After completing this chapter you should be able to

1. calculate the sine, cosine and tangent of any angle
2. know the exact trigonometrical ratios for 30°, 45° and 60°
3. sketch the graphs of the sine, cosine and tangent functions
4. sketch simple transformations of these graphs.

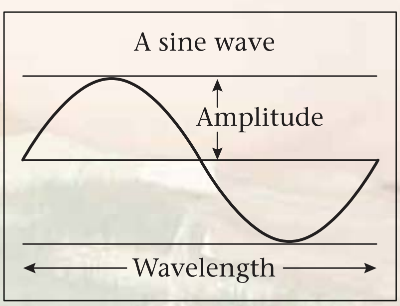
8 Graphs of trigonometric functions

It may seem unlikely but trigonometrical graphs occur quite often in everyday life.

For example sound waves move according to the equation

$$y = A\sin(kt)$$

where A represents the amplitude which is a measure of the 'loudness' and k represents the frequency which is a measure of 'pitch'

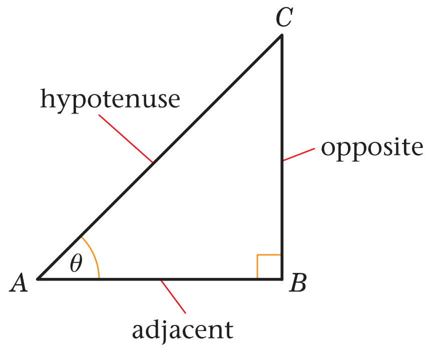


In fact any backward and forward motion such as a person on a swing or a weight on a spring can be described as sinusoidal.

8.1 You need to be able to use the three basic trigonometric functions for any angle.

You can use the trigonometric ratios

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



to find the missing sides and angles in a right-angled triangle.

To extend the work on sine, cosine and tangents to cover angles of any size, both positive and negative, we need to modify these definitions.

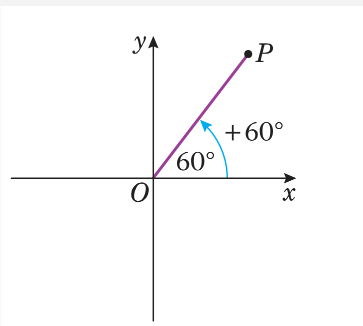
You need to know what is meant by positive and negative angles.

Example 1

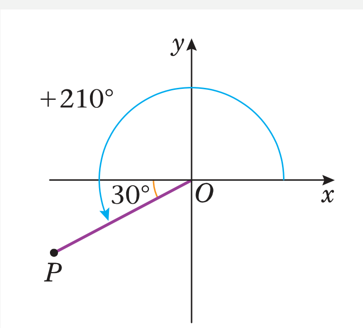
The line OP, where O is the origin, makes an angle θ with the positive x-axis.
Draw diagrams to show the position of OP where θ equals:

a $+60°$ **b** $+210°$ **c** $-60°$ **d** $-200°$

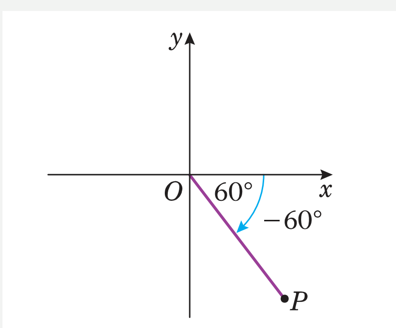
a



b



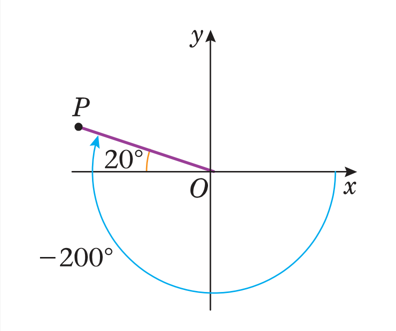
c



You could also give the angles in radians as

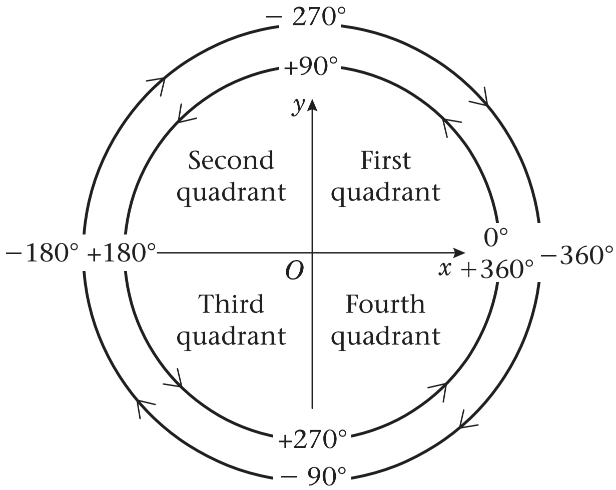
a $+\frac{\pi}{3}$ **b** $+\frac{7\pi}{6}$ **c** $-\frac{\pi}{3}$ **d** $-\frac{10\pi}{9}$

d



Remember: Anticlockwise angles are positive, clockwise angles are negative, measured from the positive x-axis.

■ **The x–y plane is divided into quadrants:**

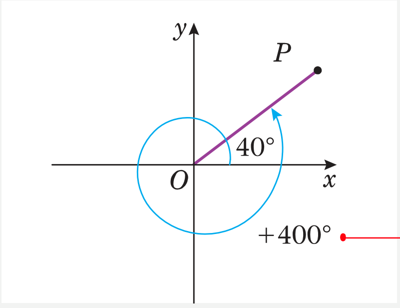


Hint: Angles may lie outside the range 0–360°, but they will always lie in one of the four quadrants.

Example 2

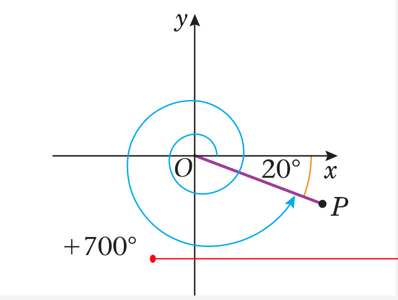
Draw diagrams to show the position of OP where $\theta =$: **a** +400°, **b** +700°, **c** −480°.

a



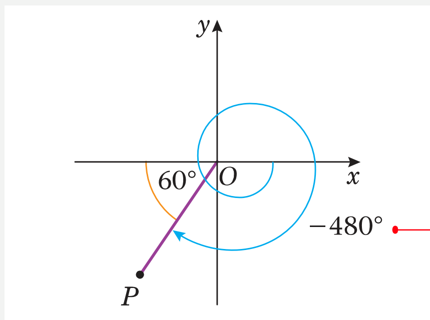
360° + 40°

b



360° + 340°

c



$-360° - 120°$

Exercise 8A

1 Draw diagrams, as in Examples **1** and **2**, to show the following angles. Mark in the acute angle that OP makes with the x-axis.

a $-80°$ **b** $100°$ **c** $200°$
d $165°$ **e** $-145°$ **f** $225°$
g $280°$ **h** $330°$ **i** $-160°$
j $-280°$ **k** $\frac{3\pi}{4}$ **l** $\frac{7\pi}{6}$
m $-\frac{5\pi}{3}$ **n** $-\frac{5\pi}{8}$ **o** $\frac{19\pi}{9}$

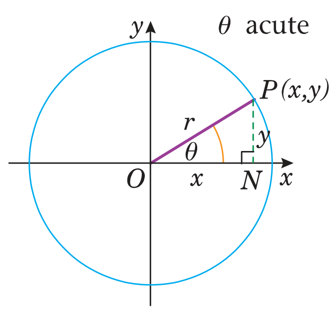
2 State the quadrant that OP lies in when the angle that OP makes with the positive x-axis is:

a $400°$ **b** $115°$ **c** $-210°$
d $255°$ **e** $-100°$ **f** $\frac{7\pi}{8}$
g $-\frac{11\pi}{6}$ **h** $\frac{13\pi}{7}$

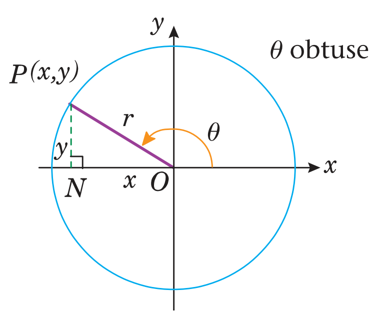
■ **For all values of θ, the definitions of sin θ, cos θ and tan θ are taken to be**

$$\sin\theta = \frac{y}{r} \quad \cos\theta = \frac{x}{r} \quad \tan\theta = \frac{y}{x}$$

where x and y are the coordinates of P and r is the length of OP.



The values of sin θ and cos θ, where θ is a multiple of 90°, follow from the definitions above.



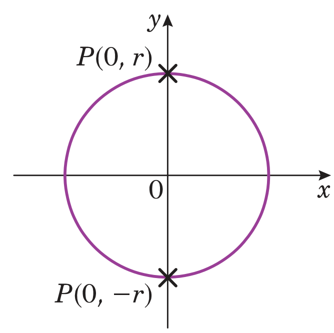
Example 3

Write down the values of **a** sin 90°, **b** sin 180°, **c** sin 270°, **d** cos 180°, **e** cos (−90)° **f** cos 450°.

a $\sin 90° = 1$	P has coordinates $(0, r)$ so $\sin 90° = \frac{r}{r}$.
b $\sin 180° = 0$	P has coordinates $(-r, 0)$ so $\sin 180° = \frac{0}{r}$.
c $\sin 270° = -1$	P has coordinates $(0, -r)$ so $\sin 270° = -\frac{r}{r}$.
d $\cos 180° = -1$	P has coordinates $(-r, 0)$ so $\cos 180° = -\frac{r}{r}$.
e $\cos (-90)° = 0$	P has coordinates $(0, -r)$ so $\cos (-90)° = \frac{0}{r}$.
f $\cos 450° = 0$	P has coordinates $(0, r)$ so $\cos 450° = \frac{0}{r}$.

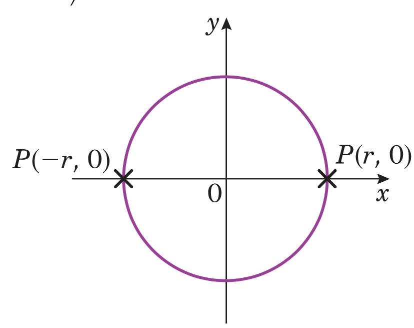
Tan $\theta = \frac{y}{x}$ so when $x = 0$ and $y \neq 0$ tan θ is indeterminate.

This is when P is at $(0, r)$ or $(0, -r)$.



- Tan θ is indeterminate when θ is an odd multiple of 90° $\left(\text{or } \frac{\pi}{2} \text{ radians}\right)$.

When $y = 0$, tan $\theta = 0$. This is when P is at $(r, 0)$ or $(-r, 0)$.



- Tan $\theta = 0$ when θ is 0° or an even multiple of 90° $\left(\text{or } \frac{\pi}{2} \text{ radians}\right)$.

Exercise 8B

(*Note*: do not use a calculator.)

1 Write down the values of:

a sin (−90)° **b** sin 450° **c** sin 540° **d** sin (−450)°
e cos (−180)° **f** cos (−270)° **g** cos 270° **h** cos 810°
i tan 360° **j** tan (−180)°

2 Write down the values of the following, where the angles are in radians:

a $\sin \frac{3\pi}{2}$ **b** $\sin \left(-\frac{\pi}{2}\right)$ **c** $\sin 3\pi$ **d** $\sin \frac{7\pi}{2}$
e $\cos 0$ **f** $\cos \pi$ **g** $\cos \frac{3\pi}{2}$ **h** $\cos \left(-\frac{3\pi}{2}\right)$
i $\tan \pi$ **j** $\tan (-2\pi)$

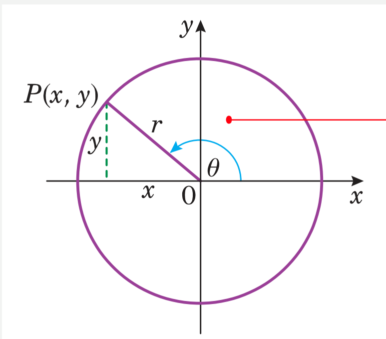
8.2 You need to know the signs of the three trigonometric functions in the four quadrants.

In the first quadrant sin, cos and tan are positive.

By considering the sign of x and y, the coordinates of P, you can find the sign of the three trigonometric functions in the other quadrants.

Example 4

Find the signs of sin θ, cos θ and tan θ in the second quadrant (θ is obtuse, $90° < \theta < 180°$).



Draw a circle, centre O and radius r, with $P(x, y)$ on the circle in the second quadrant.

As x is −ve and y is +ve in this quadrant

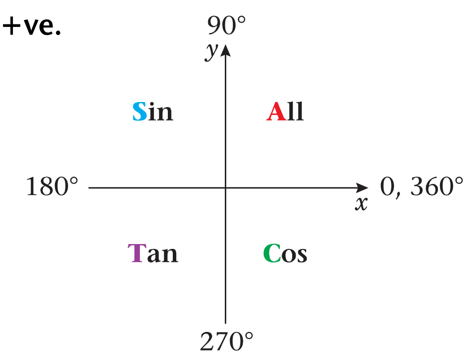
$$\sin\theta = \frac{y}{r} = \frac{+\text{ve}}{+\text{ve}} = +\text{ve}$$

$$\cos\theta = \frac{x}{r} = \frac{-\text{ve}}{+\text{ve}} = -\text{ve}$$

$$\tan\theta = \frac{y}{x} = \frac{+\text{ve}}{-\text{ve}} = -\text{ve}$$

So only sin θ is positive.

- **In the *first* quadrant sin, cos and tan are *all* +ve.**
  In the *second* quadrant only *sine* is +ve.
  In the *third* quadrant only *tan* is +ve.
  In the *fourth* quadrant only *cos* is +ve.



The diagram shows which trigonometric functions are *positive* in each quadrant.

You might find it useful to make up a mnemonic to remember these results.
For example: All Silver Toy Cars.

If you make one up, it is a good idea to keep to the order **A**, **S**, **T**, **C**.

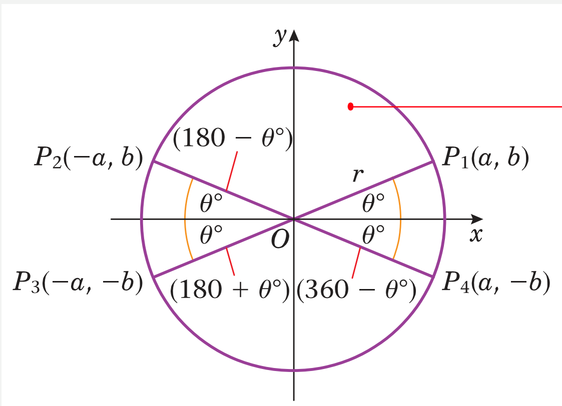
Example 5

Show that:

a $\sin(180 - \theta)° = \sin\theta°$

b $\sin(180 + \theta)° = -\sin\theta°$

c $\sin(360 - \theta)° = -\sin\theta°$



Draw a diagram to show the position of the four angles $\theta°$, $(180 - \theta)°$, $(180 + \theta)°$ and $(360 - \theta)°$.

Hint: The four lines, OP_1, OP_2, OP_3 and OP_4, representing the four angles are all inclined at $\theta°$ to the horizontal.

As $\sin\theta = \frac{y}{r}$, it follows that:

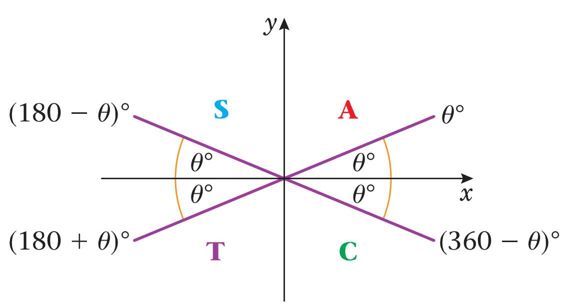
$\sin(180 - \theta)° = \frac{b}{r} = \sin\theta°$

$\sin(180 + \theta)° = -\frac{b}{r} = -\sin\theta°$

$\sin(360 - \theta)° = \frac{-b}{r} = -\sin\theta°$

- **The results for sine, cosine and tangent are:**

$\sin(180 - \theta)° = \sin\theta°$
$\sin(180 + \theta)° = -\sin\theta°$
$\sin(360 - \theta)° = -\sin\theta°$
$\cos(180 - \theta)° = -\cos\theta°$
$\cos(180 + \theta)° = -\cos\theta°$
$\cos(360 - \theta)° = \cos\theta°$
$\tan(180 - \theta)° = -\tan\theta°$
$\tan(180 + \theta)° = \tan\theta°$
$\tan(360 - \theta)° = -\tan\theta°$



Hint: For angles measured in radians, the same results hold, with 180° being replaced by π, e.g. $\sin(\pi - \theta) = \sin\theta$; $\cos(\pi + \theta) = -\cos\theta$; $\tan(2\pi - \theta) = -\tan\theta$.

Hint: All angles that are equally inclined to either the +ve x-axis or the −ve x-axis have trigonometric ratios which are equal in magnitude, but they take the sign indicated by the quadrant the angle is in.

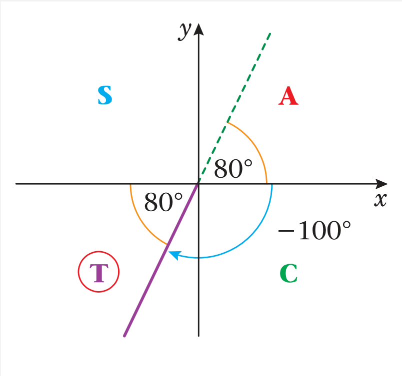
Example 6

Express in terms of trigonometric ratios of acute angles:

a sin (−100)° **b** cos 330° **c** tan 500°

For each part, draw diagrams showing the position of *OP* for the given angle and insert the acute angle that OP makes with the x-axis.

a

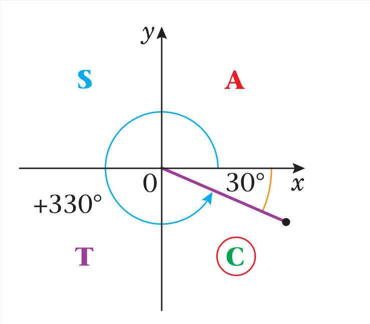


The acute angle made with x-axis is 80°.

In the third quadrant only tan is +ve, so sin is −ve.

So sin (−100)° = −sin 80°

b

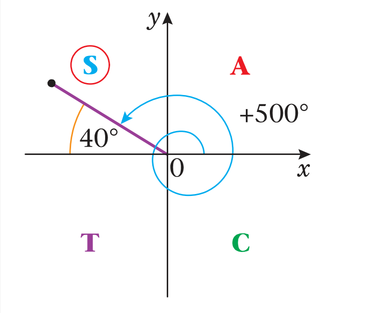


The acute angle made with x-axis is 30°.

In the fourth quadrant only cos is +ve.

So cos 330° = +cos 30°

c



The acute angle made with x-axis is 40°.

In the second quadrant only sin is +ve.

So tan 500° = −tan 40°

Exercise 8C

(*Note*: Do not use a calculator.)

1 By drawing diagrams, as in Example **6**, express the following in terms of trigonometric ratios of acute angles:

a $\sin 240°$ **b** $\sin(-80)°$ **c** $\sin(-200)°$ **d** $\sin 300°$ **e** $\sin 460°$
f $\cos 110°$ **g** $\cos 260°$ **h** $\cos(-50)°$ **i** $\cos(-200)°$ **j** $\cos 545°$
k $\tan 100°$ **l** $\tan 325°$ **m** $\tan(-30)°$ **n** $\tan(-175)°$ **o** $\tan 600°$
p $\sin\frac{7\pi}{6}$ **q** $\cos\frac{4\pi}{3}$ **r** $\cos\left(-\frac{3\pi}{4}\right)$ **s** $\tan\frac{7\pi}{5}$ **t** $\tan\left(-\frac{\pi}{3}\right)$
u $\sin\frac{15\pi}{16}$ **v** $\cos\frac{8\pi}{5}$ **w** $\sin\left(-\frac{6\pi}{7}\right)$ **x** $\tan\frac{15\pi}{8}$

2 Given that θ is an acute angle measured in degrees, express in terms of $\sin\theta$:

a $\sin(-\theta)$ **b** $\sin(180° + \theta)$ **c** $\sin(360° - \theta)$
d $\sin -(180° + \theta)$ **e** $\sin(-180° + \theta)$ **f** $\sin(-360° + \theta)$
g $\sin(540° + \theta)$ **h** $\sin(720° - \theta)$ **i** $\sin(\theta + 720°)$

3 Given that θ is an acute angle measured in degrees, express in terms of $\cos\theta$ or $\tan\theta$:

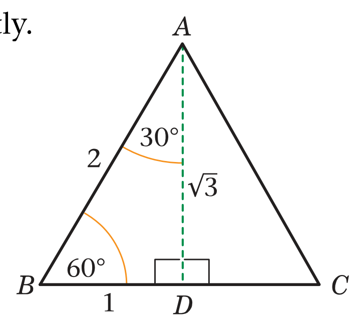
a $\cos(180° - \theta)$ **b** $\cos(180° + \theta)$ **c** $\cos(-\theta)$
d $\cos -(180° - \theta)$ **e** $\cos(\theta - 360°)$ **f** $\cos(\theta - 540°)$
g $\tan(-\theta)$ **h** $\tan(180° - \theta)$ **i** $\tan(180° + \theta)$
j $\tan(-180° + \theta)$ **k** $\tan(540° - \theta)$ **l** $\tan(\theta - 360°)$

The results obtained in questions **2** and **3** are true for all values of θ.

4 A function f is an even function if $f(-\theta) = f(\theta)$.
A function f is an odd function if $f(-\theta) = -f(\theta)$.
Using your results from questions **2a**, **3c** and **3g**, state whether $\sin\theta$, $\cos\theta$ and $\tan\theta$ are odd or even functions.

8.3 You need to be able to find the exact values of some trigonometrical ratios.

You can find the trigonometrical ratios of angles 30°, 45° and 60° exactly.
Consider an equilateral triangle *ABC* of side 2 units.
If you drop a perpendicular from *A* to meet *BC* at *D*,
then $BD = DC = 1$ unit, $\angle BAD = 30°$ and $\angle ABD = 60°$.
Using Pythagoras' theorem in $\triangle ABD$



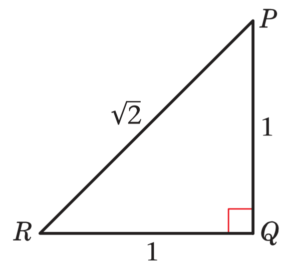
$$AD^2 = 2^2 - 1^2 = 3$$

So $$AD = \sqrt{3} \text{ units}$$

Using $\triangle ABD$, $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$,

and $\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 60° = \frac{1}{2}$, $\tan 60° = \sqrt{3}$.

If you now consider an isosceles right-angled triangle PQR, in which $PQ = QR = 1$ unit, then the ratios for 45° can be found.



Using Pythagoras' theorem

$$PR^2 = 1^2 + 1^2 = 2$$

So $PR = \sqrt{2}$ units

Then $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and $\tan 45° = 1$

Exercise 8D

1 Express the following as trigonometric ratios of either 30°, 45° or 60°, and hence find their exact values.

a sin 135°	**b** sin (−60°)	**c** sin 330°	**d** sin 420°	**e** sin (−300°)
f cos 120°	**g** cos 300°	**h** cos 225°	**i** cos (−210°)	**j** cos 495°
k tan 135°	**l** tan (−225°)	**m** tan 210°	**n** tan 300°	**o** tan (−120°)

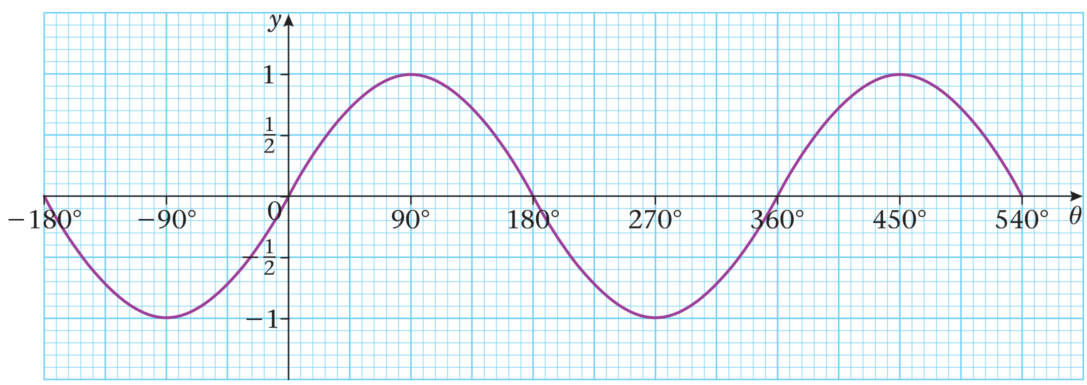
2 In Section 8.3 you saw that $\sin 30° = \cos 60°$, $\cos 30° = \sin 60°$, and $\tan 60° = \frac{1}{\tan 30°}$.

These are particular examples of the general results: $\sin(90° - \theta) = \cos\theta$, and $\cos(90° - \theta) = \sin\theta$, and $\tan(90° - \theta) = \frac{1}{\tan\theta}$, where the angle θ is measured in degrees.

Use a right-angled triangle ABC to verify these results for the case when θ is acute.

8.4 You need to be able to recognise the graphs of sin θ, cos θ and tan θ.

$y = \sin\theta$



Functions that repeat themselves after a certain interval are called periodic functions, and the interval is called the period of the function. You can see that $\sin\theta$ is periodic with a period of 360°.

Hint: The graph of $\sin\theta$, where θ is in radians, has period 2π.

There are many symmetry properties of $\sin\theta$ (some were seen in Example 5) but you can see from the graph that

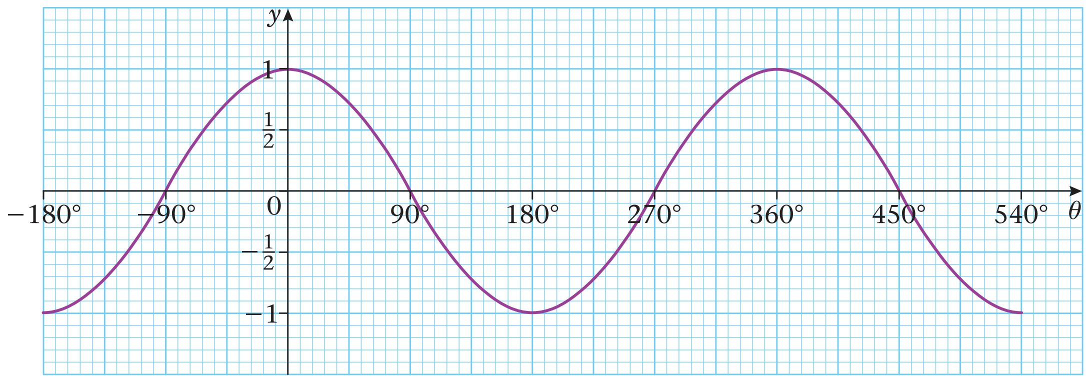
$\sin(\theta + 360°) = \sin\theta$ and $\sin(\theta - 360°) = \sin\theta$

$\sin(90° - \theta) = \sin(90° + \theta)$

Hint: Because it is periodic.

Hint: Symmetry about $\theta = 90°$.

$y = \cos\theta$



Like $\sin\theta$, $\cos\theta$ is periodic with a period of 360°. In fact, the graph of $\cos\theta$ is the same as that of $\sin\theta$ when it has been translated by 90° to the left.

Two further symmetry properties of $\cos\theta$ are

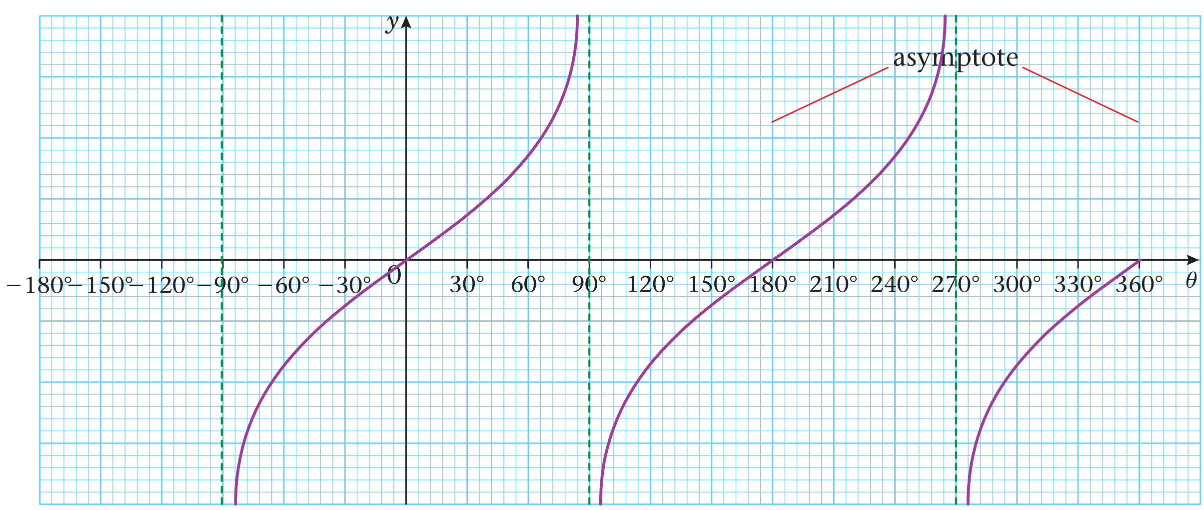
Hint: Because it is periodic.

$\cos(\theta + 360°) = \cos\theta$ and $\cos(\theta - 360°) = \cos\theta$

$\cos(-\theta) = \cos\theta$ (seen on page 125)

Hint: Symmetry about $\theta = 0°$.

$y = \tan\theta$



This function behaves very differently from the sine and cosine functions but it is still periodic, it repeats itself in cycles of 180° so its period is 180°.

The period symmetry properties of $\tan\theta$ are

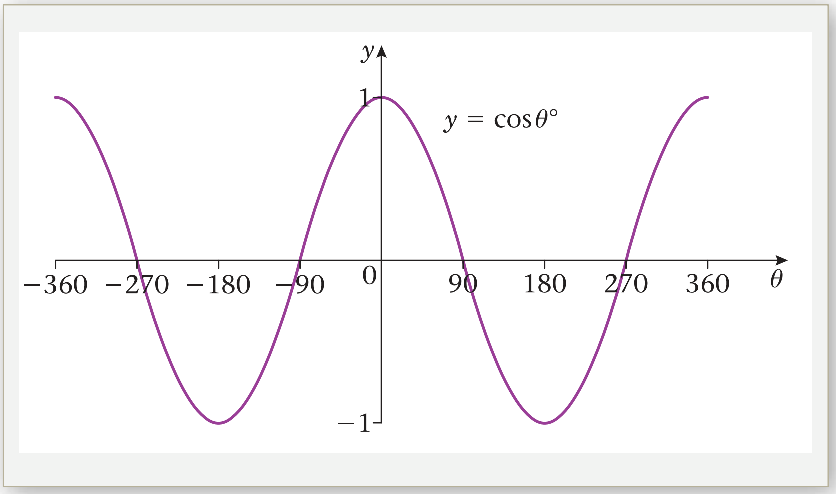
$\tan(\theta + 180°) = \tan\theta$

$\tan(\theta - 180°) = \tan\theta$

Hint: The dotted lines on the graph are called asymptotes, lines to which the curve approaches but never reaches; these occur at $\theta = (2n + 1)90°$ where n is integer.

Example 7

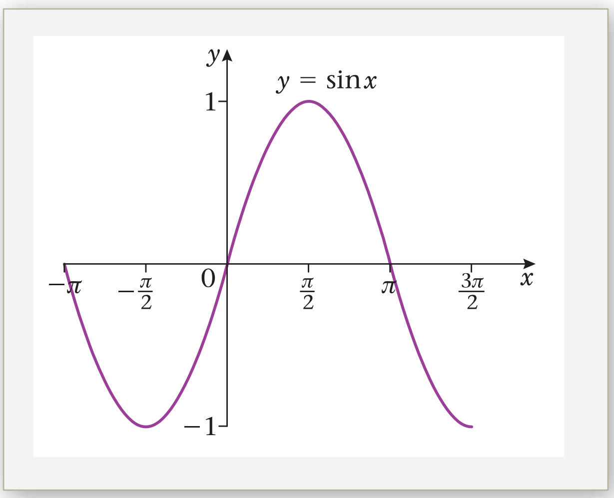
Sketch the graph of $y = \cos\theta°$ in the interval $-360 \leqslant \theta \leqslant 360$.



The axes are θ and y.
The curve meets the θ-axis at $\theta = \pm 270°$ and $\theta = \pm 90°$.
Note that the form of the equation given here means that θ is a number.
The curve crosses the y-axis at (0, 1).

Example 8

Sketch the graph of $y = \sin x$ in the interval $-\pi \leqslant x \leqslant \dfrac{3\pi}{2}$.



Here the axes are x and y, and the interval tells you that x is in radians.
The curve meets the x-axis at $x = \pm\pi$ and $x = 0$.

Exercise 8E

1 Sketch the graph of $y = \cos\theta$ in the interval $-\pi \leqslant \theta \leqslant \pi$.

2 Sketch the graph of $y = \tan\theta°$ in the interval $-180 \leqslant \theta \leqslant 180$.

3 Sketch the graph of $y = \sin\theta°$ in the interval $-90 \leqslant \theta \leqslant 270$.

8.5 You need to be able to perform simple transformations on the graphs of sin θ, cos θ and tan θ.

You can sketch the graphs of $a \sin\theta$, $a\cos\theta$, and $a\tan\theta$, where a is a constant.

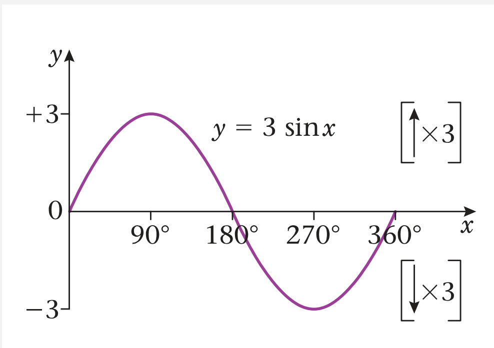
Example 9

Sketch on separate axes the graphs of:

a $y = 3\sin x$, $0 \leqslant x \leqslant 360°$

b $y = -\tan\theta$, $-\pi \leqslant \theta \leqslant \pi$

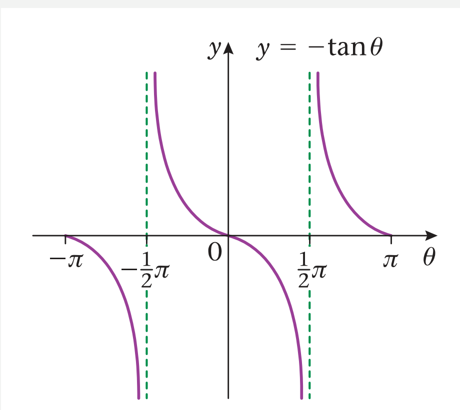
a



The effect of the multiplication factor 3, is to stretch vertically the graph of sin x by a scale factor of 3; there is no effect in the x-direction.

In this case the labelling of the intercepts on the x-axis are in degrees (°).

b



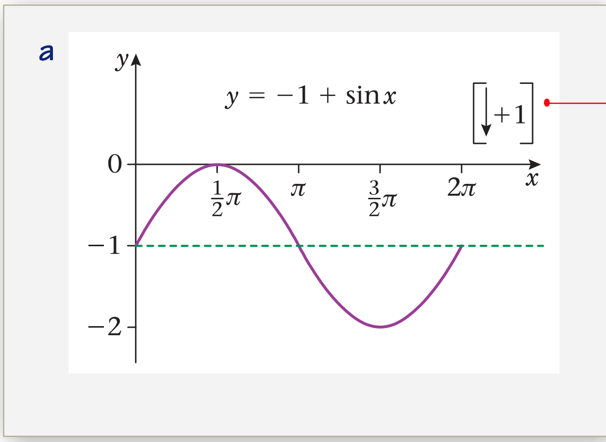
The effect of the multiplication factor -1, is to reflect the graph of tan θ in the θ-axis. Labelling on the θ-axis is in radians.

You can sketch the graphs of $\sin\theta + a$, $\cos\theta + a$ and $\tan\theta + a$.

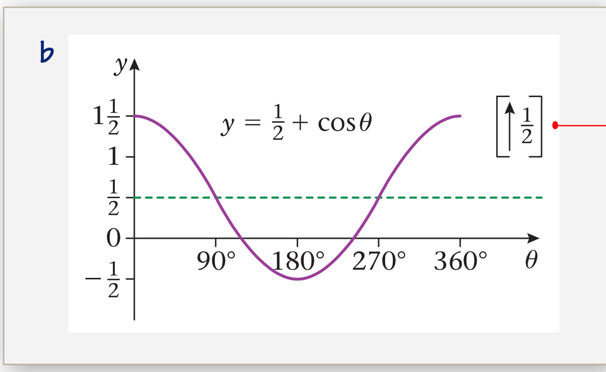
Example 10

Sketch on separate axes the graphs of:

a $y = -1 + \sin x$, $0 \leqslant x \leqslant 2\pi$ **b** $y = \frac{1}{2} + \cos\theta$, $0 \leqslant \theta \leqslant 360°$



The graph of $y = \sin x$ is translated by 1 unit in the negative y-direction.



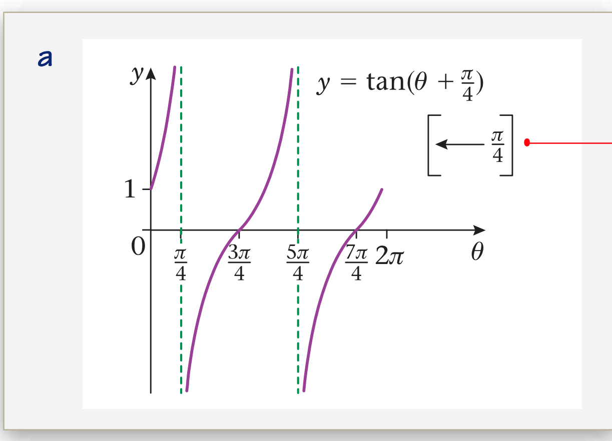
The graph of $y = \cos\theta$ is translated by $\frac{1}{2}$ unit in the positive y-direction. To find the intercepts on the θ-axis requires solving the equation $\frac{1}{2} + \cos\theta = 0$, but if you look back to the graph on page 129 you can see that these are at 120° and 240°.

You can sketch the graphs of $\sin(\theta + \alpha)$, $\cos(\theta + \alpha)$ and $\tan(\theta + \alpha)$.

Example 11

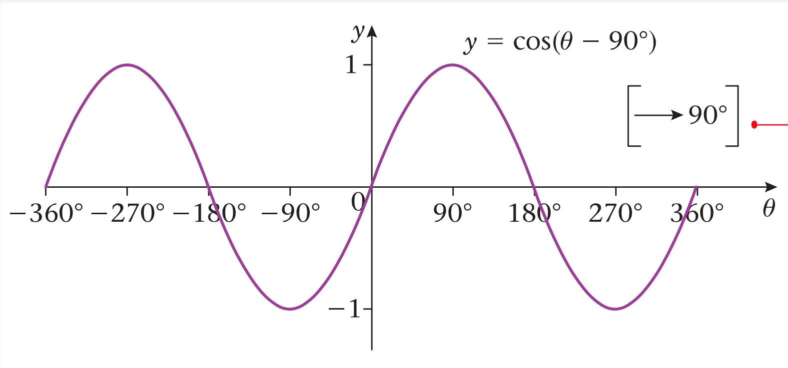
Sketch on separate axes the graphs of:

a $y = \tan\left(\theta + \frac{\pi}{4}\right)$, $0 \leqslant \theta \leqslant 2\pi$ **b** $y = \cos(\theta - 90°)$, $-360° \leqslant \theta \leqslant 360°$



The graph of $y = \tan\theta$ is translated by $\frac{\pi}{4}$ to the left. The asymptotes are now at $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$. The curve meets the y-axis where $\theta = 0$, so $y = 1$.

b



The graph of $y = \cos\theta$ is translated by 90° to the right. Note that this is exactly the same curve as $y = \sin\theta$, so another property is that $\cos(\theta - 90°) = \sin\theta$.

You can sketch the graphs of sin $n\theta$, cos $n\theta$ and tan $n\theta$.

Remember that the curve with equation $y = f(ax)$ is a horizontal stretch with scale factor $\frac{1}{a}$ of the curve $y = f(x)$. In the special case where $a = -1$, this is equivalent to a reflection in the y-axis.

