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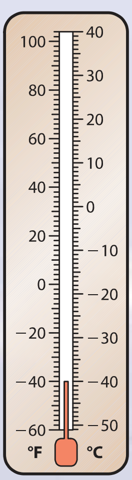
After completing this chapter you should be able to

1 solve simultaneous equations by elimination
2 solve simultaneous equations by substitution
3 solve linear and quadratic inequalities.

You will meet simultaneous equations on many occasions during the A level Mathematics course. In Core 1 you will use them to find where lines intersect. You will also use them to solve problems in sequences and series.

3 Equations and inequalities



Did you know?

...that there is only one value at which the Fahrenheit and Celsius temperatures are the same?

Substitute F = C into F = 1.8C + 32. Solve the resulting linear equation. You should find that

$$C = F = -40$$

The temperature in the Arctic can reach **−40°F** which is identical to **−40°C**. So if you meet a polar bear and he asks you how cold it is you don't have to worry about units!

3.1 You can solve simultaneous linear equations by elimination.

Example 1

Solve the equations:

a $2x + 3y = 8$
$3x - y = 23$

b $4x - 5y = 4$
$6x + 2y = 25$

a $2x + 3y = 8$
$9x - 3y = 69$
$11x = 77$
$x = 7$

$14 + 3y = 8$
$3y = 8 - 14$
$y = -2$
So solution is $x = 7$, $y = -2$

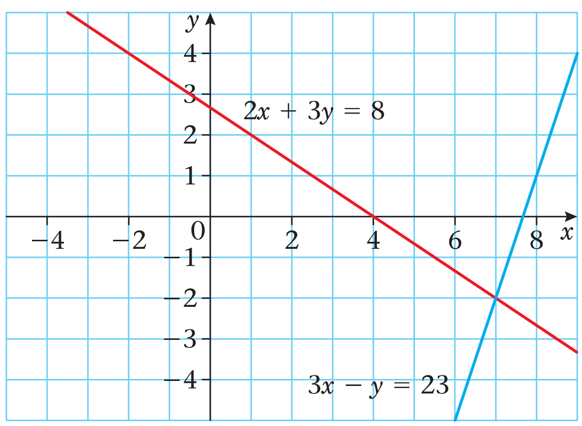
First look for a way to eliminate x or y.

Multiply the 2nd equation by 3 to get $3y$ in each equation.

Then add, since the $3y$ terms have different signs and y will be eliminated.

Use $x = 7$ in the first equation to find y.

You can consider the solution graphically.
The graph of each equation is a straight line.
The two straight lines intersect at $(7, -2)$.



b $12x - 15y = 12$
$12x + 4y = 50$
$-19y = -38$
$y = 2$

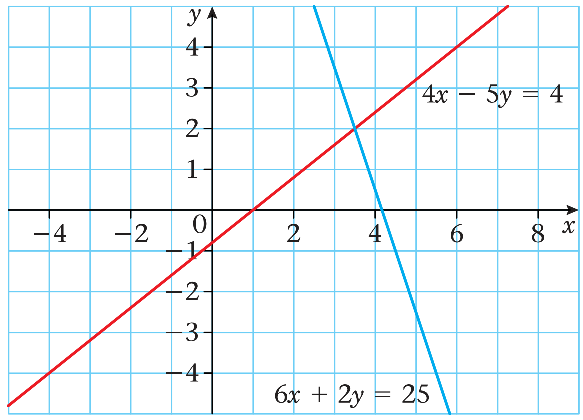
$4x - 10 = 4$
$4x = 14$
$x = 3\frac{1}{2}$
So solution is $x = 3\frac{1}{2}$, $y = 2$

Multiply the first equation by 3 and multiply the 2nd equation by 2 to get $12x$ in each equation.

Subtract, since the $12x$ terms have the same sign (both positive).

Use $y = 2$ in the first equation to find the value of x.

Graphically, each equation is a straight line.
The two straight lines intersect at (3.5, 2).



