After completing this chapter you should be able to

1 convert between radians and degrees and vice versa

2 know and use the formula in radians for the length of an arc

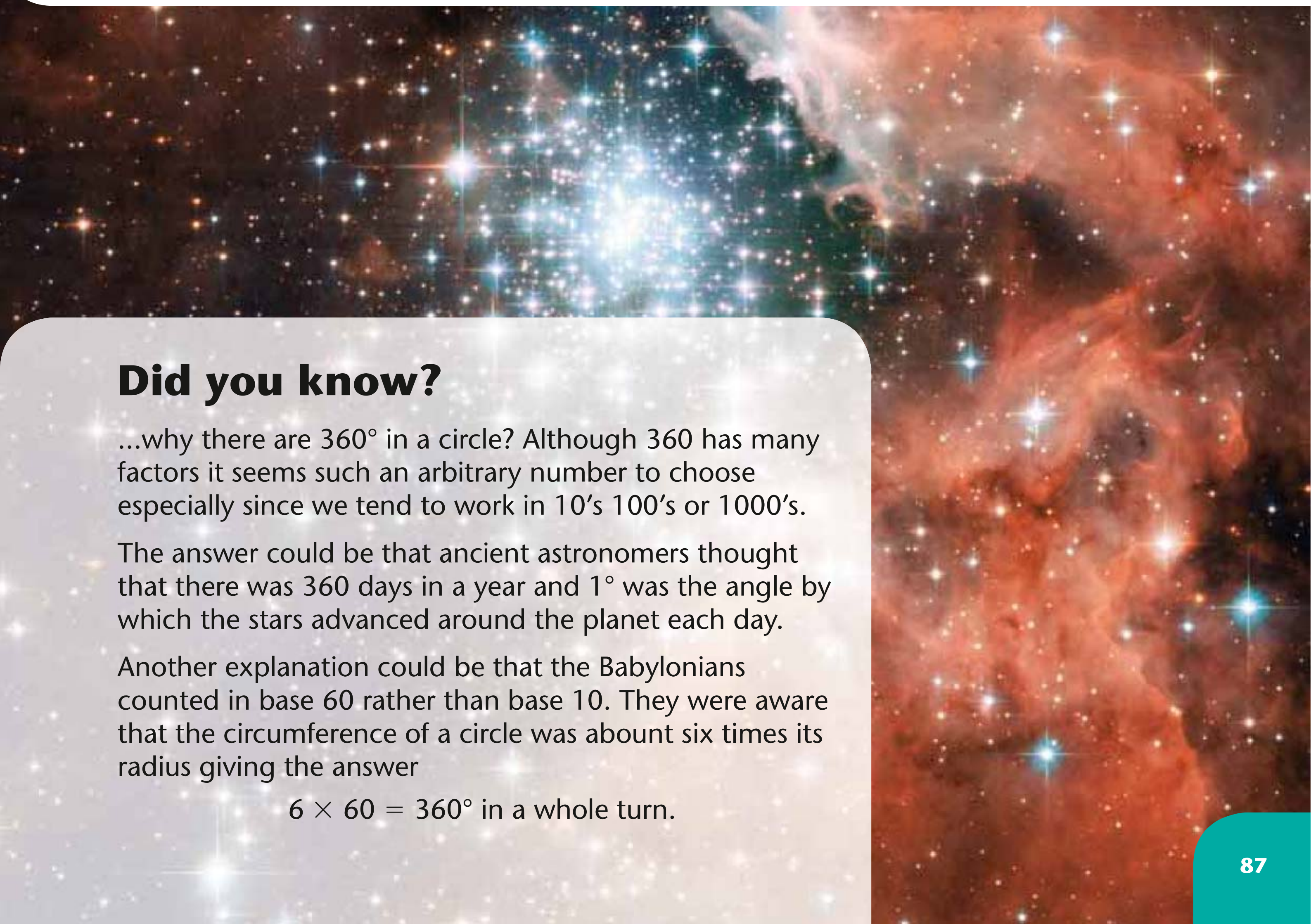
3 know and use the formula in radians for the area of a sector

4 know and use the formula in radians for the segment of a circle.

Radians are a very important way of measuring angles in A level Mathematics. You will meet them every time you encounter trigonometry, including chapter 8 in C2.

6

Radian measure and its applications



Did you know?

...why there are 360° in a circle? Although 360 has many factors it seems such an arbitrary number to choose especially since we tend to work in 10’s 100’s or 1000’s.

The answer could be that ancient astronomers thought that there was 360 days in a year and 1° was the angle by which the stars advanced around the planet each day.

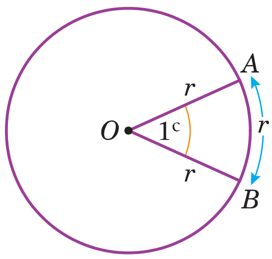
Another explanation could be that the Babylonians counted in base 60 rather than base 10. They were aware that the circumference of a circle was abount six times its radius giving the answer

$$6 \times 60 = 360° \text{ in a whole turn.}$$

6.1 You can measure angles in radians.

In Chapter 2 you worked with angles in degrees, where one degree is $\frac{1}{360}$th of a complete revolution. This convention dates back to the Babylonians. It has the advantage that 360 has a great number of factors making division of the circle that much easier, but it is still only a convention. Another and perhaps initially stranger measure of an angle is the radian.