Example 12

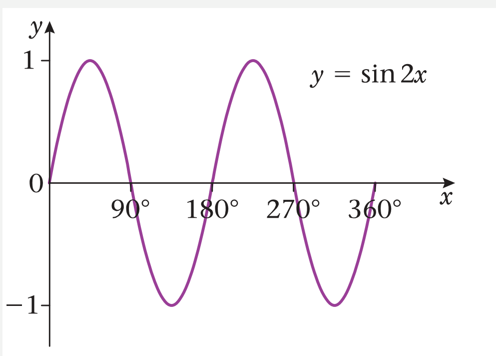
Sketch on separate axes the graphs of:

a $y = \sin 2x$, $0 \leqslant x \leqslant 360°$

b $y = \cos\frac{\theta}{3}$, $-3\pi \leqslant \theta \leqslant 3\pi$

c $y = \tan(-x)$, $-360° \leqslant x \leqslant 360°$

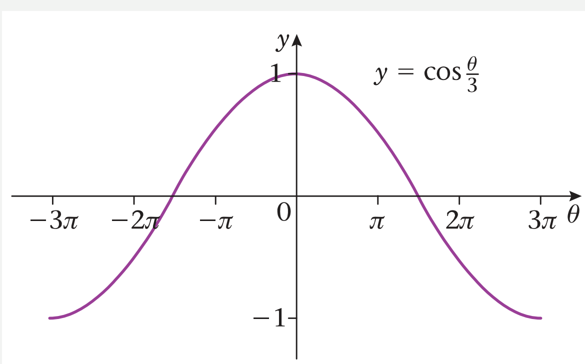
a



The graph of $y = \sin x$ is stretched horizontally with scale factor $\frac{1}{2}$.

The period is now 180° and two complete ‘waves’ are seen in the interval $0 \leqslant x \leqslant 360°$.

b

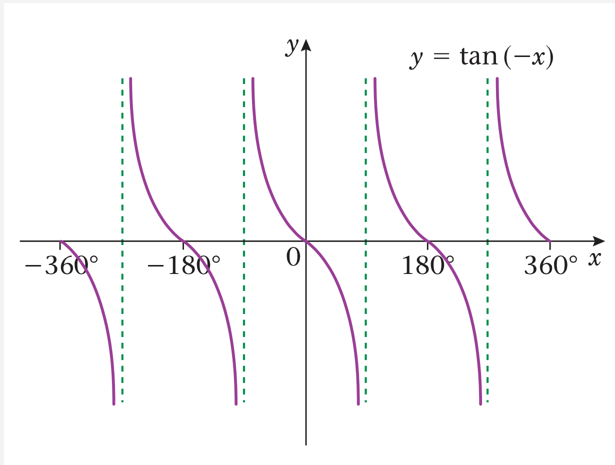


The graph of $y = \cos\theta$ is stretched horizontally with scale factor 3.

The period of $\cos\frac{\theta}{3}$ is 6π and only one complete wave is seen in $-3\pi \leqslant \theta \leqslant 3\pi$.

The curve crosses the θ-axis at $\theta = \pm\frac{3}{2}\pi$.

c



The graph of $y = \tan x$ is reflected in the y-axis.

Exercise 8F

1 Write down **i** the maximum value, and **ii** the minimum value, of the following expressions, and in each case give the smallest positive (or zero) value of x for which it occurs.

a $\cos x°$ **b** $4\sin x°$ **c** $\cos(-x)°$

d $3 + \sin x°$ **e** $-\sin x°$ **f** $\sin 3x°$

2 Sketch, on the same set of axes, in the interval $0 \leqslant \theta \leqslant 360°$, the graphs of $\cos\theta$ and $\cos 3\theta$.

3 Sketch, on separate axes, the graphs of the following, in the interval $0 \leqslant \theta \leqslant 360°$. Give the coordinates of points of intersection with the axes, and of maximum and minimum points where appropriate.

a $y = -\cos\theta$ **b** $y = \frac{1}{3}\sin\theta$

c $y = \sin\frac{1}{3}\theta$ **d** $y = \tan(\theta - 45°)$

4 Sketch, on separate axes, the graphs of the following, in the interval $-180 \leqslant \theta \leqslant 180$. Give the coordinates of points of intersection with the axes, and of maximum and minimum points where appropriate.

a $y = -2\sin\theta°$ **b** $y = \tan(\theta + 180)°$ **c** $y = \cos 4\theta°$ **d** $y = \sin(-\theta)°$

5 In this question θ is measured in radians. Sketch, on separate axes, the graphs of the following in the interval $-2\pi \leqslant \theta \leqslant 2\pi$. In each case give the periodicity of the function.

a $y = \sin\frac{1}{2}\theta$ **b** $y = -\frac{1}{2}\cos\theta$ **c** $y = \tan\left(\theta - \frac{\pi}{2}\right)$ **d** $y = \tan 2\theta$

6 **a** By considering the graphs of the functions, or otherwise, verify that:

i $\cos\theta = \cos(-\theta)$

ii $\sin\theta = -\sin(-\theta)$

iii $\sin(\theta - 90°) = -\cos\theta$

b Use the results in **a ii** and **iii** to show that $\sin(90° - \theta) = \cos\theta$.

c In Example 11 you saw that $\cos(\theta - 90°) = \sin\theta$.
Use this result with part **a i** to show that $\cos(90° - \theta) = \sin\theta$.

Mixed exercise 8G

1 Write each of the following as a trigonometric ratio of an acute angle:

a $\cos 237°$ **b** $\sin 312°$ **c** $\tan 190°$ **d** $\sin 2.3^c$ **e** $\cos\left(-\frac{\pi}{15}\right)$

2 Without using your calculator, work out the values of:

a $\cos 270°$ **b** $\sin 225°$ **c** $\cos 180°$ **d** $\tan 240°$ **e** $\tan 135°$

f $\cos 690°$ **g** $\sin \frac{5\pi}{3}$ **h** $\cos\left(-\frac{2\pi}{3}\right)$ **i** $\tan 2\pi$ **j** $\sin\left(-\frac{7\pi}{6}\right)$

3 Describe geometrically the transformations which map:

a The graph of $y = \tan x°$ onto the graph of $\tan \frac{1}{2}x°$.

b The graph of $y = \tan \frac{1}{2}x°$ onto the graph of $3 + \tan \frac{1}{2}x°$.

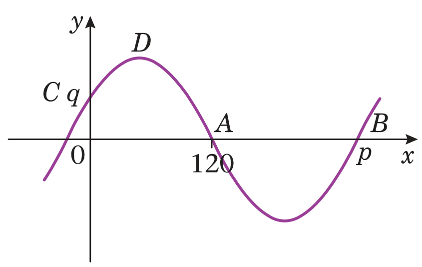
c The graph of $y = \cos x°$ onto the graph of $-\cos x°$.

d The graph of $y = \sin (x - 10)°$ onto the graph of $\sin (x + 10)°$.

4 **a** Sketch on the same set of axes, in the interval $0 \leqslant x \leqslant \pi$, the graphs of $y = \tan (x - \frac{1}{4}\pi)$ and $y = -2\cos x$, showing the coordinates of points of intersection with the axes.

b Deduce the number of solutions of the equation $\tan (x - \frac{1}{4}\pi) + 2\cos x = 0$, in the interval $0 \leqslant x \leqslant \pi$.

5 The diagram shows part of the graph of $y = \text{f}(x)$. It crosses the x-axis at $A(120, 0)$ and $B(p, 0)$. It crosses the y-axis at $C(0, q)$ and has a maximum value at D, as shown.



