After completing this chapter you should be able to

1 sketch cubic graphs
2 sketch the graph of the reciprocal function $y = \frac{k}{x}$
3 find where curves intersect
4 understand how the transformations $f(x + a)$, $f(x) + a$, $f(ax)$ and $af(x)$ affect the graph of the curve $y = f(x)$.

You will analyse graphs in greater detail when you start differentiation. It is worth remembering the techniques in this chapter, because they will provide further information about the shape of the function. Later on in the course you will be asked to sketch complex graphs which are simple transformations of a standard function..

4 Sketching curves

Did you know?

The following is a real life example of a cubic function. An open box is to be made from a sheet of card 10 cm by 10 cm.
Identical squares are cut off the four corners of the card as shown in Figure 1.

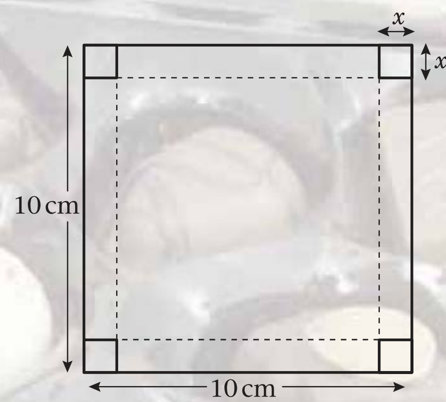


Figure 1

The card is then folded along the dotted lines to make a box as shown in Figure 2.

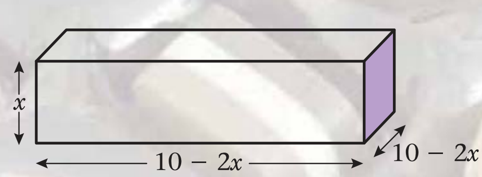
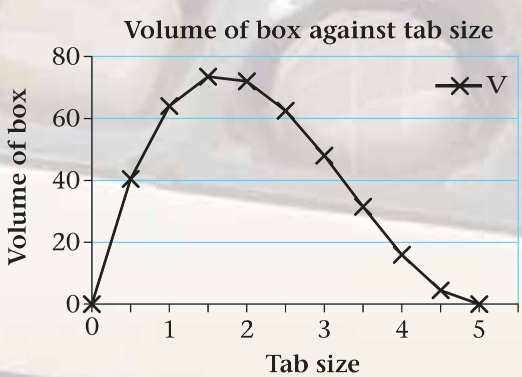


Figure 2

The volume of the box varies according to the formula

$V = (10 - 2x)^2x$



Can you give a reason why the graph has not been drawn for values of x greater than 5 and less than 0?

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4.1 You can sketch cubic curves of the form $y = ax^3 + bx^2 + cx + d$

Example 1

Sketch the curve with the equation $y = (x - 2)(x - 1)(x + 1)$

$0 = (x - 2)(x - 1)(x + 1)$

So $x = 2$ or $x = 1$ or $x = -1$

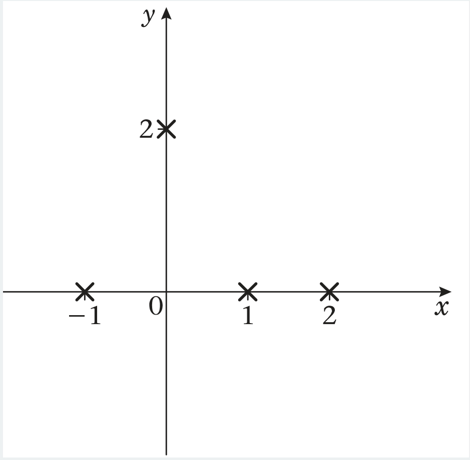
So the curve crosses the x-axis at $(2, 0)$ $(1, 0)$ and $(-1, 0)$.

Put $y = 0$ and solve for x to find the roots (the points where the curve crosses the x-axis).

When $x = 0$, $y = -2 \times -1 \times 1 = 2$

So the curve crosses the y-axis at $(0, 2)$.

Put $x = 0$ to find where the curve crosses the y-axis.