Exercise 3A

Solve these simultaneous equations by elimination:

1 $2x - y = 6$
$4x + 3y = 22$

2 $7x + 3y = 16$
$2x + 9y = 29$

3 $5x + 2y = 6$
$3x - 10y = 26$

4 $2x - y = 12$
$6x + 2y = 21$

5 $3x - 2y = -6$
$6x + 3y = 2$

6 $3x + 8y = 33$
$6x = 3 + 5y$

3.2 You can solve simultaneous linear equations by substitution.

Example 2

Solve the equations:

$2x - y = 1$
$4x + 2y = -30$

$y = 2x - 1$	Rearrange an equation to get either $x = \ldots$ or $y = \ldots$ (here $y = \ldots$).
$4x + 2(2x - 1) = -30$	Substitute this into the other equation (here in place of y).
$4x + 4x - 2 = -30$	Solve for x.
$8x = -28$	
$x = -3\frac{1}{2}$	
$y = 2(-3\frac{1}{2}) - 1 = -8$	Substitute $x = -3\frac{1}{2}$ into $y = 2x - 1$ to find the value of y.
So solution is $x = -3\frac{1}{2}$, $y = -8$.	

Exercise 3B

Solve these simultaneous equations by substitution:

1 $x + 3y = 11$
$4x - 7y = 6$

2 $4x - 3y = 40$
$2x + y = 5$

3 $3x - y = 7$
$10x + 3y = -2$

4 $2y = 2x - 3$
$3y = x - 1$

3.3 You can use the substitution method to solve simultaneous equations where one equation is linear and the other is quadratic.

Example 3

Solve the equations:

a $x + 2y = 3$
$x^2 + 3xy = 10$

b $3x - 2y = 1$
$x^2 + y^2 = 25$

a $x = 3 - 2y$

Rearrange the linear equation to get $x = \ldots$ or $y = \ldots$ (here $x = \ldots$).

$(3 - 2y)^2 + 3y(3 - 2y) = 10$

Substitute this into the quadratic equation (here in place of x).
$(3 - 2y)^2$ means $(3 - 2y)(3 - 2y)$ (see Chapter 1).

$9 - 12y + 4y^2 + 9y - 6y^2 = 10$

$-2y^2 - 3y - 1 = 0$

$2y^2 + 3y + 1 = 0$

$(2y + 1)(y + 1) = 0$

Solve for y using factorisation.

$y = -\frac{1}{2}$ or $y = -1$

So $x = 4$ or $x = 5$

Find the corresponding x-values by substituting the y-values into $x = 3 - 2y$.

Solutions are $x = 4, y = -\frac{1}{2}$

and $x = 5, y = -1$

There are two solution pairs. The graph of the linear equation (straight line) would intersect the graph of the quadratic (curve) at two points.

b $3x - 2y = 1$

$2y = 3x - 1$

$y = \frac{3x - 1}{2}$

Find $y = \ldots$ from linear equation.

$x^2 + \left(\frac{3x - 1}{2}\right)^2 = 25$

Substitute $y = \dfrac{3x - 1}{2}$ into the quadratic equation to form an equation in x.

$x^2 + \left(\frac{9x^2 - 6x + 1}{4}\right) = 25$

Now multiply by 4.

$4x^2 + 9x^2 - 6x + 1 = 100$

$13x^2 - 6x - 99 = 0$

$(13x + 33)(x - 3) = 0$

$x = -\frac{33}{13}$ or $x = 3$

Solve for x.

$y = -\frac{56}{13}$ or $y = 4$

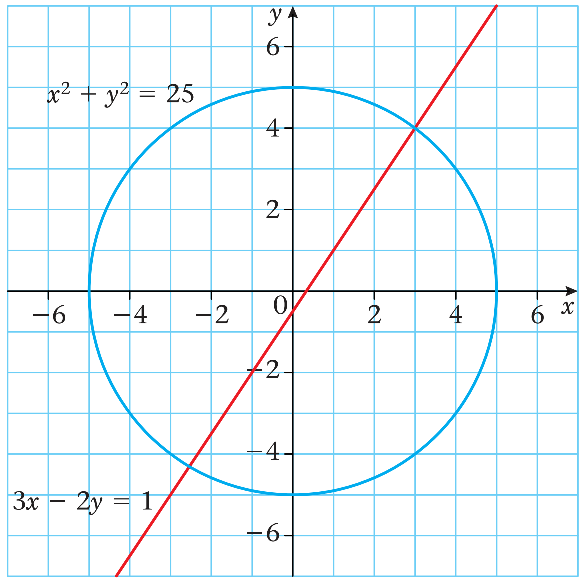
Substitute x-values into $y = \dfrac{3x - 1}{2}$.

