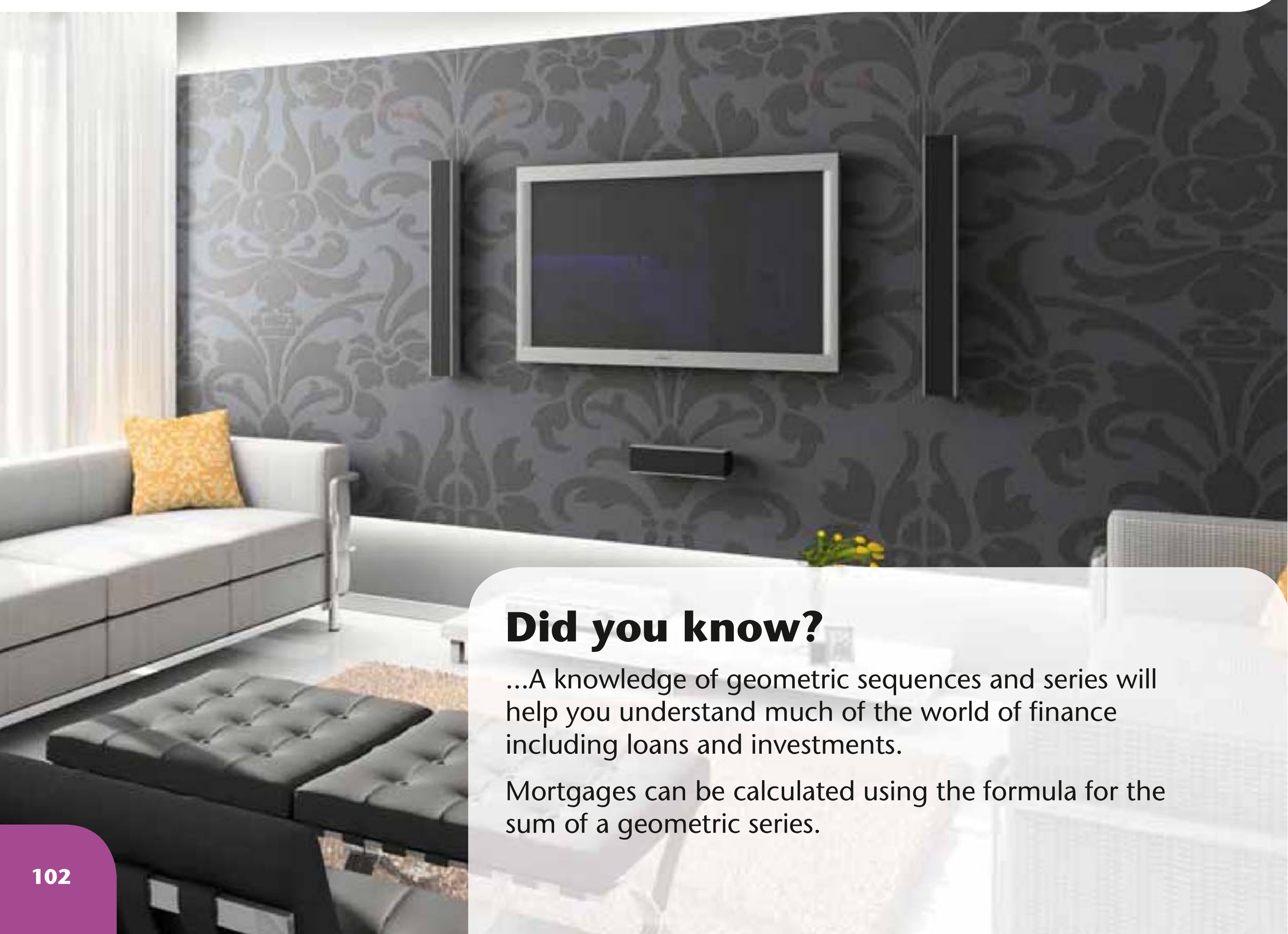
7

After completing this chapter you should be able to

1. recognise a geometric sequence and state its common ratio
2. calculate the *n*th term of a geometric sequence
3. find the sum of a geometric series
4. solve problems involving growth and decay
5. find the sum to infinity of a convergent geometric series.

Geometric sequences and series



Did you know?

...A knowledge of geometric sequences and series will help you understand much of the world of finance including loans and investments.

Mortgages can be calculated using the formula for the sum of a geometric series.

