** $(2 - x)^6$:

a $(1 + 2x)^5 = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$

Compare $(1 + x)^n$ with $(1 + 2x)^n$.

$= 1 + 5(2x) + \frac{5(4)}{2!}(2x)^2 + \frac{5(4)(3)}{3!}(2x)^3 + \dots$

Replace n by 5 and 'x' by $2x$.

$= 1 + 10x + 40x^2 + 80x^3 + \dots$

b $(2 - x)^6 = \left[2\left(1 - \frac{x}{2}\right)\right]^6$

The expansion only works for $(1 + x)^n$, so take out a common factor of 2.

$= 2^6\left(1 - \frac{x}{2}\right)^6$

$= 2^6\left(1 + 6\left(-\frac{x}{2}\right) + \frac{6 \times 5}{2!}\left(-\frac{x}{2}\right)^2 + \frac{6 \times 5 \times 4}{3!}\left(-\frac{x}{2}\right)^3 + \dots\right)$

Replace n by 6 and 'x' by '$-\frac{x}{2}$' in the expansion of $(1 + x)^n$.

$= 2^6\left(1 - 3x + \frac{15}{4}x^2 - \frac{5}{2}x^3 + \dots\right)$

Multiply terms in bracket by 2^6.

$= 64 - 192x + 240x^2 - 160x^3 + \dots$

Exercise 5D

1 Use the binomial expansion to find the first four terms of

a $(1 + x)^8$ **b** $(1 - 2x)^6$ **c** $\left(1 + \frac{x}{2}\right)^{10}$

d $(1 - 3x)^5$ **e** $(2 + x)^7$ **f** $(3 - 2x)^3$

g $(2 - 3x)^6$ **h** $(4 + x)^4$ **i** $(2 + 5x)^7$

2 If x is so small that terms of x^3 and higher can be ignored, show that:

$(2 + x)(1 - 3x)^5 \approx 2 - 29x + 165x^2$

3 If x is so small that terms of x^3 and higher can be ignored, and

$(2 - x)(3 + x)^4 \approx a + bx + cx^2$

find the values of the constants a, b and c.

4 When $(1 - 2x)^p$ is expanded, the coefficient of x^2 is 40. Given that $p > 0$, use this information to find:

a The value of the constant p.

b The coefficient of x.

c The coefficient of x^3.

5 Write down the first four terms in the expansion of $(1 + 2x)^8$. By substituting an appropriate value of x (which should be stated), find an approximate value of 1.02^8. State the degree of accuracy of your answer.

Mixed exercise 5E

1 When $(1 - \frac{3}{2}x)^p$ is expanded in ascending powers of x, the coefficient of x is -24.

a Find the value of p.

b Find the coefficient of x^2 in the expansion.

c Find the coefficient of x^3 in the expansion. **E**

2 Given that:

$$(2 - x)^{13} \equiv A + Bx + Cx^2 + \ldots$$

find the values of the integers A, B and C. **E**

3 **a** Expand $(1 - 2x)^{10}$ in ascending powers of x up to and including the term in x^3, simplifying each coefficient in the expansion.

b Use your expansion to find an approximation to $(0.98)^{10}$, stating clearly the substitution which you have used for x. **E**

4 **a** Use the binomial series to expand $(2 - 3x)^{10}$ in ascending powers of x up to and including the term in x^3, giving each coefficient as an integer.

b Use your series expansion, with a suitable value for x, to obtain an estimate for 1.97^{10}, giving your answer to 2 decimal places. **E**

5 **a** Expand $(3 + 2x)^4$ in ascending powers of x, giving each coefficient as an integer.

b Hence, or otherwise, write down the expansion of $(3 - 2x)^4$ in ascending powers of x.

c Hence by choosing a suitable value for x show that $(3 + 2\sqrt{2})^4 + (3 - 2\sqrt{2})^4$ is an integer and state its value. **E**

6 The coefficient of x^2 in the binomial expansion of $\left(1 + \dfrac{x}{2}\right)^n$, where n is a positive integer, is 7.

a Find the value of n.

b Using the value of n found in part **a**, find the coefficient of x^4. **E**

7 **a** Use the binomial theorem to expand $(3 + 10x)^4$ giving each coefficient as an integer.

b Use your expansion, with an appropriate value for x, to find the exact value of $(1003)^4$. State the value of x which you have used. **E**

8 **a** Expand $(1 + 2x)^{12}$ in ascending powers of x up to and including the term in x^3, simplifying each coefficient.

b By substituting a suitable value for x, which must be stated, into your answer to part **a**, calculate an approximate value of $(1.02)^{12}$.

c Use your calculator, writing down all the digits in your display, to find a more exact value of $(1.02)^{12}$.

d Calculate, to 3 significant figures, the percentage error of the approximation found in part **b**. *E*

9 Expand $\left(x - \frac{1}{x}\right)^5$, simplifying the coefficients. *E*

10 In the binomial expansion of $(2k + x)^n$, where k is a constant and n is a positive integer, the coefficient of x^2 is equal to the coefficient of x^3.

a Prove that $n = 6k + 2$.

b Given also that $k = \frac{2}{3}$, expand $(2k + x)^n$ in ascending powers of x up to and including the term in x^3, giving each coefficient as an exact fraction in its simplest form. *E*

11 **a** Expand $(2 + x)^6$ as a binomial series in ascending powers of x, giving each coefficient as an integer.

b By making suitable substitutions for x in your answer to part **a**, show that $(2 + \sqrt{3})^6 - (2 - \sqrt{3})^6$ can be simplified to the form $k\sqrt{3}$, stating the value of the integer k. *E*

12 The coefficient of x^2 in the binomial expansion of $(2 + kx)^8$, where k is a positive constant, is 2800.

a Use algebra to calculate the value of k.

b Use your value of k to find the coefficient of x^3 in the expansion. *E*

13 **a** Given that

$$(2 + x)^5 + (2 - x)^5 \equiv A + Bx^2 + Cx^4,$$

find the value of the constants A, B and C.

b Using the substitution $y = x^2$ and your answers to part **a**, solve

$$(2 + x)^5 + (2 - x)^5 = 349.$$

E

14 In the binomial expansion of $(2 + px)^5$, where p is a constant, the coefficient of x^3 is 135. Calculate:

a The value of p,

b The value of the coefficient of x^4 in the expansion. *E*

Summary of key points

1 You can use Pascal's Triangle to multiply out a bracket.

2 You can use combinations and factional notation to help you expand binomial expressions. For larger indices it is quicker than using Pascal's Triangle.

3 $n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$

4 The number of ways of choosing r items from a group of n items is written ${}^nC_r$ or $\binom{n}{r}$.

e.g. $^3C_2 = \dfrac{3!}{(3-2)!2!} = \dfrac{6}{1 \times 2} = 3$

5 The binomial expansion is

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b + {}^nC_3a^{n-3}b^3 + \ldots + {}^nC_nb^n$$

$$\text{or} \quad \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{n}b^n$$

6 Similarly,

$$(a+bx)^n = {}^nC_0a^n + {}^nC_1a^{n-1}bx + {}^nC_2a^{n-2}b^2x^2 + {}^nC_3a^{n-3}b^3x^3 + \ldots {}^nC_nb^nx^n$$

$$\text{or} \quad \binom{n}{0}a^n + \binom{n}{1}a^{n-1}bx + \binom{n}{2}a^{n-2}b^2x^2 + \binom{n}{3}a^{n-3}b^3x^3 + \ldots + \binom{n}{n}b^nx^n$$

7 $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \dfrac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots$