After completing this chapter you should be able to

1 know the difference between an increasing and decreasing function
2 know how to find a stationary point
3 know how to distinguish between a maximum, a minimum and a point of inflexion
4 apply your knowledge of turning points to solve problems.

9 Differentiation

Did you know?

...There are countless real life uses of differentiation, some of which you will start to discover in this chapter.

For example if you want to manufacture tins of paint to hold 2.5 litres, it seems sensible that you minimise their surface area to enable you to reduce costs.

Section 9.3 will give you answers as to how to do this.

Why do you think manufacturers don't make tins to these dimensions?

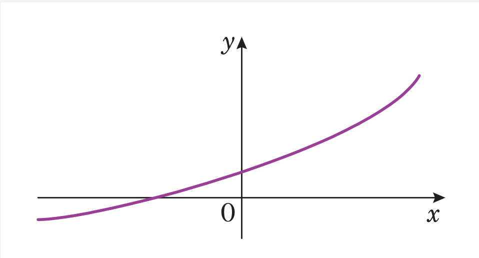
Try to make a cardboard copy of the tins with the minimum surface area.

Can you suggest a reason why it is not manufactured?

9.1 You need to know the difference between increasing and decreasing functions.

A function f which increases as x increases in the interval from $x = a$ to $x = b$ is an increasing function in the interval (a, b).

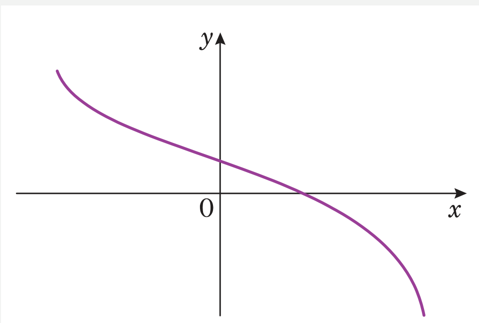
■ **For an increasing function in the interval (a, b), if x_1 and x_2 are two values of x in the interval $a \leqslant x \leqslant b$ and if $x_1 < x_2$ then $f(x_1) < f(x_2)$.**
It follows that $f'(x) > 0$ in the interval $a \leqslant x \leqslant b$.



This is a graph of an increasing function. The gradient is positive at all points on the graph.

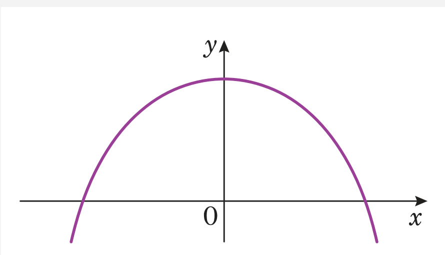
A function f which decreases as x increases in the interval from $x = a$ to $x = b$ is a decreasing function in the interval (a, b).

■ **For a decreasing function in the interval (a, b), if x_1 and x_2 are two values of x in the interval $a \leqslant x \leqslant b$ and if $x_1 < x_2$ then $f(x_1) > f(x_2)$.**
It follows that $f'(x) < 0$ in the interval $a \leqslant x \leqslant b$.



This is a graph of a decreasing function. The gradient is negative at all points on the graph.

Some functions increase as x increases in one interval and decrease as x increases in another interval.



This is the graph of a function which is increasing for $x < 0$, and is decreasing for $x > 0$.
At $x = 0$ the gradient is zero and the function is said to be stationary.

Example 1

Show that the function $f(x) = x^3 + 24x + 3$ $(x \in \mathbb{R})$ is an increasing function.

$f(x) = x^3 + 24x + 3$

$f'(x) = 3x^2 + 24$

As $x^2 \geqslant 0$ for all real x

$3x^2 + 24 > 0$

So $f(x)$ is an increasing function.

First differentiate to obtain the gradient function.

As $3x^2 > 0$, for real x, and $24 > 0$ then $f'(x) > 0$ for all values of x. So the curve always has a positive gradient.

Example 2

Find the values of x for which the function $f(x) = x^3 + 3x^2 - 9x$ is a decreasing function.

$f(x) = x^3 + 3x^2 - 9x$

$f'(x) = 3x^2 + 6x - 9$

If $f'(x) < 0 \Rightarrow 3x^2 + 6x - 9 < 0$

So $3(x^2 + 2x - 3) < 0$

$3(x + 3)(x - 1) < 0$

So $-3 < x < 1$

Find $f'(x)$ and put this expression <0.

Solve the inequality by factorisation, and by considering the three regions $x < -3$, $-3 < x < 1$ and $x > 1$, looking for sign changes.

State the answer.

Exercise 9A

1 Find the values of x for which $f(x)$ is an increasing function, given that $f(x)$ equals:

a $3x^2 + 8x + 2$ **b** $4x - 3x^2$

c $5 - 8x - 2x^2$ **d** $2x^3 - 15x^2 + 36x$

e $3 + 3x - 3x^2 + x^3$ **f** $5x^3 + 12x$

g $x^4 + 2x^2$ **h** $x^4 - 8x^3$

2 Find the values of x for which $f(x)$ is a decreasing function, given that $f(x)$ equals:

a $x^2 - 9x$ **b** $5x - x^2$

c $4 - 2x - x^2$ **d** $2x^3 - 3x^2 - 12x$

e $1 - 27x + x^3$ **f** $x + \dfrac{25}{x}$

g $x^{\frac{1}{2}} + 9x^{-\frac{1}{2}}$ **h** $x^2(x + 3)$

9.2 You need to be able to find the coordinates of a stationary point on a curve and work out whether it is a minimum point, a maximum point or a point of inflexion.

For certain curves, the function f(x) is increasing in some intervals and decreasing in others.

- **The points where f(x) stops increasing and begins to decrease are called maximum points.**

- **The points where f(x) stops decreasing and begins to increase are called minimum points.**

These points are collectively called turning points and at these points $f'(x) = 0$. You can find the coordinates of maximum and minimum points on a curve. This will help you to sketch curves accurately.

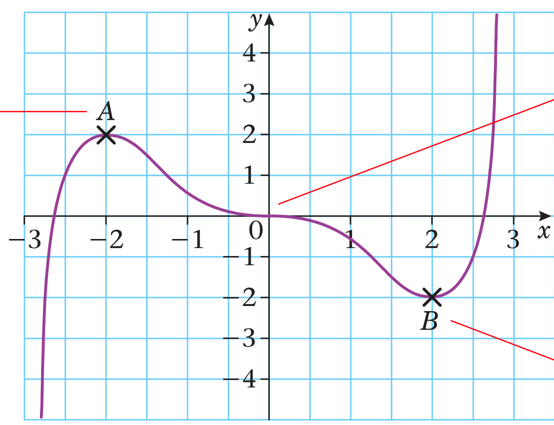
- **A point of inflexion is a point where the gradient is at a maximum or minimum value in the neighbourhood of the point.**

Hint: Points of inflexion where the gradient is not zero are not included in the C2 specification.

Some points of inflexion do not have zero gradient.

- **Points of zero gradient are called stationary points and stationary points may be maximum points, minimum points or points of inflexion.**

Hint: A is a local maximum point.



Hint: The origin is a point of inflexion on this curve.

Hint: B is a local minimum point.

- **To find the coordinates of a stationary point:**

1. **Find $\frac{dy}{dx}$, i.e. $f'(x)$, and solve the equation $f'(x) = 0$ to find the value, or values, of x.**
2. **Substitute the value(s) of x which you have found into the equation $y = f(x)$ to find the corresponding value(s) of y.**
3. **This gives the coordinates of any stationary points.**

Example 3

Find the coordinates of the turning point on the curve with equation $y = x^4 - 32x$. Establish whether it is a maximum or a minimum point or a point of inflexion by considering points either side of the turning points.

$y = x^4 - 32x$

$\frac{dy}{dx} = 4x^3 - 32$

Put $\frac{dy}{dx} = 0$

Differentiate and put $\frac{dy}{dx} = 0$.

Then $4x^3 - 32 = 0$

$4x^3 = 32$

$x^3 = 8$

$x = 2$

Solve the equation to find the value of x.

So $y = 2^4 - 32 \times 2$

$= 16 - 64$

$= -48$

Substitute the value of x into the original equation to find the value of y.

So $(2, -48)$ is a stationary point

Now consider the gradient on either side of $(2, -48)$:

Value of x	$x < 2$ e.g. $x = 1.9$	$x = 2$	$x > 2$ e.g. $x = 2.1$
Gradient	e.g. -4.56 which is $-$ve	0	e.g. 5.04 which is $+$ve
Shape of curve			

Make a table where you consider a value of x slightly less than 2 and a value slightly greater than 2.

Calculate the gradient for each of these points close to the stationary point.

Deduce the shape of the curve.

From the shape of the curve, the point $(2, -48)$ is a minimum point.

■ **You can also find out whether stationary points are maximum points, minimum points or points of inflexion by finding the value of $\frac{d^2y}{dx^2}$ and, where necessary, $\frac{d^3y}{dx^3}$ at the stationary point. This is because $\frac{d^2y}{dx^2}$ measures the change in gradient.**

- **If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$, the point is a minimum point.**

Hint: $\frac{d^2y}{dx^2}$ is the second derivative of y with respect to x. You find $\frac{d^2y}{dx^2}$ by differentiating $\frac{dy}{dx}$ with respect to x.

- **If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$, the point is a maximum point.**
- **If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, the point is either a maximum or a minimum point or a point of inflexion.**

Hint: In this case, you need to use the tabular method and consider the gradient on either side of the stationary point.

- **If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, but $\frac{d^3y}{dx^3} \neq 0$, then the point is a point of inflexion.**

Hint: $\frac{d^3y}{dx^3}$ is the third derivative of y with respect to x. You find $\frac{d^3y}{dx^3}$ by differentiating $\frac{d^2y}{dx^2}$ with respect to x.

You may also see this notation: $f''(x)$ is the second derivative of the function f with respect to x. $f'''(x)$ is the third derivative and so on.

Example 4

Find the stationary points on the curve with equation $y = 2x^3 - 15x^2 + 24x + 6$ and determine, by finding the second derivative, whether the stationary points are maximum, minimum or points of inflexion.

$y = 2x^3 - 15x^2 + 24x + 6$

$\frac{dy}{dx} = 6x^2 - 30x + 24$

Differentiate and put the derivative equal to zero.

Putting $6x^2 - 30x + 24 = 0$

$6(x - 4)(x - 1) = 0$

So $x = 4$ or $x = 1$

Solve the equation to obtain the values of x for the stationary points.

When $x = 1$,

$y = 2 - 15 + 24 + 6 = 17$

When $x = 4$,

$y = 2 \times 64 - 15 \times 16 + 24 \times 4 + 6 = -10$

So the stationary points are at (1, 17) and (4, −10)

Substitute $x = 4$ and $x = 1$ into the original equation of the curve to obtain the values of y which correspond to these values.

$\frac{d^2y}{dx^2} = 12x - 30$

Differentiate again to obtain the second derivative.

When $x = 1$, $\frac{d^2y}{dx^2} = -18$ which is <0

So (1, 17) is a maximum point.

When $x = 4$, $\frac{d^2y}{dx^2} = 18$ which is >0

So (4, 1) is a minimum point

Substitute $x = 1$ and $x = 4$ into the second derivative expression. If the second derivative is negative then the point is a maximum point, whereas if it is positive then the point is a minimum point.

Hint: You may be told whether a stationary value is a maximum or a minimum, in which case it will not be necessary for you to check by using the second derivative, or by considering the gradient on each side of the stationary value.

Example 5

Find the greatest value of $6x - x^2$. State the range of the function $f(x) = 6x - x^2$.

Let $y = 6x - x^2$

Then $\frac{dy}{dx} = 6 - 2x$

Put $\frac{dy}{dx} = 0$, then $x = 3$

So $y = 18 - 3^2 = 9$

The greatest value of this quadratic function is 9 and the range is given by $f(x) \leqslant 9$

This question may be done by completing the square, but calculus is a good alternative.

There was only one turning point on this parabola and the question said that there was a greatest value, so you did not need to make a check.

Put the value of x into the original equation. The greatest value is the value of y at the stationary point.

The range of the function is the set of values which y can take.

Exercise 9B

1 Find the least value of each of the following functions:

a $f(x) = x^2 - 12x + 8$ **b** $f(x) = x^2 - 8x - 1$ **c** $f(x) = 5x^2 + 2x$

2 Find the greatest value of each of the following functions:

a $f(x) = 10 - 5x^2$ **b** $f(x) = 3 + 2x - x^2$ **c** $f(x) = (6 + x)(1 - x)$

3 Find the coordinates of the points where the gradient is zero on the curves with the given equations. Establish whether these points are maximum points, minimum points or points of inflexion, by considering the second derivative in each case.

a $y = 4x^2 + 6x$ **b** $y = 9 + x - x^2$ **c** $y = x^3 - x^2 - x + 1$

d $y = x(x^2 - 4x - 3)$ **e** $y = x + \frac{1}{x}$ **f** $y = x^2 + \frac{54}{x}$

g $y = x - 3\sqrt{x}$ **h** $y = x^{\frac{1}{2}}(x - 6)$ **i** $y = x^4 - 12x^2$

4 Sketch the curves with equations given in question **3** parts **a**, **b**, **c** and **d**, labelling any stationary values.

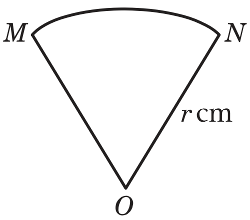
5 By considering the gradient on either side of the stationary point on the curve $y = x^3 - 3x^2 + 3x$, show that this point is a point of inflexion.
Sketch the curve $y = x^3 - 3x^2 + 3x$.

6 Find the maximum value and hence the range of values for the function $f(x) = 27 - 2x^4$.

9.3 You need to be able to apply what you have learned about turning points to solve problems.

Example 6

The diagram shows a minor sector OMN of a circle with centre O and radius r cm. The perimeter of the sector is 100 cm and the area of the sector is A cm^2.



a Show that $A = 50r - r^2$.

Given that r varies, find:

b The value of r for which A is a maximum and show that A is a maximum.

c The value of $\angle MON$ for this maximum area.

d The maximum area of the sector OMN.

a Let the perimeter of the sector be P, so

$P = 2r + r\theta$

This is the sum of two radii ($2r$) and an arc length MN ($r\theta$).

Rearrange and substitute $P = 100$ to give

$$\theta = \frac{100 - 2r}{r} \quad (1)$$

The area of the sector, $A = \frac{1}{2}r^2\theta \quad (2)$

Substitute (1) in (2)

The area formula is in terms of two variables r and θ, so you need to substitute for θ so that the formula is in terms of one variable r.

$$A = \frac{1}{2}r^2\left(\frac{100 - 2r}{r}\right)$$

So $A = 50r - r^2$

b $\frac{dA}{dr} = 50 - 2r$

When $\frac{dA}{dr} = 0$, $r = 25$

Also $\frac{d^2A}{dr^2} = -2$, which is negative

So the area is a maximum when $r = 25$.

Use the method which you learned to find stationary values: put the first derivative equal to zero, then check the sign of the second derivative.

c Substitute $r = 25$ into (1)

$$\theta = \frac{100 - 50}{25} = 2$$

So angle $MON = 2$ radians

Answer the final two parts of the question by using the appropriate equations and give the units in your answer.

d The maximum value of the area is

$50 \times 25 - 25^2 = 625$ cm^2

Use $A = 50r - r^2$.

Example 7

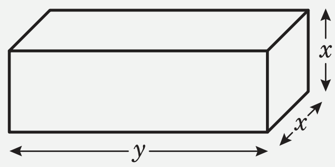
A large tank in the shape of a cuboid is to be made from 54 m^2 of sheet metal. The tank has a horizontal base and no top. The height of the tank is x metres. Two of the opposite vertical faces are squares.

a Show that the volume, V m^3, of the tank is given by $V = 18x - \frac{2}{3}x^3$.

b Given that x can vary, use differentiation to find the maximum or minimum value of V.

c Justify that the value of V you have found is a maximum.

a Let the length of the tank be y metres.



Draw a sketch.

Total area, $A = 2x^2 + 3xy$

So $54 = 2x^2 + 3xy$

$y = \frac{54 - 2x^2}{3x}$

Rearrange to find y in terms of x.

But $V = x^2y$

So $V = x^2\left(\frac{54 - 2x^2}{3x}\right)$

$= \frac{x}{3}(54 - 2x^2)$

Substitute into the equation the expression for y.

So $V = 18x - \frac{2}{3}x^3$

Simplify.

b So $\frac{dV}{dx} = 18 - 2x^2$

Put $\frac{dV}{dx} = 0$

Differentiate V with respect to x and put $\frac{dV}{dx} = 0$.

$0 = 18 - 2x^2$

So $x^2 = 9$

$x = -3$ or 3

Rearrange to find x.

But x is a length so $x = 3$

When $x = 3$, $V = 18 \times 3 - \frac{2}{3} \times 3^3$

Substitute value of x into expression for V.

$= 54 - 18$

$= 36$

$V = 36$ is a maximum or minimum value of V.

c $\dfrac{d^2V}{dx^2} = -4x$ — Find the second derivative of V.

When $x = 3$, $\dfrac{d^2V}{dx^2} = -4 \times 3 = -12$

This is negative, so $V = 36$ is the maximum value of V.

Exercise 9C

1 A rectangular garden is fenced on three sides, and the house forms the fourth side of the rectangle.

Given that the total length of the fence is 80 m show that the area, A, of the garden is given by the formula $A = y(80 - 2y)$, where y is the distance from the house to the end of the garden.

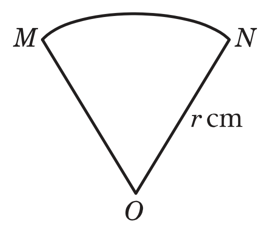
Given that the area is a maximum for this length of fence, find the dimensions of the enclosed garden, and the area which is enclosed.

2 A closed cylinder has total surface area equal to 600π. Show that the volume, V cm^3, of this cylinder is given by the formula $V = 300\pi r - \pi r^3$, where r cm is the radius of the cylinder.

Find the maximum volume of such a cylinder.

3 A sector of a circle has area 100 cm^2. Show that the perimeter of this sector is given by the formula $P = 2r + \dfrac{200}{r}$, $r > \sqrt{\dfrac{100}{\pi}}$.

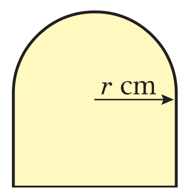
Find the minimum value for the perimeter of such a sector.



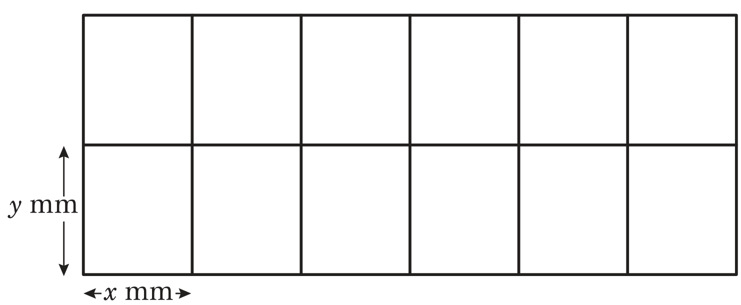
4 A shape consists of a rectangular base with a semicircular top, as shown. Given that the perimeter of the shape is 40 cm, show that its area, A cm^2, is given by the formula

$$A = 40r - 2r^2 - \frac{\pi r^2}{2}$$

where r cm is the radius of the semicircle. Find the maximum value for this area.



5 The shape shown is a wire frame in the form of a large rectangle split by parallel lengths of wire into 12 smaller equal-sized rectangles.



Given that the total length of wire used to complete the whole frame is 1512 mm, show that the area of the whole shape is A mm^2, where $A = 1296x - \dfrac{108x^2}{7}$, where x mm is the width of one of the smaller rectangles.

Find the maximum area which can be enclosed in this way.

Mixed exercise 9D

1 Given that: $y = x^{\frac{3}{2}} + \frac{48}{x}$ $(x > 0)$

a Find the value of x and the value of y when $\frac{dy}{dx} = 0$.

b Show that the value of y which you found in **a** is a minimum.

2 A curve has equation $y = x^3 - 5x^2 + 7x - 14$. Determine, by calculation, the coordinates of the stationary points of the curve C.