7.1 **The following sequences are called geometric sequences. To get from one term to the next we multiply by the same number each time. This number is called the common ratio, *r*.**

1, 2, 4, 8, 16, …
100, 25, 6.25, 1.5625, …
2, −6, 18, −54, 162, …

Example 1

Find the common ratios in the following geometric sequences:

a 2, 10, 50, 250, …

b 90, −30, 10, $-3\frac{1}{3}$

a 2, 10, 50, 250, …

Use u_1, u_2 etc. to refer to the individual terms in a sequence. Here $u_1 = 2$, $u_2 = 10$, $u_3 = 50$.

Common ratio $= \frac{10}{2} = 5$

To find the common ratio calculate $\frac{u_2}{u_1}$ or $\frac{u_3}{u_2}$.

a 90, −30, 10, $-3\frac{1}{3}$

Common ratio $= \frac{u_2}{u_1}$.

Common ratio $= \frac{-30}{90} = -\frac{1}{3}$

A common ratio can be negative or a fraction (or both).

Exercise 7A

1 Which of the following are geometric sequences? For the ones that are, give the value of '*r*' in the sequence:

a 1, 2, 4, 8, 16, 32, …
b 2, 5, 8, 11, 14, …
c 40, 36, 32, 28, …
d 2, 6, 18, 54, 162, …
e 10, 5, 2.5, 1.25, …
f 5, −5, 5, −5, 5, …
g 3, 3, 3, 3, 3, 3, 3, …
h 4, −1, 0.25, −0.0625, …

2 Continue the following geometric sequences for three more terms:

a 5, 15, 45, …
b 4, −8, 16, …
c 60, 30, 15, …
d 1, $\frac{1}{4}$, $\frac{1}{16}$, …
e 1, p, p^2, …
f x, $-2x^2$, $4x^3$, …

3 If 3, x and 9 are the first three terms of a geometric sequence. Find:

a the exact value of x,

b the exact value of the 4th term.

Hint for question 3:
In a geometric sequence the common ratio can be calculated by $\frac{u_2}{u_1}$ or $\frac{u_3}{u_2}$.

7.2 You can define a geometric sequence using the first term a and the common ratio r:

a,	ar,	ar^2,	ar^3, …	ar^{n-1}
↑	↑	↑	↑	↑
1st term	2nd term	3rd term	4th term	nth term

Sometimes a geometric sequence is called a geometric progression.

Example 2

Find the **i** 10th and **ii** nth terms in the following geometric sequences:

a 3, 6, 12, 24, …

b 40, −20, 10, −5, …

Hint: Look at the relationship between the position of the term in the sequence and the index of the term. You should be able to see that the index of r is one less than its position in the sequence. So the nth term of a geometric sequence is ar^{n-1}.

a 3, 6, 12, 24, …

For this sequence $a = 3$ and $r = \frac{6}{3} = 2$.

i 10th term $= 3 \times (2)^9$
$= 3 \times 512$
$= 1536$

For the 10th term use ar^{n-1} with $a = 3$, $r = 2$ and $n = 10$.

ii nth term $= 3 \times 2^{n-1}$

For the nth term use ar^{n-1} with $a = 3$ and $r = 2$.

b 40, −20, 10, −5, …

For this sequence $a = 40$ and $r = -\frac{20}{40} = -\frac{1}{2}$.

i 10th term $= 40 \times (-\frac{1}{2})^9$
$= 40 \times -\frac{1}{512}$
$= -\frac{5}{64}$

Use ar^{n-1} with $a = 40$, $r = -\frac{1}{2}$ and $n = 10$.

ii nth term $= 40 \times (-\frac{1}{2})^{n-1}$

Use ar^{n-1} with $a = 40$, $r = -\frac{1}{2}$ and $n = n$.

$= 5 \times 8 \times (-\frac{1}{2})^{n-1}$
$= 5 \times 2^3 \times (-\frac{1}{2})^{n-1}$
$= (-1)^{n-1} \times \frac{5}{2^{n-4}}$

Use laws of indices $\frac{x^m}{x^n} = \frac{1}{x^{n-m}}$.
So $2^3 \times \frac{1}{2^{n-1}} = \frac{1}{2^{n-1-3}}$.

Example 3

The second term of a geometric sequence is 4 and the 4th term is 8. Find the exact values of **a** the common ratio, **b** first term and **c** the 10th term:

a 2nd term = 4, $ar = 4$ ①

4th term = 6, $ar^3 = 8$ ②

② ÷ ① $\quad r^2 = 2$

$r = \sqrt{2}$

So common ratio $= \sqrt{2}$

Using nth term $= ar^{n-1}$ with $n = 2$ and $n = 4$.

b Substitute back in ① $a\sqrt{2} = 4$

$$a = \frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2}$$

$a = 2\sqrt{2}$

So first term $= 2\sqrt{2}$

Divide equation by $\sqrt{2}$.

To rationalise $\frac{4}{\sqrt{2}}$, multiply top and bottom by $\sqrt{2}$.

c 10th term $= ar^9$

$= 2\sqrt{2}(\sqrt{2})^9$

$= 2(\sqrt{2})^{10}$

$= 2 \times 2^5$

$= 2^6$

$= 64$

So 10th term = 64

Substitute the values of $a(= 2\sqrt{2})$ and $r(= \sqrt{2})$ back into ar^{n-1} with $n = 10$.

$(\sqrt{2})^{10} = (2^{\frac{1}{2}})^{10} = 2^{\frac{1}{2}\times 10} = 2^5$

Example 4

The numbers 3, x and $(x + 6)$ form the first three terms of a positive geometric sequence. Find:

a the possible values of x,

b the 10th term of the sequence.

a

$$\frac{u_2}{u_1} = \frac{u_3}{u_2}$$

The sequence is geometric so $\frac{u_2}{u_1} = \frac{u_3}{u_2}$.

$$\frac{x}{3} = \frac{x+6}{x}$$

Cross multiply.

$$x^2 = 3(x + 6)$$
$$x^2 = 3x + 18$$
$$x^2 - 3x - 18 = 0$$
$$(x - 6)(x + 3) = 0$$

Factorise.

$$x = 6 \text{ or } -3$$

So x is either 6 or -3, but there are no negative terms so $x = 6$.

Accept $x = 6$, as terms are positive.

If there are no negative terms then -3 cannot be an answer.

b 10th term $= ar^9$

$$= 3 \times 2^9$$

Use the formula nth term $= ar^{n-1}$ with $n = 9$, $a = 3$ and $r = \frac{x}{3} = \frac{6}{3} = 2$.

$$= 3 \times 512$$
$$= 1536$$

The 10th term is 1536.

Exercise 7B

1 Find the sixth, tenth and nth terms of the following geometric sequences:

a 2, 6, 18, 54, …

b 100, 50, 25, 12.5, …

c 1, −2, 4, −8, …

d 1, 1.1, 1.21, 1.331, …

2 The nth term of a geometric sequence is $2 \times (5)^n$. Find the first and 5th terms.

3 The sixth term of a geometric sequence is 32 and the 3rd term is 4. Find the first term and the common ratio.

4 Given that the first term of a geometric sequence is 4, and the third is 1, find possible values for the 6th term.

5 The expressions $x - 6$, $2x$ and x^2 form the first three terms of a geometric progression. By calculating two different expressions for the common ratio, form and solve an equation in x to find possible values of the first term.

7.3 **You can use geometric sequences to solve problems involving growth and decay, e.g. interest rates, population growth and decline.**

Example 5

Andy invests £A at a rate of interest 4% per annum.
After 5 years it will be worth £10 000.
How much (to the nearest penny) will it be worth after 10 years?

After 1 year it will be worth £$A \times 1.04$

After 2 years it will be worth

£$A \times 1.04 \times 1.04 =$ £$A \times 1.04^2$

If property values are increasing at 4% per annum, the multiplication factor is 1.04 (100% + 4%). So you multiply by 1.04 for each year you have this rise. However, if unemployment is coming down by 4% per annum, then the factor is 0.96 (100% − 4%).

So after 5 years investment is worth

£$A \times 1.04^5$

This is a geometric sequence where $a =$ £A and $r = 1.04$.

$A \times 1.04^5 = £10\,000$

After 5 years the investment is worth £10 000.

$A = \frac{£10\,000}{1.04^5}$

Divide by 1.04^5.

$= £8219.27$

The initial investment $A = £8219.27$

After 10 years the investment is worth

$A \times 1.04^{10}$

Use the exact value of A.

$A \times r^{10} = \frac{10\,000}{1.04^5} \times 1.04^{10}$

$= 10\,000 \times 1.04^5$

Use laws of indices $\frac{x^m}{x^n} = x^{m-n}$.

$= 12\,166.529\,02$

$= £12\,166.53$

Put to the nearest penny.

Example 6

What is the first term in the geometric progression 3, 6, 12, 24, … to exceed 1 million?

nth term $= ar^{n-1}$

$= 3 \times (2)^{n-1}$

Sequence has $a = 3$ and $r = 2$.

We want nth term $> 1\,000\,000$

So $3 \times (2)^{n-1} > 1\,000\,000$

$$(2)^{n-1} > \frac{1\,000\,000}{3}$$

Divide by 3.

$$\log(2)^{n-1} > \log\left(\frac{1\,000\,000}{3}\right)$$

To solve this equation take logs of both sides.

$$(n-1)\log(2) > \log\left(\frac{1\,000\,000}{3}\right)$$

Log $a^n = n \log a$.

$$(n-1) > \frac{\log\left(\frac{1\,000\,000}{3}\right)}{\log(2)}$$

Divide by log 2.

$n - 1 > 18.35$ (2 d.p.)

$n > 19.35$

$n = 20$

n has to be an integer.

The 20th term is the first to exceed 1 000 000.

Exercise 7C

1 A population of ants is growing at a rate of 10% a year. If there were 200 ants in the initial population, write down the number after

a 1 year, **b** 2 years, **c** 3 years and **d** 10 years.

2 A motorcycle has four gears. The maximum speed in bottom gear is 40 km h^{-1} and the maximum speed in top gear is 120 km h^{-1}. Given that the maximum speeds in each successive gear form a geometric progression, calculate, in km h^{-1} to one decimal place, the maximum speeds in the two intermediate gears. *E*

3 A car depreciates in value by 15% a year. If it is worth £11 054.25 after 3 years, what was its new price and when will it first be worth less than £5000?

4 The population decline in a school of whales can be modelled by a geometric progression. Initially there were 80 whales in the school. Four years later there were 40. Find out how many there will be at the end of the fifth year. (Round to the nearest whole number.)

5 Find which term in the progression 3, 12, 48, … is the first to exceed 1 000 000.

6 A virus is spreading such that the number of people infected increases by 4% a day. Initially 100 people were diagnosed with the virus. How many days will it be before 1000 are infected?

7 I invest £*A* in the bank at a rate of interest of 3.5% per annum. How long will it be before I double my money?

8 The fish in a particular area of the North Sea are being reduced by 6% each year due to overfishing. How long would it be before the fish stocks are halved?

7.4 You need to be able to find the sum of a geometric series.

Example 7

Find the general term for the sum of the first n terms of a geometric series $a, ar, ar^2, \ldots, ar^n$.

Let $S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$ ①

$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$ ②

① − ② gives $S_n - rS_n = a - ar^n$

$S_n(1 - r) = a(1 - r^n)$

$S_n = \dfrac{a(1 - r^n)}{1 - r}$

Multiply by r.

Subtract rS_n from S_n.

Take out the common factor.

Divide by $(1 - r)$.

■ **The general rule for the sum of a geometric series is** $S_n = \dfrac{a(r^n - 1)}{r - 1}$ **or** $\dfrac{a(1 - r^n)}{1 - r}$

Example 8

Find the sum of the following series:

a 2 + 6 + 18 + 54 + … (for 10 terms)

b 1024 − 512 + 256 − 128 + … + 1

a Series is

$2 + 6 + 18 + 54 + \ldots$ (for 10 terms)

So $a = 2$, $r = \frac{6}{2} = 3$ and $n = 10$

So $S_{10} = \dfrac{2(3^{10} - 1)}{3 - 1} = 59\,048$

As in all questions, write down what is given.

As $r = 3$ (>1), use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$.

b Series is

$1024 - 512 + 256 - 128 + \ldots + 1$

So $a = 1024$, $r = -\frac{512}{1024} = -\frac{1}{2}$

and nth term $= 1$

$$1024(-\tfrac{1}{2})^{n-1} = 1$$

First solve $ar^{n-1} = 1$ to find n.

$$(-2)^{n-1} = 1024$$

$$2^{n-1} = 1024$$

$(-2)^{n-1} = (-1)^{n-1}(2^{n-1}) = 1024$, so $(-1)^{n-1}$ must be positive and $2^{n-1} = 1024$.

$$n - 1 = \frac{\log 1024}{\log 2}$$

$$n - 1 = 10$$

$1024 = 2^{10}$

$$n = 11$$

So $$S_{11} = \frac{1024[1 - (-\frac{1}{2})^{11}]}{1 - (-\frac{1}{2})}$$

As $r = -\frac{1}{2}$ (<1) we use the formula $S_n = \dfrac{a(1 - r^n)}{1 - r}$.

$$= \frac{1024(1 + \frac{1}{2048})}{1 + \frac{1}{2}}$$

$$= \frac{1024.5}{\frac{3}{2}} = 683$$

Example 9

An investor invests £2000 on January 1st every year in a savings account that guarantees him 4% per annum for life. If interest is calculated on the 31st of December each year, how much will be in the account at the end of the 10th year?

End of year 1, amount $= 2000 \times 1.04$

A rate of 4% means $\times 1.04$.

Start of year 2, amount $= 2000 \times 1.04 + 2000$

Every new year he invests £2000.

End of year 2, amount $= (2000 \times 1.04 + 2000) \times 1.04$

$= 2000 \times 1.04^2 + 2000 \times 1.04$

At the end of every year the total amount in the account is multiplied by 1.04.

Start of year 3,

amount $= 2000 \times 1.04^2 + 2000 \times 1.04 + 2000$

End of year 3,

amount $= (2000 \times 1.04^2 + 2000 \times 1.04 + 2000) \times 1.04$

$= 2000 \times 1.04^3 + 2000 \times 1.04^2 + 2000 \times 1.04$

So by end of year 10,

Look at the values for the end of year 3 and extend this for 10 years.

amount $= 2000 \times 1.04^{10} + 2000 \times 1.04^9 + \ldots + 2000 \times 1.04$

$= 2000(1.04^{10} + 1.04^9 + \ldots + 1.04)$

This is a geometric series. Substitute $a = 1.04$, $r = 1.04$ and $n = 10$ in $S = \dfrac{a(r^n - 1)}{r - 1}$

$$= 2000 \times \frac{1.04(1.04^{10} - 1)}{1.04 - 1}$$

$= 2000 \times 12.486\ldots = £24\,972.70$

Example 10

Find the least value of n such that the sum of $1 + 2 + 4 + 8 + \ldots$ to n terms would exceed 2 000 000.

Sum to n terms is $S_n = 1\dfrac{(2^n - 1)}{2 - 1}$

$= 2^n - 1$

Substitute $a = 1$, $r = 2$ into $S_n = \dfrac{a(r^n - 1)}{r - 1}$.

If this is to exceed 2 000 000 then

$S_n > 2\,000\,000$

$2^n - 1 > 2\,000\,000$

$2^n > 2\,000\,001$

Add 1.

$n \log(2) > \log(2\,000\,001)$

$n > \dfrac{\log(2\,000\,001)}{\log(2)}$

Use laws of logs: $\log a^n = n \log a$.

$n > 20.9$

It needs 21 terms to exceed 2 000 000

Round up n to the nearest integer.

Example 11

Find $\displaystyle\sum_{r=1}^{10} (3 \times 2^r)$.

$S_{10} = \displaystyle\sum_{r=1}^{10} (3 \times 2^r)$

'Σ' means 'sum of' – in this case the sum of (3×2^r) from $r = 1$ to $r = 10$.

$= 3 \times 2^1 + 3 \times 2^2 + 3 \times 2^3 + \ldots + 3 \times 2^{10}$

$= 3(2^1 + 2^2 + 2^3 + \ldots + 2^{10})$

$= 3 \times 2\dfrac{(2^{10} - 1)}{2 - 1}$

This is a geometric series with $a = 2$, $r = 2$ and $n = 10$.
Use $s = \dfrac{a(r^n - 1)}{r - 1}$

So $S_{10} = 6138$

Exercise 7D

1 Find the sum of the following geometric series (to 3 d.p. if necessary):

a $1 + 2 + 4 + 8 + \ldots$ (8 terms)

b $32 + 16 + 8 + \ldots$ (10 terms)

c $4 - 12 + 36 - 108 \ldots$ (6 terms)

d $729 - 243 + 81 - \ldots -\frac{1}{3}$

e $\displaystyle\sum_{r=1}^{6} 4^r$

f $\displaystyle\sum_{r=1}^{8} 2 \times (3)^r$

g $\displaystyle\sum_{r=1}^{10} 6 \times (\tfrac{1}{2})^r$

h $\displaystyle\sum_{r=0}^{5} 60 \times (-\tfrac{1}{3})^r$

2 The sum of the first three terms of a geometric series is 30.5. If the first term is 8, find possible values of r.

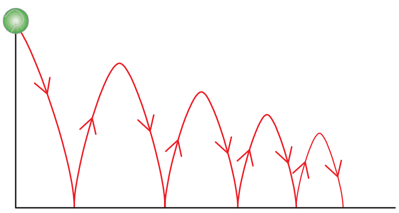
3 The man who invented the game of chess was asked to name his reward. He asked for 1 grain of corn to be placed on the first square of his chessboard, 2 on the second, 4 on the third and so on until all 64 squares were covered. He then said he would like as many grains of corn as the chessboard carried. How many grains of corn did he claim as his prize?

4 Jane invests £4000 at the start of every year. She negotiates a rate of interest of 4% per annum, which is paid at the end of the year. How much is her investment worth at the end of **a** the 10th year and **b** the 20th year?

5 A ball is dropped from a height of 10 m. It bounces to a height of 7 m and continues to bounce. Subsequent heights to which it bounces follow a geometric sequence. Find out:

a how high it will bounce after the fourth bounce,

b the total distance travelled until it hits the ground for the sixth time.