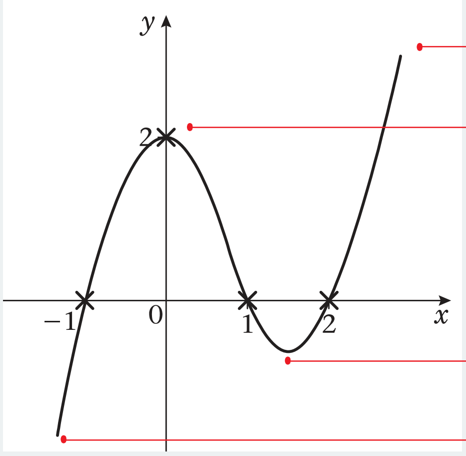


When x is large and positive, y is large and positive.

When x is large and negative, y is large and negative.

Check what happens to y for large positive and negative values of x.

You can write this as
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



$x \to \infty, y \to \infty$

This is called a maximum point because the gradient changes from +ve to 0 to −ve.

This is called a minimum point because the gradient changes from −ve to 0 to +ve.

$x \to -\infty, y \to -\infty$

In your exam you will not be expected to work out the coordinates of the maximum or minimum points without further work, but you should mark points where the curve meets the axes.

Example 2

Sketch the curves with the following equations and show the points where they cross the coordinate axes.

a $y = (x - 2)(1 - x)(1 + x)$ **b** $y = x(x + 1)(x + 2)$

a $0 = (x - 2)(1 - x)(1 + x)$

So $x = 2$, $x = 1$ or $x = -1$

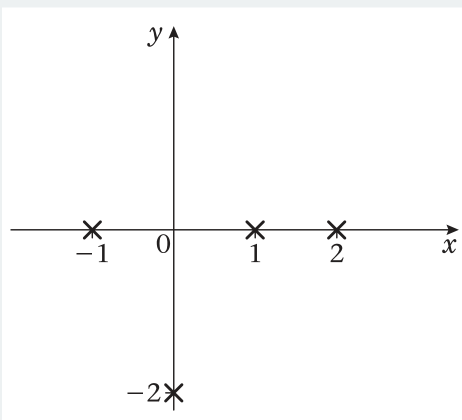
So the curve crosses the x-axis at $(2, 0)$, $(1, 0)$ and $(-1, 0)$.

Put $y = 0$ and solve for x.

When $x = 0$, $y = -2 \times 1 \times 1 = -2$

So the curve crosses the y-axis at $(0, -2)$.

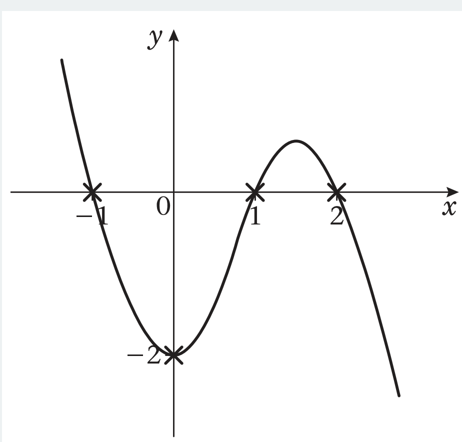
Find the value of y when $x = 0$.



$x \to \infty, y \to -\infty$

$x \to -\infty, y \to \infty$

Check what happens to y for large positive and negative values of x.



Notice that this curve is a reflection in the x-axis of the curve in Example 1.

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b $y = x(x+1)(x+2)$

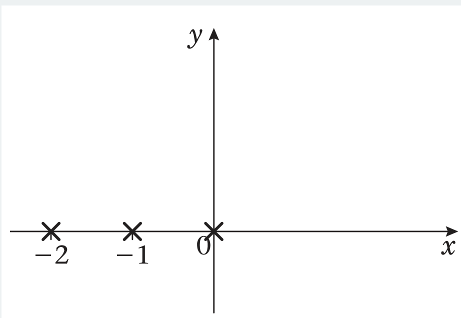
$0 = x(x+1)(x+2)$

So $x = 0$, $x = -1$ or $x = -2$

Put $y = 0$ and solve for x.

So the curve crosses the x-axis at $(0, 0)$, $(-1, 0)$ and $(-2, 0)$.

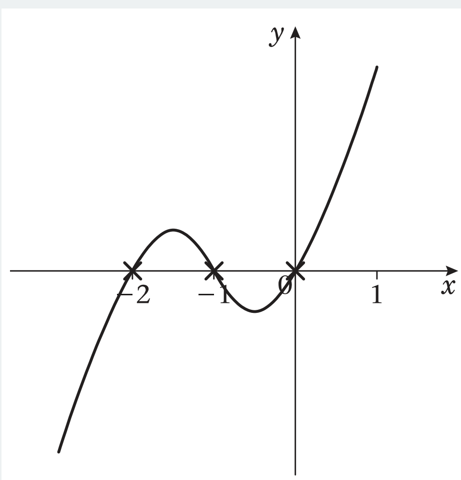
So the curve crosses the y-axis at (0, 0).



$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

Check what happens to y for large positive and negative values of x.



Example 3

Sketch the following curves.

a $y = (x-1)^2(x+1)$

b $y = x^3 - 2x^2 - 3x$

a $y = (x-1)^2(x+1)$

$0 = (x-1)^2(x+1)$

So $x = 1$ or $x = -1$.

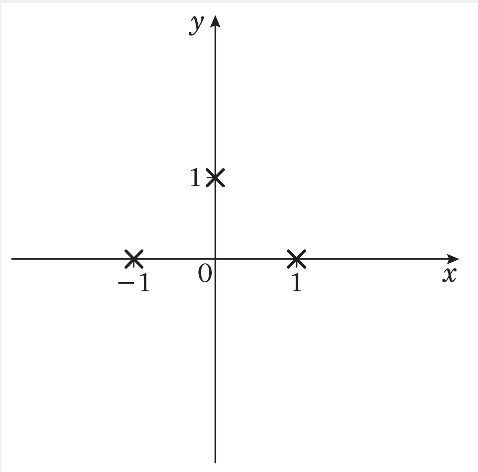
So the curve crosses the x-axis at $(1, 0)$ and $(-1, 0)$.

When $x = 0$ $y = (-1)^2 \times 1 = 1$

So the curve crosses the y-axis at $(0, 1)$.

Put $y = 0$ and solve for x.

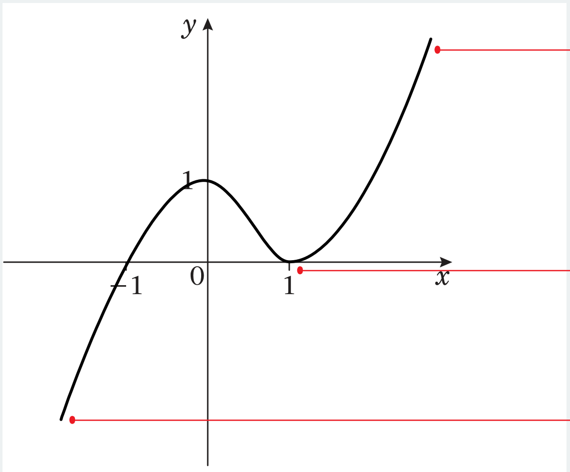
Find the value of y when $x = 0$.



$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

Check what happens to y for large positive and negative values of x.



$x \to \infty, y \to \infty$

$x = 1$ is a 'double' root.

$x \to -\infty, y \to -\infty$

b $y = x^3 - 2x^2 - 3x$

$= x(x^2 - 2x - 3)$

$= x(x - 3)(x + 1)$

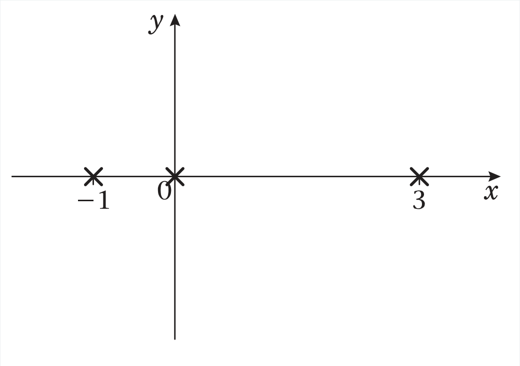
First factorise.

$0 = x(x - 3)(x + 1)$

So $x = 0$, $x = 3$ or $x = -1$

So the curve crosses the x-axis at (0, 0) (3, 0) and (−1, 0).

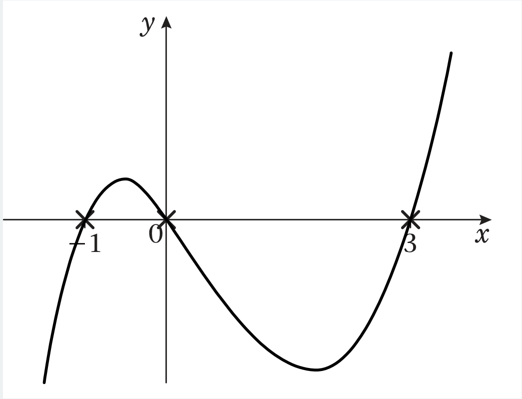
So the curve crosses the y-axis at (0, 0).



$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

Check what happens to y for large positive and negative values of x.



Exercise 4A

1 Sketch the following curves and indicate clearly the points of intersection with the axes:

a $y = (x - 3)(x - 2)(x + 1)$

b $y = (x - 1)(x + 2)(x + 3)$

c $y = (x + 1)(x + 2)(x + 3)$

d $y = (x + 1)(1 - x)(x + 3)$

e $y = (x - 2)(x - 3)(4 - x)$

f $y = x(x - 2)(x + 1)$

g $y = x(x + 1)(x - 1)$

h $y = x(x + 1)(1 - x)$

i $y = (x - 2)(2x - 1)(2x + 1)$

j $y = x(2x - 1)(x + 3)$

2 Sketch the curves with the following equations:

a $y = (x + 1)^2(x - 1)$ **b** $y = (x + 2)(x - 1)^2$

c $y = (2 - x)(x + 1)^2$ **d** $y = (x - 2)(x + 1)^2$

e $y = x^2(x + 2)$ **f** $y = (x - 1)^2x$

g $y = (1 - x)^2(3 + x)$ **h** $y = (x - 1)^2(3 - x)$

i $y = x^2(2 - x)$ **j** $y = x^2(x - 2)$

3 Factorise the following equations and then sketch the curves:

a $y = x^3 + x^2 - 2x$ **b** $y = x^3 + 5x^2 + 4x$

c $y = x^3 + 2x^2 + x$ **d** $y = 3x + 2x^2 - x^3$

e $y = x^3 - x^2$ **f** $y = x - x^3$

g $y = 12x^3 - 3x$ **h** $y = x^3 - x^2 - 2x$

i $y = x^3 - 9x$ **j** $y = x^3 - 9x^2$

4.2 You need to be able to sketch and interpret graphs of cubic functions of the form $y = x^3$.

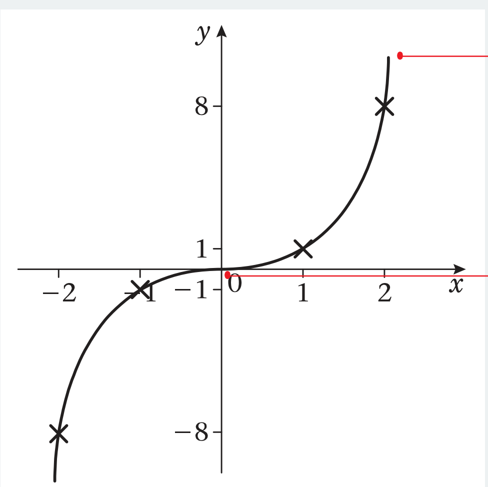
Example 4

Sketch the curve with equation $y = x^3$.

$0 = x^3$

So the curve crosses both axes at (0, 0).

x	−2	−1	0	1	2
$y = x^3$	−8	−1	0	1	8



Put $y = 0$ and solve for x.

As the curve passes the axes at only one point, find its shape by plotting a few points.

Notice that as x increases, y increases rapidly.

The curve is 'flat' at (0, 0). This point is called a point of inflexion. The gradient is positive just before (0, 0) and positive just after (0, 0).

Notice that the shape of this curve is the same as the curve with equation $y = (x + 1)^3$, which is shown in Example 5.

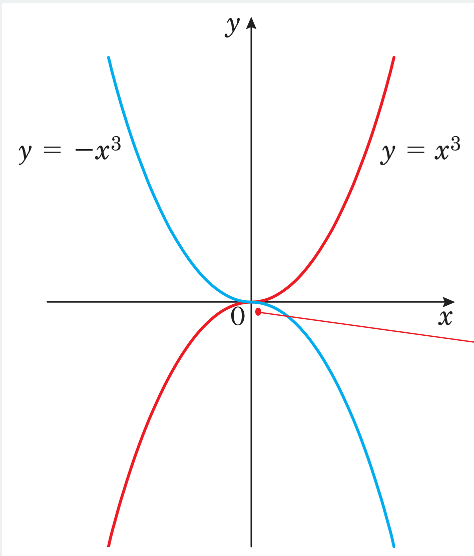
Example 5

Sketch the curve with equations:

a $y = -x^3$ **b** $y = (x + 1)^3$ **c** $y = (3 - x)^3$

Show their positions relative to the curve with equation $y = x^3$.

a $y = -x^3$



You do not need to plot any points. It is quicker if you realise the curve $y = -x^3$ is a reflection in the x-axis of the curve $y = x^3$. You can check this by looking at the values used to sketch $y = x^3$. So, for example, $x = 2$ will now correspond to $y = -8$ on the curve $y = -x^3$.