3 The function f, defined for $x \in R$, $x > 0$, is such that:

$$f'(x) = x^2 - 2 + \frac{1}{x^2}$$

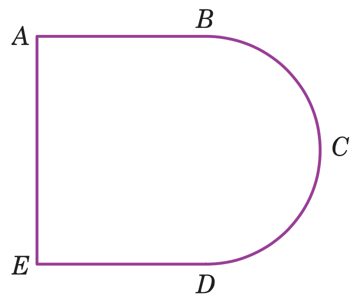
a Find the value of $f''(x)$ at $x = 4$.

b Given that $f(3) = 0$, find $f(x)$.

c Prove that f is an increasing function.

4 A curve has equation $y = x^3 - 6x^2 + 9x$.
Find the coordinates of its maximum turning point.

5 A wire is bent into the plane shape $ABCDEA$ as shown. Shape $ABDE$ is a rectangle and BCD is a semicircle with diameter BD. The area of the region enclosed by the wire is $R\,\text{m}^2$, $AE = x$ metres, $AB = ED = y$ metres. The total length of the wire is 2 m.



a Find an expression for y in terms of x.

b Prove that $R = \frac{x}{8}(8 - 4x - \pi x)$

Given that x can vary, using calculus and showing your working,

c find the maximum value of R. (You do not have to prove that the value you obtain is a maximum.)

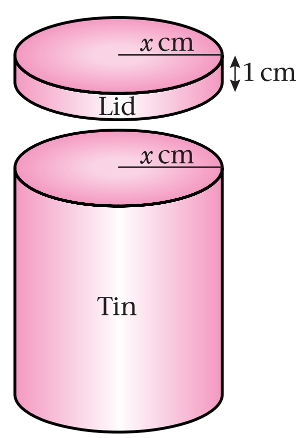
6 The fixed point A has coordinates $(8, -6, 5)$ and the variable point P has coordinates $(t, t, 2t)$.

a Show that $AP^2 = 6t^2 - 24t + 125$.

b Hence find the value of t for which the distance AP is least.

c Determine this least distance.

7 A cylindrical biscuit tin has a close-fitting lid which overlaps the tin by 1 cm, as shown. The radii of the tin and the lid are both x cm. The tin and the lid are made from a thin sheet of metal of area $80\pi\,\text{cm}^2$ and there is no wastage. The volume of the tine is $V\,\text{cm}^3$.



a Show that $V = \pi(40x - x^2 - x^3)$.

Given that x can vary:

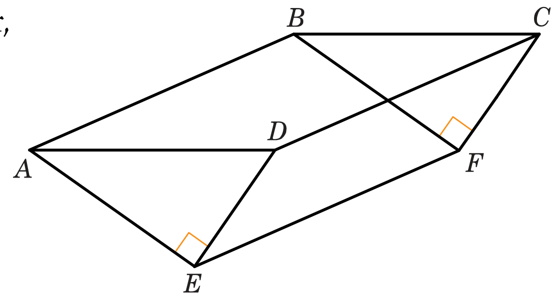
b use differentiation to find the positive value of x for which V is stationary.

c Prove that this value of x gives a maximum value of V.

d Find this maximum value of V.

e Determine the percentage of the sheet metal used in the lid when V is a maximum.

8 The diagram shows an open tank for storing water, *ABCDEF*. The sides *ABFE* and *CDEF* are rectangles. The triangular ends *ADE* and *BCF* are isosceles, and $\angle AED = \angle BFC = 90°$. The ends *ADE* and *BCF* are vertical and *EF* is horizontal.



Given that $AD = x$ metres:

a show that the area of triangle *ADE* is $\frac{1}{4}x^2$ m^2.

Given also that the capacity of the container is 4000 m^3 and that the total area of the two triangular and two rectangular sides of the container is S m^2:

b show that $S = \dfrac{x^2}{2} + \dfrac{16\,000\sqrt{2}}{x}$.