■ **If the arc *AB* has length *r*, then ∠AOB is 1 radian (1^c or 1 rad).**

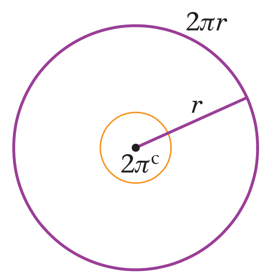


Hint: The symbol for radians is ^{c}, so θ^c means that θ is in radians. If there is no symbol with an angle you should assume that it is in radians, unless the context makes it clear that it is in degrees.

You can put this into words.

■ **A radian is the angle subtended at the centre of a circle by an arc whose length is equal to that of the radius of the circle.**

As an arc of length r subtends 1 radian at the centre of the circle, it follows that the circumference (an arc of length $2\pi r$) subtends 2π radians at the centre.



As the circumference subtends an angle of 360° at the centre,

2π radians = 360°

so π radians = 180°

It follows that 1 rad = 57.295 …°.

■ **1 radian = $\frac{180°}{\pi}$**

Example 1

Convert the following angles into degrees:

a $\frac{7\pi}{8}$ rad **b** $\frac{4\pi}{15}$ rad

a $\frac{7\pi}{8}$ rad

$= \frac{7}{8} \times 180°$

$= 157.5°$

b $\frac{4\pi}{15}$ rad

$= 4 \times \frac{180°}{15}$

$= 48°$

Remember that π rad = 180°. Check using your calculator.

Example 2

Convert the following angles into radians:

a 150° **b** 110°

a $150° = 150 \times \frac{\pi}{180}$ rad

$= \frac{5\pi}{6}$ rad

b $110° = 110 \times \frac{\pi}{180}$ rad

$= \frac{11}{18}\pi$ rad

Since $180° = \pi$ rad, $1° = \frac{\pi}{180}$ rad.

It is worth remembering that $30° = \frac{\pi}{6}$ rad.

Your calculator will give the decimal answer 1.919 86 …

These answers, in terms of π, are exact.

Caution: If you have been working in radian mode on your calculator make sure you return to degree mode when working with questions involving degrees.

Exercise 6A

1 Convert the following angles in radians to degrees:

a $\frac{\pi}{20}$ **b** $\frac{\pi}{15}$ **c** $\frac{5\pi}{12}$

d $\frac{\pi}{2}$ **e** $\frac{7\pi}{9}$ **f** $\frac{7\pi}{6}$

g $\frac{5\pi}{4}$ **h** $\frac{3\pi}{2}$ **i** 3π

2 Use your calculator to convert the following angles to degrees, giving your answer to the nearest 0.1°:

a 0.46^c **b** 1^c **c** 1.135^c **d** $\sqrt{3}^c$

e 2.5^c **f** 3.14^c **g** 3.49^c

3 Use your calculator to write down the value, to 3 significant figures, of the following trigonometric functions.

a $\sin 0.5^c$ **b** $\cos \sqrt{2}^c$ **c** $\tan 1.05^c$ **d** $\sin 2^c$ **e** $\cos 3.6^c$

4 Convert the following angles to radians, giving your answers as multiples of π:

a 8° **b** 10° **c** 22.5° **d** 30°

e 45° **f** 60° **g** 75° **h** 80°

i 112.5° **j** 120° **k** 135° **l** 200°

m 240° **n** 270° **o** 315° **p** 330°

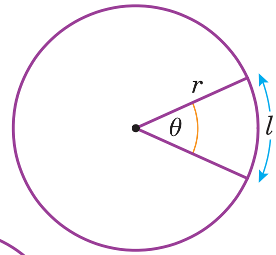
5 Use your calculator to convert the following angles to radians, giving your answers to 3 significant figures:

a 50° **b** 75° **c** 100°

d 160° **e** 230° **f** 320°.

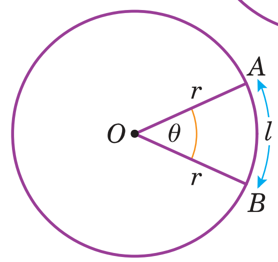
6.2 The formula for the length of an arc of a circle is simpler when you use radians.

■ **To find the arc length l of a circle use the formula $l = r\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.**



Example 3

Show that the length of an arc is $l = r\theta$.



The circle has centre O and radius r.

The arc AB has length l.

So $\dfrac{l}{2\pi r} = \dfrac{\theta}{2\pi}$

$l = r\theta$

$\dfrac{\text{Length of arc}}{\text{Circumference}} = \dfrac{\text{angle } AOB}{\text{total angle around } O}$ (both angles are in radians).

Multiply throughout by $2\pi r$.
If you know two of r, θ and l, the third can be found.

Example 4

Find the length of the arc of a circle of radius 5.2 cm, given that the arc subtends an angle of 0.8 rad at the centre of the circle.

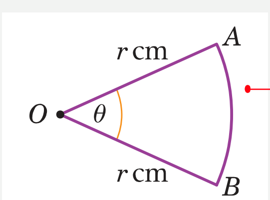
Arc length $= 5.2 \times 0.8$ cm

$= 4.16$ cm

Use $l = r\theta$, with $r = 5.2$ and $\theta = 0.8$.

Example 5

An arc AB of a circle, with centre O and radius r cm, subtends an angle of θ radians at O. The perimeter of the sector AOB is P cm. Express r in terms of θ.



Draw a diagram to display the data.

$P = r\theta + 2r$

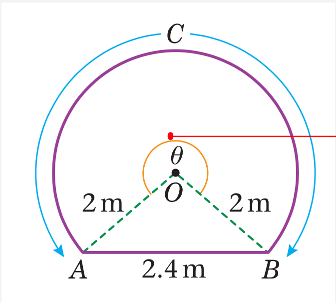
$= r(2 + \theta)$

So $r = \dfrac{P}{(2 + \theta)}$

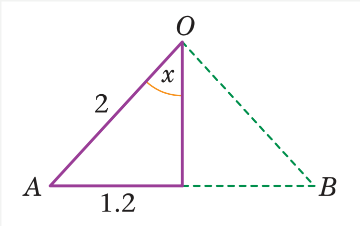
The perimeter = arc AB + OA + OB, where arc $AB = r\theta$.
Factorising.

Example 6

The border of a garden pond consists a straight edge AB of length 2.4 m, and a curved part C, as shown in the diagram below. The curved part is an arc of a circle, centre O and radius 2 m. Find the length of C.



C subtends the reflex angle θ at O, so length of $C = 2\theta$.



You can use the isosceles triangle AOB to find the angle AOB inside the triangle. (*Use your calculator in radian mode.*)

$\sin x = \frac{1.2}{2}$

$x = 0.6435$ rad

Acute $\angle AOB = 2x$ rad

$= 2(0.6435)$

$= 1.287$ rad

So $\theta = (2\pi - 1.287)$ rad

$= 4.996$ rad

So length of $C = 9.99$ m (3 s.f.)

θ + acute $\angle AOB = 2\pi$ rad.

$C = 2\theta$

Exercise 6B

1 An arc AB of a circle, centre O and radius r cm, subtends an angle θ radians at O. The length of AB is l cm.

a Find l when **i** $r = 6, \theta = 0.45$ **ii** $r = 4.5, \theta = 0.45$ **iii** $r = 20, \theta = \frac{3}{8}\pi$

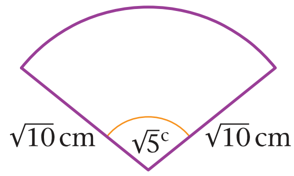
b Find r when **i** $l = 10, \theta = 0.6$ **ii** $l = 1.26, \theta = 0.7$ **iii** $l = 1.5\pi, \theta = \frac{5}{12}\pi$

c Find θ when **i** $l = 10, r = 7.5$ **ii** $l = 4.5, r = 5.625$ **iii** $l = \sqrt{12}, r = \sqrt{3}$

2 A minor arc AB of a circle, centre O and radius 10 cm, subtends an angle x at O. The major arc AB subtends an angle $5x$ at O. Find, in terms of π, the length of the minor arc AB.

3 An arc AB of a circle, centre O and radius 6 cm, has length l cm. Given that the chord AB has length 6 cm, find the value of l, giving your answer in terms of π.

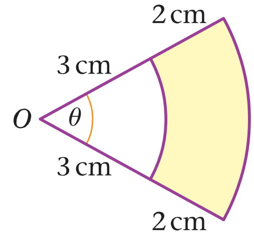
4 The sector of a circle of radius $\sqrt{10}$ cm contains an angle of $\sqrt{5}$ radians, as shown in the diagram. Find the length of the arc, giving your answer in the form $p\sqrt{q}$ cm, where p and q are integers.



5 Referring to the diagram, find:

a The perimeter of the shaded region when $\theta = 0.8$ radians.

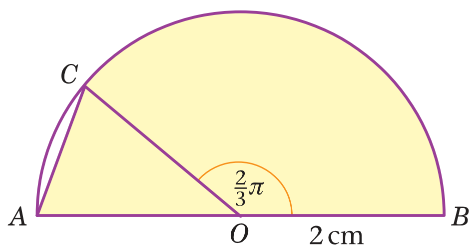
b The value of θ when the perimeter of the shaded region is 14 cm.



6 A sector of a circle of radius r cm contains an angle of 1.2 radians. Given that the sector has the same perimeter as a square of area 36 cm^2, find the value of r.

7 A sector of a circle of radius 15 cm contains an angle of θ radians. Given that the perimeter of the sector is 42 cm, find the value of θ.

8 In the diagram AB is the diameter of a circle, centre O and radius 2 cm. The point C is on the circumference such that $\angle COB = \frac{2}{3}\pi$ radians.



a State the value, in radians, of $\angle COA$.

The shaded region enclosed by the chord AC, arc CB and AB is the template for a brooch.

b Find the exact value of the perimeter of the brooch.

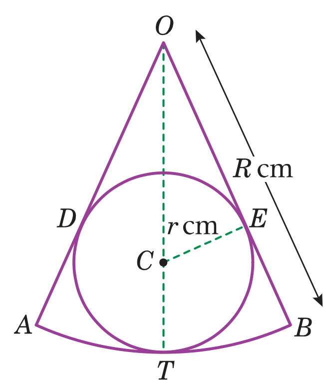
9 The points A and B lie on the circumference of a circle with centre O and radius 8.5 cm. The point C lies on the major arc AB. Given that $\angle ACB = 0.4$ radians, calculate the length of the minor arc AB.

10 In the diagram OAB is a sector of a circle, centre O and radius R cm, and $\angle AOB = 2\theta$ radians. A circle, centre C and radius r cm, touches the arc AB at T, and touches OA and OB at D and E respectively, as shown.

a Write down, in terms of R and r, the length of OC.