6 Find the least value of n such that the sum $3 + 6 + 12 + 24 + \ldots$ to n terms would first exceed 1.5 million.

7 Find the least value of n such that the sum $5 + 4.5 + 4.05 + \ldots$ to n terms would first exceed 45.

8 Richard is sponsored to cycle 1000 miles over a number of days. He cycles 10 miles on day 1, and increases this distance by 10% a day. How long will it take him to complete the challenge? What was the greatest number of miles he completed in a single day?

9 A savings scheme is offering a rate of interest of 3.5% per annum for the lifetime of the plan. Alan wants to save up £20 000. He works out that he can afford to save £500 every year, which he will deposit on 1 January. If interest is paid on 31 December, how many years will it be before he has saved up his £20 000?

7.5 You need to be able to find the sum to infinity of a convergent geometric series.

Consider the series $S = 3 + 1.5 + 0.75 + 0.375 + \ldots$

No matter how many terms of the series you take, the sum never exceeds a certain number. We call this number the limit of the sum, or more often, its sum to infinity.

We can find out what this limit is.

As $a = 3$ and $r = \frac{1}{2}$, $S = \dfrac{a(1 - r^n)}{1 - r} = \dfrac{3(1 - (\frac{1}{2})^n)}{1 - \frac{1}{2}} = 6(1 - (\frac{1}{2})^n)$

If we replace n with certain values to find the sum we find that

when $n = 3$, $S_3 = 5.25$
when $n = 5$, $S_5 = 5.8125$
when $n = 10$, $S_{10} = 5.9994$
when $n = 20$, $S_{20} = 5.999\,994$

You can see that as n gets larger, S becomes closer and closer to 6.

We say that this infinite series is **convergent**, and has a sum to infinity of 6. Convergent means the series tends towards a specific value as more terms are added.

Not all series converge. The reason that this one does is that the terms of the sequence are getting smaller.

This happens because $-1 < r < 1$.

The sum to infinity of a series exists only if $-1 < r < 1$.

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

If $-1 < r < 1$, $r^n \to 0$ as $n \to \infty$

$$S_\infty = \frac{a(1 - 0)}{1 - r} = \frac{a}{1 - r}$$

Hint: You can write 'the sum to infinity' is S_∞.

- **The sum to infinity of a geometric series is $\frac{a}{1 - r}$ if $|r| < 1$.**

Hint: $|r|$ means $-1 < r < 1$.

Example 12

Find the sums to infinity of the following series:

a $40 + 10 + 2.5 + 0.625 + \ldots$

b $1 + \frac{1}{p} + \frac{1}{p^2} + \ldots$

a $40 + 10 + 2.5 + 0.625 + \ldots$

In this series $a = 40$ and $r = \frac{10}{40} = \frac{1}{4}$

$-1 < r < 1$, so S_∞ exists

$$S = \frac{a}{1 - r} = \frac{40}{1 - \frac{1}{4}} = \frac{40}{\frac{3}{4}} = \frac{160}{3}$$

Always write down the values of a and r, using $\frac{u_2}{u_1}$ for r.

Substitute $a = 40$ and $r = \frac{10}{40} = \frac{1}{4}$ into $S = \frac{a}{1 - r}$.

b $1 + \frac{1}{p} + \frac{1}{p^2} + \ldots$

In this series $a = 1$ and $r = \frac{u_2}{u_1} = \frac{\frac{1}{p}}{1} = \frac{1}{p}$

S will exist if $\left|\frac{1}{p}\right| < 1$ so $p > 1$.

If $p > 1$, $S_\infty = \frac{1}{1 - \frac{1}{p}}$

$$= \frac{p}{p - 1}$$

Multiply top and bottom by p.

Example 13

The sum to 4 terms of a geometric series is 15 and the sum to infinity is 16.

a Find the possible values of r.

b Given that the terms are all positive, find the first term in the series.

a $\frac{a(1-r^4)}{1-r} = 15$ ①

$\frac{a}{1-r} = 16$ ②

$16(1-r^4) = 15$

$1 - r^4 = \frac{15}{16}$

$r^4 = \frac{1}{16}$

$r = \pm\frac{1}{2}$

$S_4 = 15$ so use the formula $S_n = \frac{a(1-r^n)}{1-r}$ with $n = 4$.

$S_\infty = 16$ so use the formula $S_\infty = \frac{a}{1-r}$ with $S_\infty = 16$.

Solve equations simultaneously.

Replace $\frac{a}{1-r}$ by 16 in equation ①

Divide by 16.