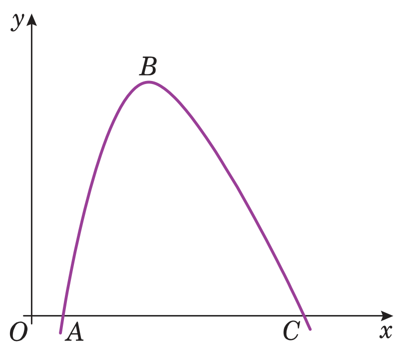
Given that x can vary:

c use calculus to find the minimum value of S.

d Justify that the value of S you have found is a minimum. *E*

9 The diagram shows part of the curve with equation $y = f(x)$, where:

$$f(x) \equiv 200 - \frac{250}{x} - x^2, \; x > 0$$

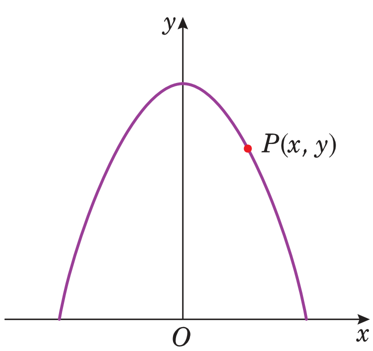


The curve cuts the x-axis at the points A and C. The point B is the maximum point of the curve.

a Find $f'(x)$.

b Use your answer to part **a** to calculate the coordinates of B. *E*

10 The diagram shows the part of the curve with equation $y = 5 - \frac{1}{2}x^2$ for which $y \geqslant 0$.
The point $P(x, y)$ lies on the curve and O is the origin.

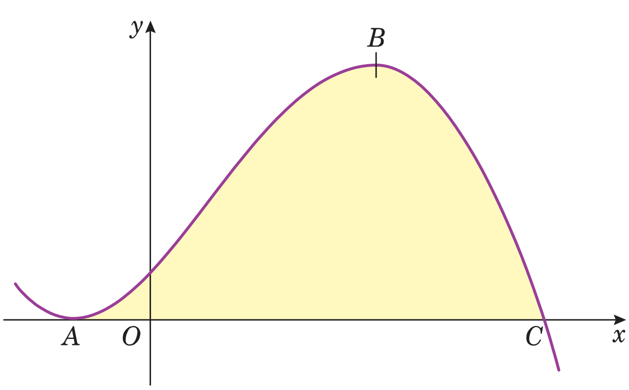


a Show that $OP^2 = \frac{1}{4}x^4 - 4x^2 + 25$.
Taking $f(x) \equiv \frac{1}{4}x^4 - 4x^2 + 25$:

b find the values of x for which $f'(x) = 0$.

c Hence, or otherwise, find the minimum distance from O to the curve, showing that your answer is a minimum. *E*

11 The diagram shows part of the curve with equation $y = 3 + 5x + x^2 - x^3$. The curve touches the x-axis at A and crosses the x-axis at C. The points A and B are stationary points on the curve.



a Show that C has coordinates (3, 0).

b Using calculus and showing all your working, find the coordinates of A and B.

Summary of key points

1 For an increasing function $f(x)$ in the interval (a, b), $f'(x) > 0$ in the interval $a \leqslant x \leqslant b$.

2 For a decreasing function $f(x)$ in the interval (a, b), $f'(x) < 0$ in the interval $a \leqslant x \leqslant b$.

3 The points where $f(x)$ stops increasing and begins to decrease are called maximum points.

4 The points where $f(x)$ stops decreasing and begins to increase are called minimum points.

5 A point of inflexion is a point where the gradient is at a maximum or minimum value in the neighbourhood of the point.

6 A stationary point is a point of zero gradient. It may be a maximum, a minimum or a point of inflexion.

7 To find the coordinates of a stationary point find $\frac{dy}{dx}$, i.e. $f'(x)$, and solve the equation $f'(x) = 0$ to find the value, or values, of x and then substitute into $y = f(x)$ to find the corresponding values of y.

8 The stationary value of a function is the value of y at the stationary point. You can sometimes use this to find the range of a function.

9 You may determine the nature of a stationary point by using the second derivative.

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$, the point is a minimum point.

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$, the point is a maximum point.

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, the point is either a maximum or a minimum point or a point of inflexion.

If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, but $\frac{d^3y}{dx^3} \neq 0$, then the point is a point of inflexion.

Hint: In this case you need to use the tabular method and consider the gradient on either side of the stationary point.

10 In problems where you need to find the maximum or minimum value of a variable y, first establish a formula for y in terms of x, then differentiate and put the derived function equal to zero to find x and then y.