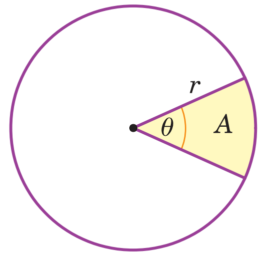
b Using $\triangle OCE$, show that $R\sin\theta = r(1 + \sin\theta)$.

c Given that $\sin\theta = \frac{3}{4}$ and that the perimeter of the sector OAB is 21 cm, find r, giving your answer to 3 significant figures.



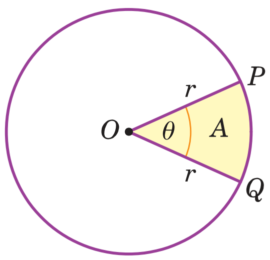
6.3 The formula for the area of a sector of a circle is simpler when you use radians.

- **To find the area A of a sector of a circle use the formula $A = \frac{1}{2}r^2\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.**



Example 7

Show that the area of the sector of a circle with radius r is $A = \frac{1}{2}r^2\theta$.



The circle has centre O and radius r.

The sector POQ has area A.

So $\dfrac{A}{\pi r^2} = \dfrac{\theta}{2\pi}$

$A = \dfrac{1}{2}r^2\theta$

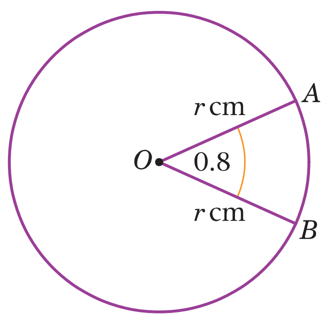
$\dfrac{\text{area of sector}}{\text{area of circle}} = \dfrac{\text{angle } POQ}{\text{total angle around } O}$

Multiply throughout by πr^2.
If you know two of r, θ and A, the third can be found.

Example 8

In the diagram, the area of the minor sector AOB is 28.9 cm^2.

Given that $\angle AOB = 0.8$ radians, calculate the value of r.



$28.9 = \dfrac{1}{2}r^2(0.8) = 0.4r^2$

So $r^2 = \dfrac{28.9}{0.4} = 72.25$

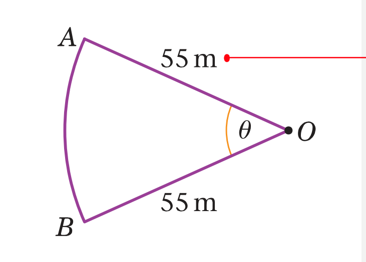
$r = 8.5$

Let area of sector be A cm^2, and use $A = \frac{1}{2}r^2\theta$.

Find r^2 and then take the square root.

Example 9

A plot of land is in the shape of a sector of a circle of radius 55 m. The length of fencing that is erected along the edge of the plot to enclose the land is 176 m. Calculate the area of the plot of land.



Draw a diagram to include all the data and let the angle of the sector be θ.

Arc $AB = 176 - (55 + 55)$

$= 66$ m

As the perimeter is given, first find length of arc AB.

$66 = 55\theta$

Use the formula for arc length, $l = r\theta$.

So $\theta = 1.2$ radians

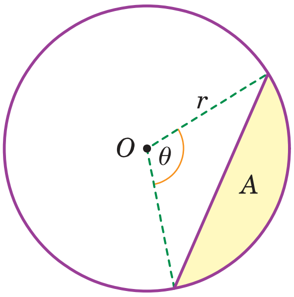
Area of plot $= \frac{1}{2}(55)^2(1.2)$

$= 1815$ m^2

Use the formula for area of a sector, $A = \frac{1}{2}r^2\theta$.

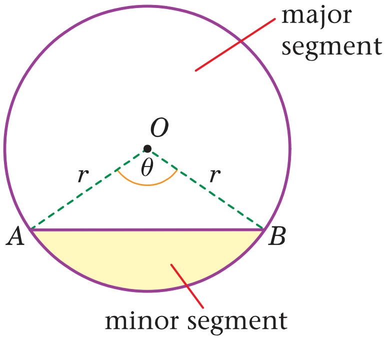
6.4 You can work out the area of a segment using radians.

- **The area of a segment in a circle of radius r is**
  $A = \frac{1}{2}r^2(\theta - \sin\theta)$



Example 10

Show that the area of the shaded segment in the circle shown is $\frac{1}{2}r^2(\theta - \sin\theta)$



Area of shaded minor segment

= area of sector AOB − area of triangle AOB.

Area of sector $AOB = \frac{1}{2}r^2\theta$

Use $A = \frac{1}{2}r^2\theta$.

Area of triangle $AOB = \frac{1}{2}r^2\sin\theta$

Use $\frac{1}{2}ab\sin C$ from Section 2.7.

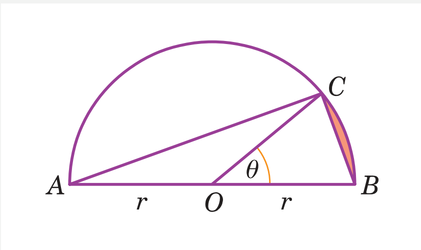
So area of shaded minor segment

$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$= \frac{1}{2}r^2(\theta - \sin\theta)$

Example 11

In the diagram AB is the diameter of a circle of radius r cm, and $\angle BOC = \theta$ radians. Given that the area of $\triangle AOC$ is three times that of the shaded segment, show that $3\theta - 4\sin\theta = 0$.