Solutions are $x = 3, y = 4$

and $x = -\frac{33}{13}, y = -\frac{56}{13}$

Graphically, the linear equation (straight line) intersects the quadratic equation (curve) at two points.

(This curve is a circle. You will learn about its equation in Book C2.)



It is possible, of course, that a given straight line and a given curve do *not* intersect. In this case, the quadratic equation that has to be solved would have no real roots (in this case $b^2 - 4ac < 0$). (See Section 2.6.)

Exercise 3C

1 Solve the simultaneous equations:

a $x + y = 11$
$xy = 30$

b $2x + y = 1$
$x^2 + y^2 = 1$

c $y = 3x$
$2y^2 - xy = 15$

d $x + y = 9$
$x^2 - 3xy + 2y^2 = 0$

e $3a + b = 8$
$3a^2 + b^2 = 28$

f $2u + v = 7$
$uv = 6$

2 Find the coordinates of the points at which the line with equation $y = x - 4$ intersects the curve with equation $y^2 = 2x^2 - 17$.

3 Find the coordinates of the points at which the line with equation $y = 3x - 1$ intersects the curve with equation $y^2 - xy = 15$.

4 Solve the simultaneous equations:

a $3x + 2y = 7$
$x^2 + y = 8$

b $2x + 2y = 7$
$x^2 - 4y^2 = 8$

5 Solve the simultaneous equations, giving your answers in their simplest surd form:

a $x - y = 6$
$xy = 4$

b $2x + 3y = 13$
$x^2 + y^2 = 78$

3.4 You can solve linear inequalities using similar methods to those for solving linear equations.

You need to be careful when you multiply or divide an inequality by a negative number. You need to turn round the inequality sign:

$$5 > 2$$

Multiply by −2 $\quad -10 < -4$

■ **When you multiply or divide an inequality by a negative number, you need to change the inequality sign to its opposite.**

Example 4

Find the set of values of x for which:

a $2x - 5 < 7$

b $5x + 9 \geqslant x + 20$

c $12 - 3x < 27$

d $3(x - 5) > 5 - 2(x - 8)$

a $2x - 5 < 7$

$2x < 12$ — Add 5 to both sides.

$x < 6$ — Divide both sides by 2.

b $5x + 9 \geqslant x + 20$

$4x + 9 \geqslant 20$ — Subtract x from both sides.

$4x \geqslant 11$ — Subtract 9 from both sides.

$x \geqslant 2.75$ — Divide both sides by 4.

For **c**, two approaches are shown:

c $12 - 3x < 27$

$-3x < 15$ — Subtract 12 from both sides.

$x > -5$ — Divide both sides by -3. (You therefore need to turn round the inequality sign.)

$12 - 3x < 27$

$12 < 27 + 3x$ — Add $3x$ to both sides.

$-15 < 3x$ — Subtract 27 from both sides.

$-5 < x$ — Divide both sides by 3.

$x > -5$ — Rewrite with x on LHS.

d $3(x - 5) > 5 - 2(x - 8)$

$3x - 15 > 5 - 2x + 16$ — Multiply out (note: $-2 \times -8 = +16$).

$5x > 5 + 16 + 15$ — Add 15 to both sides.

$5x > 36$

$x > 7.2$ — Divide both sides by 5.

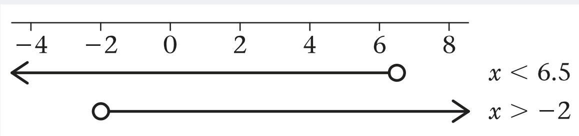
You may sometimes need to find the set of values of x for which two inequalities are true together. Number lines are helpful here.

Example 5

Find the set of values of x for which:

$$3x - 5 < x + 8 \text{ and } 5x > x - 8$$

$3x - 5 < x + 8$	$5x > x - 8$
$2x - 5 < 8$	$4x > -8$
$2x < 13$	$x > -2$
$x < 6.5$	



So the required set of values is $-2 < x < 6.5$.

Draw a number line to illustrate the two inequalities.

The 'hollow dots' at the end of each line show that the end value is not included in the set of values.

Show an included end value ($\leqslant$ or $\geqslant$) by using a 'solid dot' (●).

The two sets of values overlap (or intersect) where $-2 < x < 6.5$.

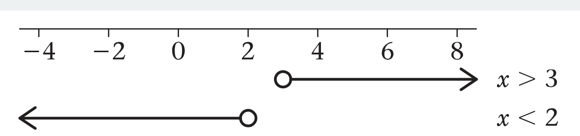
Notice here how this is written when x lies between two values.

Example 6

Find the set of values of x for which:

$$x - 5 > 1 - x \text{ and } 15 - 3x > 5 + 2x$$

$x - 5 > 1 - x$	$15 - 3x > 5 + 2x$
$2x - 5 > 1$	$10 - 3x > 2x$
$2x > 6$	$10 > 5x$
$x > 3$	$2 > x$
	$x < 2$



So there are no values of x for which both inequalities are true together.

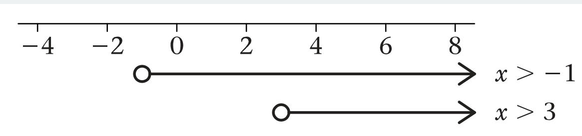
Draw a number line. Note that there is no overlap between the two sets of values.

Example 7

Find the set of values of x for which:

$4x + 7 > 3$ and $17 < 11 + 2x$

$4x + 7 > 3$	$17 < 11 + 2x$
$4x > -4$	$17 - 11 < 2x$
$x > -1$	$6 < 2x$
	$3 < x$
	$x > 3$



Draw a number line. Note that the two sets of values overlap where $x > 3$.

So the required set of values is $x > 3$.

Exercise 3D

1 Find the set of values of x for which:

a $2x - 3 < 5$ **b** $5x + 4 \geqslant 39$

c $6x - 3 > 2x + 7$ **d** $5x + 6 \leqslant -12 - x$

e $15 - x > 4$ **f** $21 - 2x > 8 + 3x$

g $1 + x < 25 + 3x$ **h** $7x - 7 < 7 - 7x$

i $5 - 0.5x \geqslant 1$ **j** $5x + 4 > 12 - 2x$

2 Find the set of values of x for which:

a $2(x - 3) \geqslant 0$ **b** $8(1 - x) > x - 1$

c $3(x + 7) \leqslant 8 - x$ **d** $2(x - 3) - (x + 12) < 0$

e $1 + 11(2 - x) < 10(x - 4)$ **f** $2(x - 5) \geqslant 3(4 - x)$

g $12x - 3(x - 3) < 45$ **h** $x - 2(5 + 2x) < 11$

i $x(x - 4) \geqslant x^2 + 2$ **j** $x(5 - x) \geqslant 3 + x - x^2$

3 Find the set of values of x for which:

a $3(x - 2) > x - 4$ and $4x + 12 > 2x + 17$

b $2x - 5 < x - 1$ and $7(x + 1) > 23 - x$

c $2x - 3 > 2$ and $3(x + 2) < 12 + x$

d $15 - x < 2(11 - x)$ and $5(3x - 1) > 12x + 19$

e $3x + 8 \leqslant 20$ and $2(3x - 7) \geqslant x + 6$

3.5 **To solve a quadratic inequality you**
- **solve the corresponding quadratic equation, then**
- **sketch the graph of the quadratic function, then**
- **use your sketch to find the required set of values.**

Example 8

Find the set of values of x for which $x^2 - 4x - 5 < 0$ and draw a sketch to show this.

$x^2 - 4x - 5 = 0$

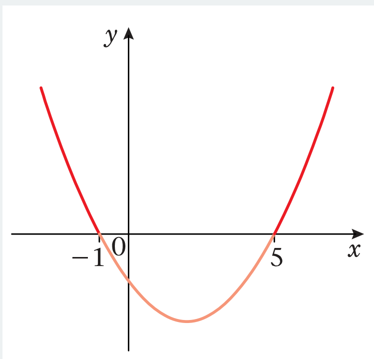
Quadratic equation.

$(x + 1)(x - 5) = 0$

Factorise (or use the quadratic formula). (See Section 2.5.)

$x = -1$ or $x = 5$

-1 and 5 are called critical values.



So the required set of values is $-1 < x < 5$.

Your sketch does not need to be accurate. All you really need to know is that the graph is '∪-shaped' and crosses the x-axis at -1 and 5. (See Section 2.6.)

$x^2 - 4x - 5 < 0$ ($y < 0$) for the part of the graph below the x-axis, as shown by the paler part in the rough sketch.

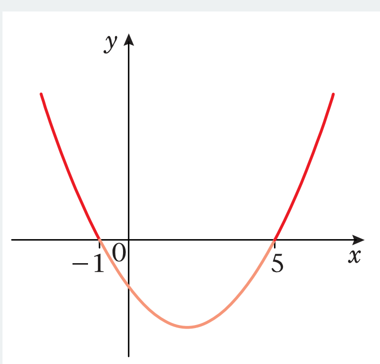
Example 9

Find the set of values of x for which $x^2 - 4x - 5 > 0$.

$x^2 - 4x - 5 = 0$

$(x + 1)(x - 5) = 0$

$x = -1$ or $x = 5$



The required set of values is $x < -1$ or $x > 5$.

The only difference between this example and the previous example is that it has to be greater than 0 (> 0). The solution would be exactly the same apart from the final stage.

$x^2 - 4x - 5 > 0$ ($y > 0$) for the part of the graph above the x-axis, as shown by the darker parts of the rough sketch in Example 8.

Be careful how you write down solutions like those on page 33.

$-1 < x < 5$ is fine, showing that x is between -1 and 5.

But it is wrong to write something like $5 < x < -1$ or $-1 > x > 5$ because x cannot be less than -1 and greater than 5 at the same time.

This type of solution (the darker parts of the graph) needs to be written in two separate parts, $x < -1$, $x > 5$.

Example 10

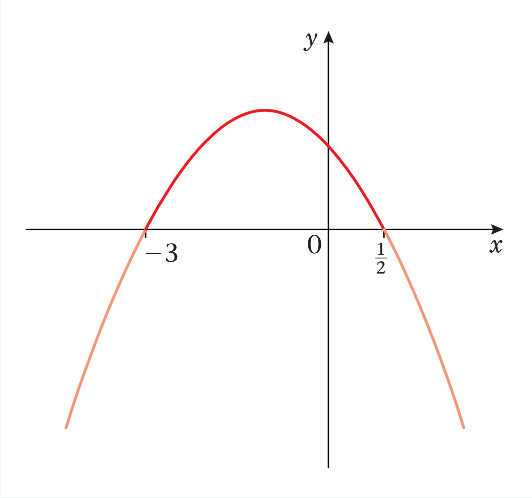
Find the set of values of x for which $3 - 5x - 2x^2 < 0$ and sketch the graph of $y = 3 - 5x - 2x^2$.

$3 - 5x - 2x^2 = 0$ — Quadratic equation.

$2x^2 + 5x - 3 = 0$ — Multiply by -1 (so it's easier to factorise).

$(2x - 1)(x + 3) = 0$

$x = \frac{1}{2}$ or $x = -3$ — $\frac{1}{2}$ and -3 are the critical values.



Since the coefficient of x^2 is negative, the graph is 'upside-down ⋃-shaped' and crosses the x-axis at -3 and $\frac{1}{2}$ (see Section 2.6).
$3 - 5x - 2x^2 < 0$ ($y < 0$) for the outer parts of the graph, below the x-axis, as shown by the paler parts in the rough sketch.

So the required set of values is

$x < -3$ or $x > \frac{1}{2}$.

You may have to rearrange the quadratic inequality to get all the terms 'on one side' before you can solve it, as shown in the next example.

Example 11

Find the set of values of x for which $12 + 4x > x^2$.

Method 1: sketch graph

$12 + 4x > x^2$

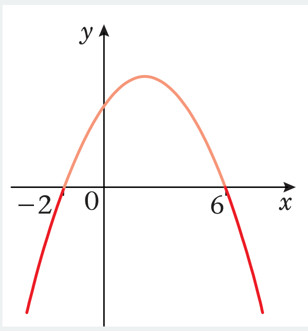
$12 + 4x - x^2 > 0$

$x^2 - 4x - 12 = 0$

$(x + 2)(x - 6) = 0$

$x = -2$ or $x = 6$

Sketch of $y = 12 + 4x - x^2$



$12 + 4x - x^2 > 0$

Solution: $-2 < x < 6$

$12 + 4x > x^2$

$0 > x^2 - 4x - 12$

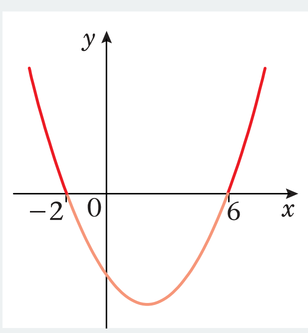
$x^2 - 4x - 12 < 0$

$x^2 - 4x - 12 = 0$

$(x + 2)(x - 6) = 0$

$x = -2$ or $x = 6$

Sketch of $y = x^2 - 4x - 12$



$x^2 - 4x - 12 < 0$

Solution: $-2 < x < 6$

There are two possible approaches for Method 1, depending on which side of the inequality sign you put the expression.

Find the set of values of x for which $12 + 4x > x^2$.

Method 2: table

$12 + 4x > x^2$

$0 > x^2 - 4x - 12$

$x^2 - 4x - 12 < 0$

$x^2 - 4x - 12 = 0$

$(x + 2)(x - 6) = 0$

$x = -2$ or $x = 6$

Use the critical values to split the real number line into sets.

	$x < -2$	$-2 < x < 6$	$x > 6$
$(x + 2)$	−	+	+
$(x - 6)$	−	−	+
$(x + 2)(x - 6)$	+	−	+

For each set, check whether the set of values makes the value of the bracket positive or negative.

For example, if $x < -2$, $(x + 2)$ is negative, $(x - 6)$ is negative, $(x + 2)(x - 6)$ is (neg) × (neg) = positive.

$x^2 - 4x - 12 < 0$

$(x + 2)(x - 6) < 0$

$(x + 2)(x - 6)$ is negative for $-2 < x < 6$

Solution: $-2 < x < 6$

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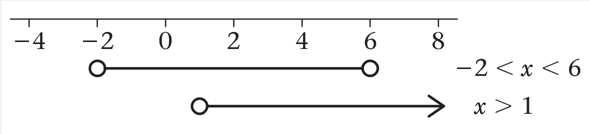
Example 12

Find the set of values of x for which $12 + 4x > x^2$ and $5x - 3 > 2$.

Solving $12 + 4x > x^2$ gives $-2 < x < 6$ (see Example 11).

Solving $5x - 3 > 2$ gives $x > 1$.



The two sets of values overlap where $1 < x < 6$.

So the solution is $1 < x < 6$.