Rearrange.

Take the 4th root of $\frac{1}{16}$.

b As all terms positive, $r = +\frac{1}{2}$

Substitute $r = +\frac{1}{2}$ back into equation ② to find a

$\frac{a}{1-\frac{1}{2}} = 16$

$16(1-\frac{1}{2}) = a$

$a = 8$

The first term in the series is 8.

Exercise 7E

1 Find the sum to infinity, if it exists, of the following series:

a $1 + 0.1 + 0.01 + 0.001 + \ldots$

b $1 + 2 + 4 + 8 + 16 + \ldots$

c $10 - 5 + 2.5 - 1.25 + \ldots$

d $2 + 6 + 10 + 14$

e $1 + 1 + 1 + 1 + 1 + \ldots$

f $3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots$

g $0.4 + 0.8 + 1.2 + 1.6 + \ldots$

h $9 + 8.1 + 7.29 + 6.561 + \ldots$

i $1 + r + r^2 + r^3 + \ldots$

j $1 - 2x + 4x^2 - 8x^3 + \ldots$

2 Find the common ratio of a geometric series with a first term of 10 and a sum to infinity of 30.

3 Find the common ratio of a geometric series with a first term of -5 and a sum to infinity of -3.

4 Find the first term of a geometric series with a common ratio of $\frac{2}{3}$ and a sum to infinity of 60.

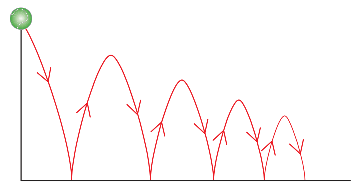
5 Find the first term of a geometric series with a common ratio of $-\frac{1}{3}$ and a sum to infinity of 10.

6 Find the fraction equal to the recurring decimal 0.232 323 232 3.

Hint for question 6: Write 0.232 323 232 3 as $\frac{23}{100} + \frac{23}{10\,000} + \frac{23}{1\,000\,000} + \ldots$

7 Find $\sum_{r=1}^{\infty} 4(0.5)^r$.

8 A ball is dropped from a height of 10 m. It bounces to a height of 6 m, then 3.6, and so on following a geometric sequence.
Find the total distance travelled by the ball.



9 The sum to three terms of geometric series is 9 and its sum to infinity is 8. What could you deduce about the common ratio. Why? Find the first term and common ratio.

10 The sum to infinity of a geometric series is three times the sum to 2 terms. Find all possible values of the common ratio.

Mixed exercise 7F

1 State which of the following series are geometric. For the ones that are, give the value of the common ratio r.

a $4 + 7 + 10 + 13 + 16 + \ldots$ **b** $4 + 6 + 9 + 13.5 + \ldots$

c $20 + 10 + 5 + 2.5 + \ldots$ **d** $4 - 8 + 16 - 32 + \ldots$

e $4 - 2 - 8 - 14 - \ldots$ **f** $1 + 1 + 1 + 1 + \ldots$

2 Find the 8th and nth terms of the following geometric sequences:

a 10, 7, 4.9, … **b** 5, 10, 20, …

c 4, −4, 4, … **d** 3, −1.5, 0.75, …

3 Find the sum to 10 terms of the following geometric series:

a $4 + 8 + 16 + \ldots$ **b** $30 - 15 + 7.5, \ldots$

c $5 + 5 + 5, \ldots$ **d** $2 + 0.8 + 0.32, \ldots$

4 Determine which of the following geometric series converge. For the ones that do, give the limiting value of this sum (i.e. S_∞).

a $6 + 2 + \frac{2}{3} + \ldots$ **b** $4 - 2 + 1 - \ldots$

c $5 + 10 + 20 + \ldots$ **d** $4 + 1 + 0.25 + \ldots$

5 A geometric series has third term 27 and sixth term 8:

a Show that the common ratio of the series is $\frac{2}{3}$.

b Find the first term of the series.

c Find the sum to infinity of the series.

d Find, to 3 significant figures, the difference between the sum of the first 10 terms of the series and the sum to infinity of the series. *E*

6 The second term of a geometric series is 80 and the fifth term of the series is 5.12:

a Show that the common ratio of the series is 0.4.

Calculate:

b the first term of the series,

c the sum to infinity of the series, giving your answer as an exact fraction,

d the difference between the sum to infinity of the series and the sum of the first 14 terms of the series, giving your answer in the form $a \times 10^n$, where $1 \leqslant a < 10$ and n is an integer. *E*

7 The nth term of a sequence is u_n, where $u_n = 95(\frac{4}{5})^n$, $n = 1, 2, 3, \ldots$

a Find the value of u_1 and u_2.