Area of segment $= \frac{1}{2}r^2(\theta - \sin\theta)$

Area of $\triangle AOC = \frac{1}{2}r^2\sin(\pi - \theta)$

$= \frac{1}{2}r^2\sin\theta$

So $\quad \frac{1}{2}r^2\sin\theta = 3 \times \frac{1}{2}r^2(\theta - \sin\theta)$

$\sin\theta = 3(\theta - \sin\theta)$

So $\quad 3\theta - 4\sin\theta = 0$

Area of segment area of sector − area of triangle.

$\angle AOB = \pi$ radians.
Remember from Section 2.3 that $\sin(180° - \theta°) = \sin\theta°$ so $\sin(\pi - \theta) = \sin\theta$.

Area of $\triangle AOC = 3 \times$ area of shaded segment.

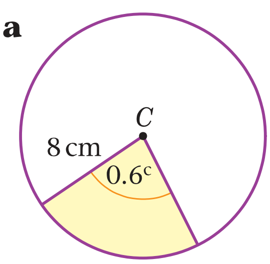
(In Chapter 4 of Book C3 you will be able to find an approximation for θ.)

Exercise 6C

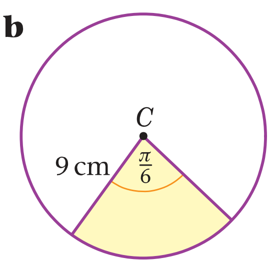
(*Note:* give non-exact answers to 3 significant figures.)

1 Find the area of the shaded sector in each of the following circles with centre C. Leave your answer in terms of π, where appropriate.

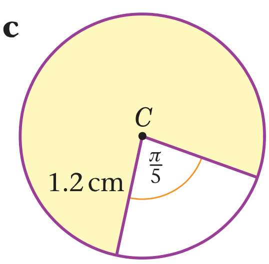
a



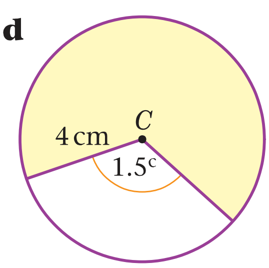
b



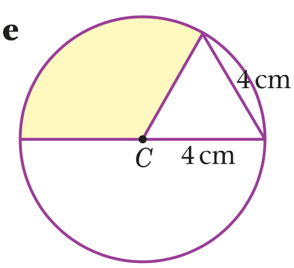
c



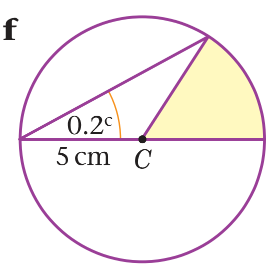
d



e

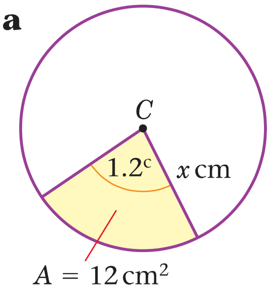


f

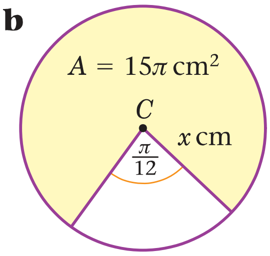


2 For the following circles with centre C, the area A of the shaded sector is given. Find the value of x in each case.

a



b



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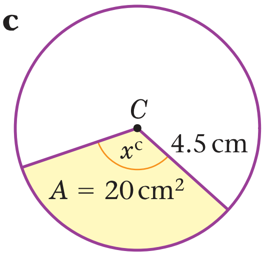


3 The arc AB of a circle, centre O and radius 6 cm, has length 4 cm. Find the area of the minor sector AOB.

4 The chord AB of a circle, centre O and radius 10 cm, has length 18.65 cm and subtends an angle of θ radians at O.

a Show that $\theta = 2.40$ (to 3 significant figures).

b Find the area of the minor sector AOB.

5 The area of a sector of a circle of radius 12 cm is 100 cm^2. Find the perimeter of the sector.

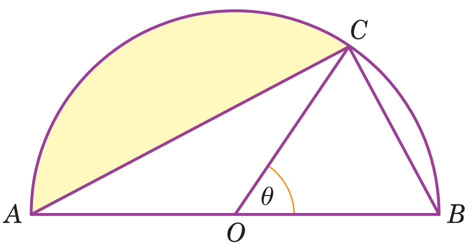
6 The arc AB of a circle, centre O and radius r cm, is such that $\angle AOB = 0.5$ radians. Given that the perimeter of the minor sector AOB is 30 cm:

a Calculate the value of r.

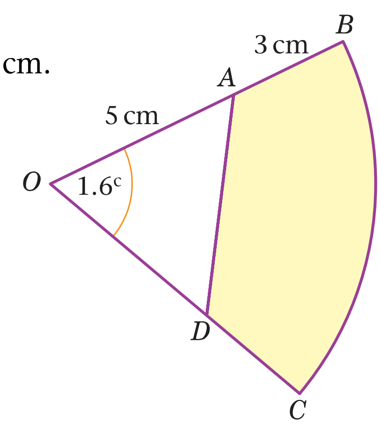
b Show that the area of the minor sector AOB is 36 cm^2.

c Calculate the area of the segment enclosed by the chord AB and the minor arc AB.