Exercise 3E

1 Find the set of values of x for which:

a $x^2 - 11x + 24 < 0$ **b** $12 - x - x^2 > 0$ **c** $x^2 - 3x - 10 > 0$

d $x^2 + 7x + 12 \geqslant 0$ **e** $7 + 13x - 2x^2 > 0$ **f** $10 + x - 2x^2 < 0$

g $4x^2 - 8x + 3 \leqslant 0$ **h** $-2 + 7x - 3x^2 < 0$ **i** $x^2 - 9 < 0$

j $6x^2 + 11x - 10 > 0$ **k** $x^2 - 5x > 0$ **l** $2x^2 + 3x \leqslant 0$

2 Find the set of values of x for which:

a $x^2 < 10 - 3x$ **b** $11 < x^2 + 10$

c $x(3 - 2x) > 1$ **d** $x(x + 11) < 3(1 - x^2)$

3 Find the set of values of x for which:

a $x^2 - 7x + 10 < 0$ and $3x + 5 < 17$

b $x^2 - x - 6 > 0$ and $10 - 2x < 5$

c $4x^2 - 3x - 1 < 0$ and $4(x + 2) < 15 - (x + 7)$

d $2x^2 - x - 1 < 0$ and $14 < 3x - 2$

e $x^2 - x - 12 > 0$ and $3x + 17 > 2$

f $x^2 - 2x - 3 < 0$ and $x^2 - 3x + 2 > 0$

4 **a** Find the range of values of k for which the equation $x^2 - kx + (k + 3) = 0$ has real roots.

b Find the range of values of p for which the roots of the equation $px^2 + px - 2 = 0$ are real.

Mixed exercise 3F

1 Solve the simultaneous equations:

$$x + 2y = 3$$
$$x^2 - 4y^2 = -33$$

E

2 Show that the elimination of x from the simultaneous equations

$$x - 2y = 1$$
$$3xy - y^2 = 8$$

produces the equation

$$5y^2 + 3y - 8 = 0.$$

Solve this quadratic equation and hence find the pairs (x, y) for which the simultaneous equations are satisfied.

3 **a** Given that $3^x = 9^{y-1}$, show that $x = 2y - 2$.

b Solve the simultaneous equations:

$$x = 2y - 2$$
$$x^2 = y^2 + 7$$

4 Solve the simultaneous equations:

$$x + 2y = 3$$
$$x^2 - 2y + 4y^2 = 18$$

5 **a** Solve the inequality $3x - 8 > x + 13$.

b Solve the inequality $x^2 - 5x - 14 > 0$.

6 Find the set of values of x for which $(x - 1)(x - 4) < 2(x - 4)$.

7 **a** Use algebra to solve $(x - 1)(x + 2) = 18$.

b Hence, or otherwise, find the set of values of x for which $(x - 1)(x + 2) > 18$.

8 Find the set of values of x for which:

a $6x - 7 < 2x + 3$

b $2x^2 - 11x + 5 < 0$

c both $6x - 7 < 2x + 3$ and $2x^2 - 11x + 5 < 0$.

9 Find the values of k for which $kx^2 + 8x + 5 = 0$ has real roots.

10 Find algebraically the set of values of x for which $(2x - 3)(x + 2) > 3(x - 2)$.

11 **a** Find, as surds, the roots of the equation $2(x + 1)(x - 4) - (x - 2)^2 = 0$.

b Hence find the set of values of x for which $2(x + 1)(x - 4) - (x - 2)^2 > 0$.

12 **a** Use algebra to find the set of values of x for which $x(x - 5) > 36$.

b Using your answer to part **a**, find the set of values of y for which $y^2(y^2 - 5) > 36$.

13 The specification for a rectangular car park states that the length x m is to be 5 m more than the breadth. The perimeter of the car park is to be greater than 32 m.

a Form a linear inequality in x.

The area of the car park is to be less than 104 m^2.

b Form a quadratic inequality in x.

c By solving your inequalities, determine the set of possible values of x.

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Summary of key points

1 You can solve linear simultaneous equations by elimination or substitution.

2 You can use the substitution method to solve simultaneous equations, where one equation is linear and the other is quadratic. You usually start by finding an expression for x or y from the linear equation.

3 When you multiply or divide an inequality by a negative number, you need to change the inequality sign to its opposite.

4 To solve a quadratic inequality you

- solve the corresponding quadratic equation, then
- sketch the graph of the quadratic function, then
- use your sketch to find the required set of values.