Given that $\text{f}(x) = \sin (x + k)°$, where $k > 0$, write down:

a the value of p

b the coordinates of D

c the smallest value of k

d the value of q

6 Consider the function $\text{f}(x) = \sin px$, $p \in \mathbb{R}$, $0 \leqslant x \leqslant 2\pi$.

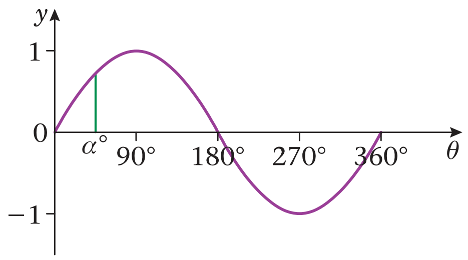
The closest point to the origin that the graph of $\text{f}(x)$ crosses the x-axis has x-coordinate $\frac{\pi}{5}$.

a Sketch the graph of $\text{f}(x)$.

b Write down the period of $\text{f}(x)$.

c Find the value of p.

7 The graph below shows $y = \sin\theta$, $0 \leqslant \theta \leqslant 360°$, with one value of θ ($\theta = \alpha°$) marked on the axis.



a Copy the graph and mark on the θ-axis the positions of $(180 - \alpha)°$, $(180 + \alpha)°$, and $(360 - \alpha)°$.

b Establish the result $\sin\alpha° = \sin(180 - \alpha)° = -\sin(180 + \alpha)° = -\sin(360 - \alpha)°$.

8 **a** Sketch on separate axes the graphs of $y = \cos\theta$ $(0 \leqslant \theta \leqslant 360°)$ and $y = \tan\theta$ $(0 \leqslant \theta \leqslant 360°)$, and on each θ-axis mark the point $(\alpha°, 0)$ as in question **7**.

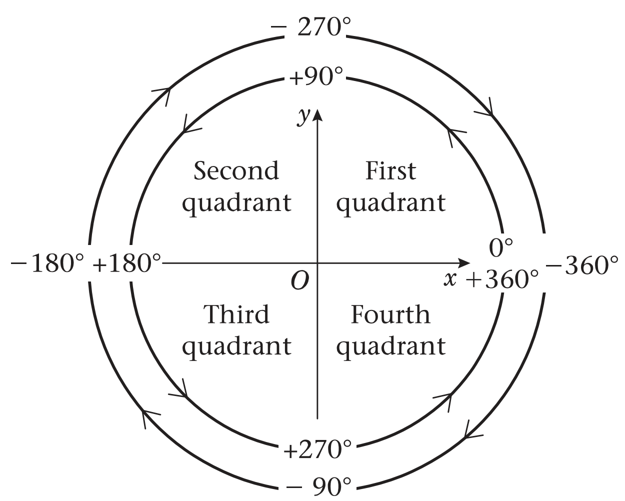
b Verify that:

i $\cos\alpha° = -\cos(180 - \alpha)° = -\cos(180 + \alpha)° = \cos(360 - \alpha)°$.

ii $\tan\alpha° = -\tan(180 - \alpha)° = -\tan(180 + \alpha)° = -\tan(360 - \alpha)°$.

Summary of key points

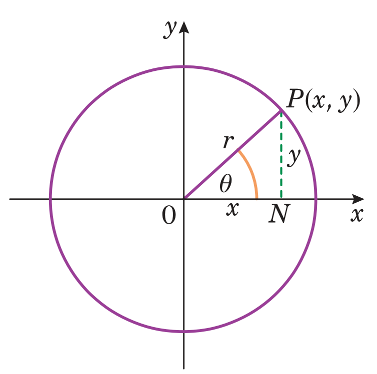
1 The x–y plane is divided into quadrants:



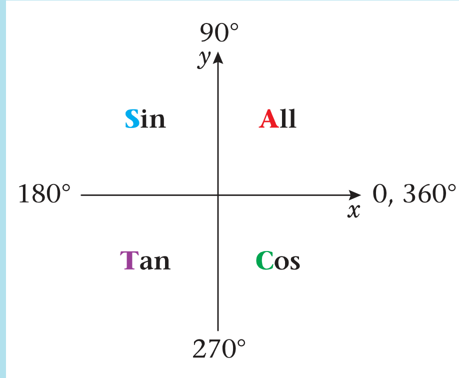
2 For all values of θ, the definitions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are taken to be

$$\sin\theta = \frac{y}{r} \quad \cos\theta = \frac{x}{r} \quad \tan\theta = \frac{y}{x}$$

where x and y are the coordinates of P and r is the radius of the circle.

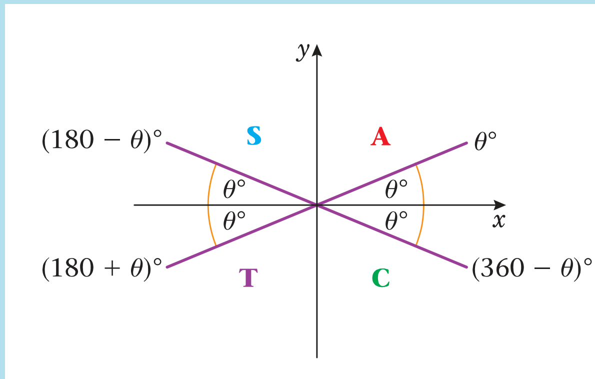


3 In the first quadrant, where θ is acute, **A**ll trigonometric functions are positive.
In the second quadrant, where θ is obtuse, only **S**ine is positive.
In the third quadrant, where θ is reflex, $180° < \theta < 270°$, only **T**angent is positive.
In the fourth quadrant, where θ is reflex, $270° < \theta < 360°$, only **C**osine is positive.



4 The trigonometric ratios of angles equally inclined to the horizontal are related:

$\sin(180-\theta)° = \sin\theta°$
$\sin(180+\theta)° = -\sin\theta°$
$\sin(360-\theta)° = -\sin\theta°$
$\cos(180-\theta)° = -\cos\theta°$
$\cos(180+\theta)° = -\cos\theta°$
$\cos(360-\theta)° = \cos\theta°$
$\tan(180-\theta)° = -\tan\theta°$
$\tan(180+\theta)° = \tan\theta°$
$\tan(360-\theta)° = -\tan\theta°$



5 The trigonometric ratios of 30°, 45° and 60° have exact forms, given below:

$\sin 30° = \dfrac{1}{2}$ $\cos 30° = \dfrac{\sqrt{3}}{2}$ $\tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$\sin 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ $\cos 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ $\tan 45° = 1$

$\sin 60° = \dfrac{\sqrt{3}}{2}$ $\cos 60° = \dfrac{1}{2}$ $\tan 60° = \sqrt{3}$

6 The sine and cosine functions have a period of 360°, (or 2π radians). Periodic properties are

$$\sin(\theta \pm 360°) = \sin\theta \quad \text{and} \quad \cos(\theta \pm 360°) = \cos\theta$$

respectively.

7 The tangent function has a period of 180°, (or π radians).

Periodic property is $\tan(\theta \pm 180°) = \tan\theta$.

8 Other useful properties are

$$\sin(-\theta) = -\sin\theta;\ \cos(-\theta) = \cos\theta;\ \tan(-\theta) = -\tan\theta;$$
$$\sin(90° - \theta) = \cos\theta;\ \cos(90° - \theta) = \sin\theta$$