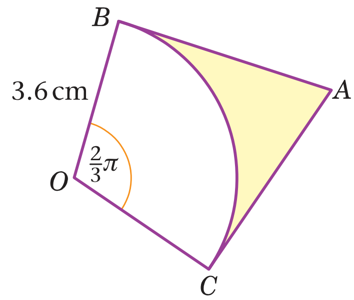
7 In the diagram, AB is the diameter of a circle of radius r cm and $\angle BOC = \theta$ radians. Given that the area of $\triangle COB$ is equal to that of the shaded segment, show that $\theta + 2\sin\theta = \pi$.



8 In the diagram, BC is the arc of a circle, centre O and radius 8 cm. The points A and D are such that $OA = OD = 5$ cm. Given that $\angle BOC = 1.6$ radians, calculate the area of the shaded region.



9 In the diagram, AB and AC are tangents to a circle, centre O and radius 3.6 cm. Calculate the area of the shaded region, given that $\angle BOC = \frac{2}{3}\pi$ radians.

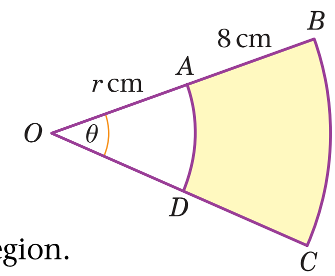


10 A chord AB subtends an angle of θ radians at the centre O of a circle of radius 6.5 cm. Find the area of the segment enclosed by the chord AB and the minor arc AB, when:

a $\theta = 0.8$ **b** $\theta = \frac{2}{3}\pi$ **c** $\theta = \frac{4}{3}\pi$

11 An arc AB subtends an angle of 0.25 radians at the *circumference* of a circle, centre O and radius 6 cm. Calculate the area of the minor sector OAB.

12 In the diagram, AD and BC are arcs of circles with centre O, such that $OA = OD = r$ cm, $AB = DC = 8$ cm and $\angle BOC = \theta$ radians.

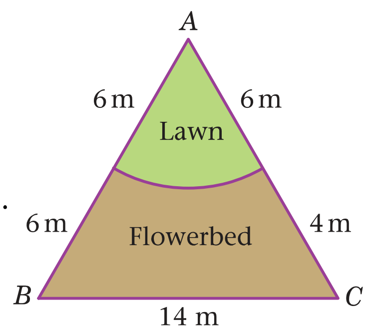


a Given that the area of the shaded region is 48 cm^2, show that $r = \dfrac{6}{\theta} - 4$.

b Given also that $r = 10\theta$, calculate the perimeter of the shaded region.

13 A sector of a circle of radius 28 cm has perimeter P cm and area A cm^2. Given that $A = 4P$, find the value of P.

14 The diagram shows a triangular plot of land. The sides AB, BC and CA have lengths 12 m, 14 m and 10 m respectively. The lawn is a sector of a circle, centre A and radius 6 m.



a Show that $\angle BAC = 1.37$ radians, correct to 3 significant figures.

b Calculate the area of the flowerbed.

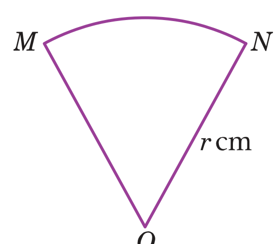
Mixed exercise 6D

1 Triangle ABC is such that $AB = 5$ cm, $AC = 10$ cm and $\angle ABC = 90°$. An arc of a circle, centre A and radius 5 cm, cuts AC at D.

a State, in radians, the value of $\angle BAC$.

b Calculate the area of the region enclosed by BC, DC and the arc BD.

2 The diagram shows a minor sector OMN of a circle centre O and radius r cm. The perimeter of the sector is 100 cm and the area of the sector is A cm^2.



a Show that $A = 50r - r^2$.

b Given that r varies, find:

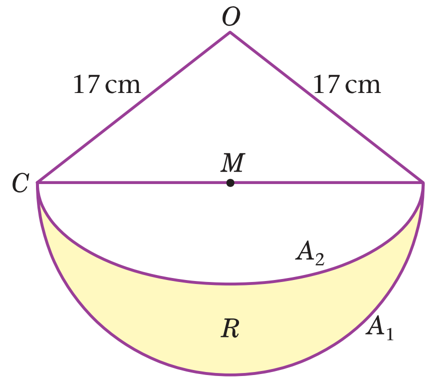
i The value of r for which A is a maximum and show that A is a maximum.

ii The value of $\angle MON$ for this maximum area.

iii The maximum area of the sector OMN.

E

3 The diagram shows the triangle OCD with $OC = OD = 17$ cm and $CD = 30$ cm. The mid-point of CD is M. With centre M, a semicircular arc A_1 is drawn on CD as diameter. With centre O and radius 17 cm, a circular arc A_2 is drawn from C to D. The shaded region R is bounded by the arcs A_1 and A_2. Calculate, giving answers to 2 decimal places:



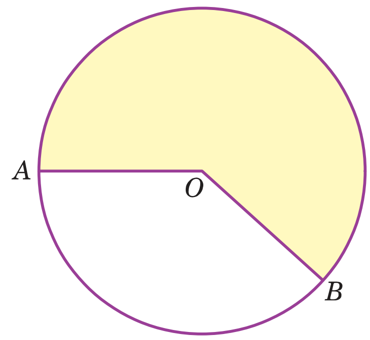
a The area of the triangle OCD.

b The angle COD in radians.

c The area of the shaded region R.