Giving your answers to 3 significant figures, calculate:

b the value of u_{21},

c $\sum_{n=1}^{15} u_n$,

d Find the sum to infinity of the series whose first term is u_1 and whose nth term is u_n. *E*

8 A sequence of numbers $u_1, u_2, \ldots, u_n, \ldots$ is given by the formula $u_n = 3(\frac{2}{3})^n - 1$ where n is a positive integer.

a Find the values of u_1, u_2 and u_3.

b Show that $\sum_{n=1}^{15} u_n = -9.014$ to 4 significant figures.

c Prove that $u_{n+1} = 2(\frac{2}{3})^n - 1$. *E*

9 The third and fourth terms of a geometric series are 6.4 and 5.12 respectively. Find:

a the common ratio of the series,

b the first term of the series,

c the sum to infinity of the series.

d Calculate the difference between the sum to infinity of the series and the sum of the first 25 terms of the series. *E*

10 The price of a car depreciates by 15% per annum. If its new price is £20 000, find:

a a formula linking its value £V with its age a years,

b its value after 5 years,

c the year in which it will be worth less than £4000.

11 The first three terms of a geometric series are $p(3q + 1)$, $p(2q + 2)$ and $p(2q - 1)$ respectively, where p and q are non-zero constants.

a Use algebra to show that one possible value of q is 5 and to find the other possible value of q.

b For each possible value of q, calculate the value of the common ratio of the series.

Given that $q = 5$ and that the sum to infinity of the geometric series is 896, calculate:

c the value of p,

d the sum, to 2 decimal places, of the first twelve terms of the series. *E*

12 A savings scheme pays 5% per annum compound interest. A deposit of £100 is invested in this scheme at the start of each year.

a Show that at the start of the third year, after the annual deposit has been made, the amount in the scheme is £315.25.

b Find the amount in the scheme at the start of the fortieth year, after the annual deposit has been made. *E*

13 A competitor is running in a 25 km race. For the first 15 km, she runs at a steady rate of 12 km h^{-1}. After completing 15 km, she slows down and it is now observed that she takes 20% longer to complete each kilometre than she took to complete the previous kilometre.

a Find the time, in hours and minutes, the competitor takes to complete the first 16 km of the race.

The time taken to complete the rth kilometre is u_r hours.

b Show that, for $16 \leqslant r \leqslant 25$, $u_r = \frac{1}{12}(1.2)^{r-15}$.

c Using the answer to **b**, or otherwise, find the time, to the nearest minute, that she takes to complete the race. *E*

14 A liquid is kept in a barrel. At the start of a year the barrel is filled with 160 litres of the liquid. Due to evaporation, at the end of every year the amount of liquid in the barrel is reduced by 15% of its volume at the start of the year.

a Calculate the amount of liquid in the barrel at the end of the first year.

b Show that the amount of liquid in the barrel at the end of ten years is approximately 31.5 litres.

At the start of each year a new barrel is filled with 160 litres of liquid so that, at the end of 20 years, there are 20 barrels containing liquid.

c Calculate the total amount of liquid, to the nearest litre, in the barrels at the end of 20 years. *E*

15 At the beginning of the year 2000 a company bought a new machine for £15 000. Each year the value of the machine decreases by 20% of its value at the start of the year.

a Show that at the start of the year 2002, the value of the machine was £9600.

b When the value of the machine falls below £500, the company will replace it. Find the year in which the machine will be replaced.

c To plan for a replacement machine, the company pays £1000 at the start of each year into a savings account. The account pays interest of 5% per annum. The first payment was made when the machine was first bought and the last payment will be made at the start of the year in which the machine is replaced. Using your answer to part **b**, find how much the savings account will be worth when the machine is replaced. *E*

16 A mortgage is taken out for £80 000. It is to be paid by annual instalments of £5000 with the first payment being made at the end of the first year that the mortgage was taken out. Interest of 4% is then charged on any outstanding debt. Find the total time taken to pay off the mortgage.

Hint for question 16: Find an expression for the debt remaining after n years and solve using the fact that if it is paid off, the debt = 0.

Summary of key points

1 In a geometric series you get from one term to the next by multiplying by a constant called the common ratio.

2 The formula for the nth term $= ar^{n-1}$ where a = first term and r = common ratio.

3 The formula for the sum to n terms is

$$S_n = \frac{a(1 - r^n)}{1 - r} \text{ or } S_n = \frac{a(r^n - 1)}{r - 1}$$

4 The sum to infinity exists if $|r| < 1$ and is $S_\infty = \dfrac{a}{1 - r}$