E

4 The diagram shows a circle, centre O, of radius 6 cm. The points A and B are on the circumference of the circle. The area of the shaded major sector is 80 cm^2. Given that $\angle AOB = \theta$ radians, where $0 < \theta < \pi$, calculate:

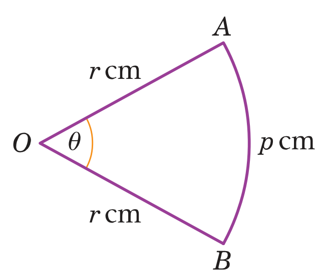


a The value, to 3 decimal places, of θ.

b The length in cm, to 2 decimal places, of the minor arc AB.

E

5 The diagram shows a sector OAB of a circle, centre O and radius r cm. The length of the arc AB is p cm and $\angle AOB$ is θ radians.



a Find θ in terms of p and r.

b Deduce that the area of the sector is $\frac{1}{2}pr$ cm^2.

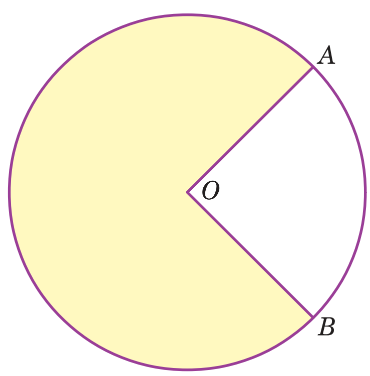
Given that $r = 4.7$ and $p = 5.3$, where each has been measured to 1 decimal place, find, giving your answer to 3 decimal places:

c The least possible value of the area of the sector.

d The range of possible values of θ.

E

6 The diagram shows a circle centre O and radius 5 cm. The length of the minor arc AB is 6.4 cm.



a Calculate, in radians, the size of the acute angle AOB.

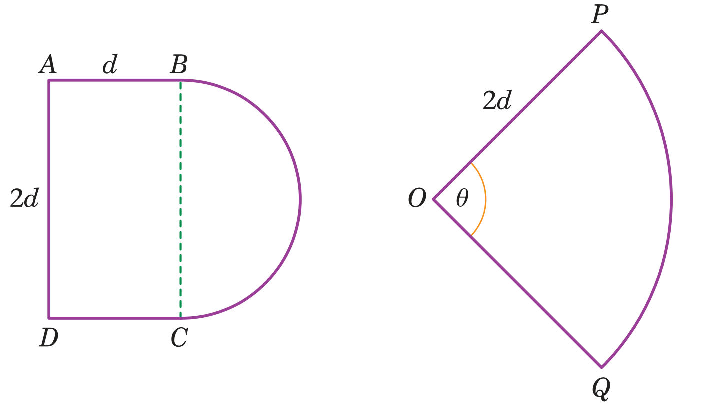
The area of the minor sector AOB is R_1 cm^2 and the area of the shaded major sector AOB is R_2 cm^2.

b Calculate the value of R_1.

c Calculate $R_1 : R_2$ in the form $1 : p$, giving the value of p to 3 significant figures.

E

7



Shape X Shape Y

The diagrams show the cross-sections of two drawer handles.
Shape X is a rectangle $ABCD$ joined to a semicircle with BC as diameter. The length $AB = d$ cm and $BC = 2d$ cm. Shape Y is a sector OPQ of a circle with centre O and radius $2d$ cm. Angle POQ is θ radians.
Given that the areas of shapes X and Y are equal:

a prove that $\theta = 1 + \frac{1}{4}\pi$.

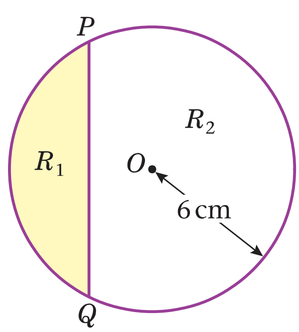
Using this value of θ, and given that $d = 3$, find in terms of π:

b the perimeter of shape X,

c the perimeter of shape Y.

d Hence find the difference, in mm, between the perimeters of shapes X and Y. *E*

8 The diagram shows a circle with centre O and radius 6 cm. The chord PQ divides the circle into a minor segment R_1 of area A_1 cm^2 and a major segment R_2 of area A_2 cm^2. The chord PQ subtends an angle θ radians at O.



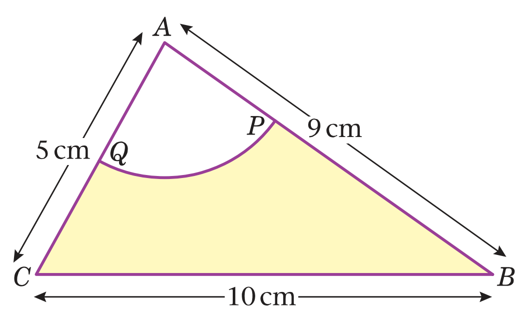
a Show that $A_1 = 18(\theta - \sin\theta)$.

Given that $A_2 = 3A_1$ and $f(\theta) = 2\theta - 2\sin\theta - \pi$:

b prove that $f(\theta) = 0$.

c Evaluate f(2.3) and f(2.32) and deduce that $2.3 < \theta < 2.32$. *E*

9 Triangle ABC has $AB = 9$ cm, $BC = 10$ cm and $CA = 5$ cm. A circle, centre A and radius 3 cm, intersects AB and AC at P and Q respectively, as shown in the diagram.



a Show that, to 3 decimal places, $\angle BAC = 1.504$ radians.

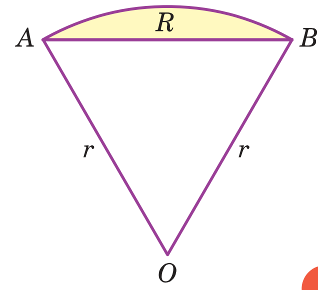
b Calculate:

i the area, in cm^2, of the sector APQ,

ii the area, in cm^2, of the shaded region $BPQC$,

iii the perimeter, in cm, of the shaded region $BPQC$. *E*

10 The diagram shows the sector OAB of a circle of radius r cm. The area of the sector is 15 cm^2 and $\angle AOB = 1.5$ radians.



a Prove that $r = 2\sqrt{5}$.

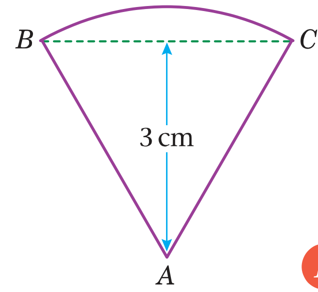
b Find, in cm, the perimeter of the sector OAB.

The segment R, shaded in the diagram, is enclosed by the arc AB and the straight line AB.

c Calculate, to 3 decimal places, the area of R.

E

11 The shape of a badge is a sector ABC of a circle with centre A and radius AB, as shown in the diagram. The triangle ABC is equilateral and has perpendicular height 3 cm.



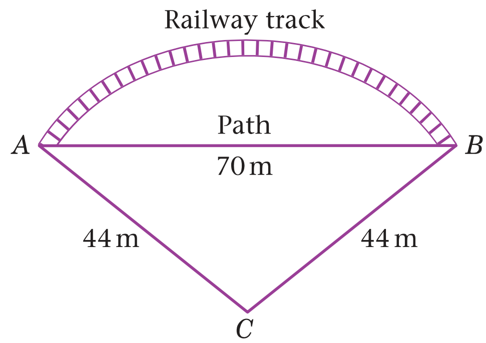
a Find, in surd form, the length of AB.

b Find, in terms of π, the area of the badge.

c Prove that the perimeter of the badge is $\frac{2\sqrt{3}}{3}(\pi + 6)$ cm.

E

12 There is a straight path of length 70 m from the point A to the point B. The points are joined also by a railway track in the form of an arc of the circle whose centre is C and whose radius is 44 m, as shown in the diagram.



a Show that the size, to 2 decimal places, of $\angle ACB$ is 1.84 radians.

b Calculate:

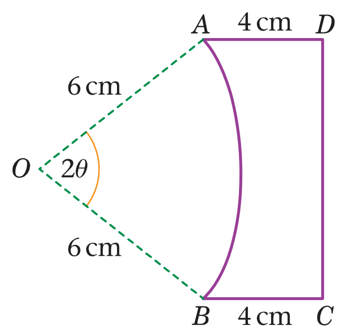
i the length of the railway track,

ii the shortest distance from C to the path,

iii the area of the region bounded by the railway track and the path.

E

13 The diagram shows the cross-section $ABCD$ of a glass prism. $AD = BC = 4$ cm and both are at right angles to DC. AB is the arc of a circle, centre O and radius 6 cm. Given that $\angle AOB = 2\theta$ radians, and that the perimeter of the cross-section is $2(7 + \pi)$ cm:



a show that $(2\theta + 2\sin\theta - 1) = \frac{\pi}{3}$,

b verify that $\theta = \frac{\pi}{6}$,

c find the area of the cross-section.

14 Two circles C_1 and C_2, both of radius 12 cm, have centres O_1 and O_2 respectively. O_1 lies on the circumference of C_2; O_2 lies on the circumference of C_1. The circles intersect at A and B, and enclose the region R.

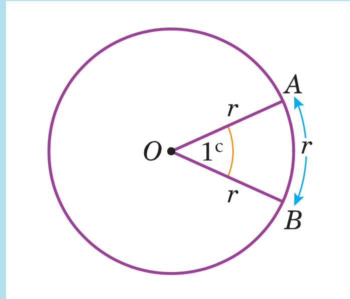
a Show that $\angle AO_1B = \frac{2}{3}\pi$ radians.

b Hence write down, in terms of π, the perimeter of R.

c Find the area of R, giving your answer to 3 significant figures.

Summary of key points

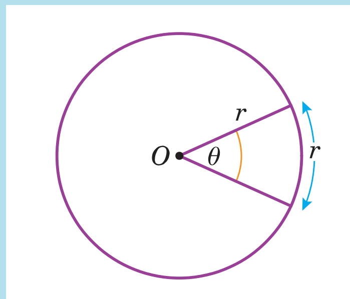
1 If the arc AB has length r, then $\angle AOB$ is 1 radian (1^c or 1 rad).



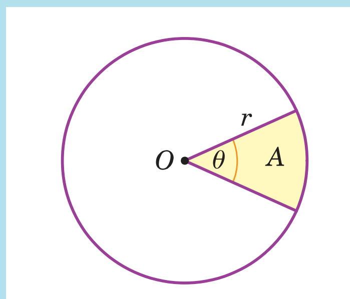
2 A radian is the angle subtended at the centre of a circle by an arc whose length is equal to that of the radius of the circle.

3 1 radian $= \dfrac{180°}{\pi}$.

4 The length of an arc of a circle is $l = r\theta$.



5 The area of a sector is $A = \frac{1}{2}r^2\theta$.



6 The area of a segment in a circle is $A = \frac{1}{2}r^2(\theta - \sin\theta)$.