The curve is still flat at (0, 0).

b $y = (x + 1)^3$

$0 = (x + 1)^3$

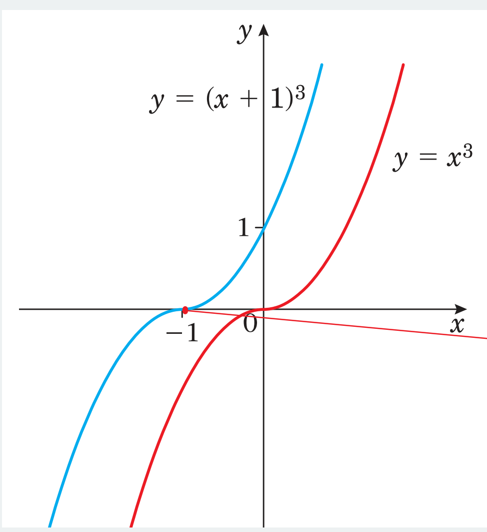
So $x = -1$

So the curve crosses the x-axis at $(-1, 0)$.

When $x = 0$, $y = 1^3 = 1$

So the curve crosses the y-axis at $(0, 1)$.

Put $y = 0$ to find where the curve crosses the x-axis.
Put $x = 0$ to find where the curve crosses the y-axis.



The curve has the same shape as $y = x^3$.

You do not need to do any working if you realise the curve $y = (x + 1)^3$ is a translation of -1 along the x-axis of the curve $y = x^3$.

The point of inflexion is at $(-1, 0)$.

c $y = (3 - x)^3$

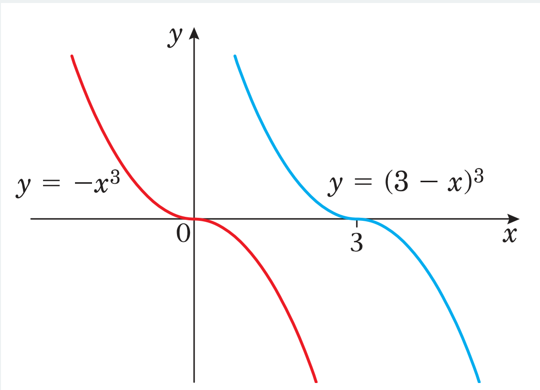
$0 = (3 - x)^3$

So $x = 3$

So the curve crosses the x-axis at (3, 0).

When $x = 0$, $y = 3^3 = 27$

So the curve crosses the y-axis at (0, 27).



Put $y = 0$ to find where the curve crosses the x-axis.

Put $x = 0$ to find where the curve crosses the y-axis.

You can write the equation for the curve as $y = [-(x - 3)]^3$ so $y = -(x - 3)^3$ so the curve will have the same shape as $y = -x^3$.

You do not need to do any working if you realise the curve $y = (3 - x)^3 = -(x - 3)^3$ is a translation of $+3$ along the x-axis of the curve $y = -x^3$.

The point of inflexion is at (3, 0).

Exercise 4B

1 Sketch the following curves and show their positions relative to the curve $y = x^3$:

a $y = (x - 2)^3$ **b** $y = (2 - x)^3$ **c** $y = (x - 1)^3$

d $y = (x + 2)^3$ **e** $y = -(x + 2)^3$

2 Sketch the following and indicate the coordinates of the points where the curves cross the axes:

a $y = (x + 3)^3$ **b** $y = (x - 3)^3$ **c** $y = (1 - x)^3$

d $y = -(x - 2)^3$ **e** $y = -(x - \frac{1}{2})^3$

4.3 You need to be able to sketch the reciprocal function $y = \frac{k}{x}$ where k is a constant.

Example 6

Sketch the curve $y = \frac{1}{x}$ and its asymptotes.

$y = \frac{1}{x}$

When $x = 0$, y is not defined.

When $y = 0$, x is not defined.

$x \to +\infty,\ y \to 0$

$x \to -\infty,\ y \to 0$

$y \to +\infty,\ x \to 0$

$y \to -\infty,\ x \to 0$

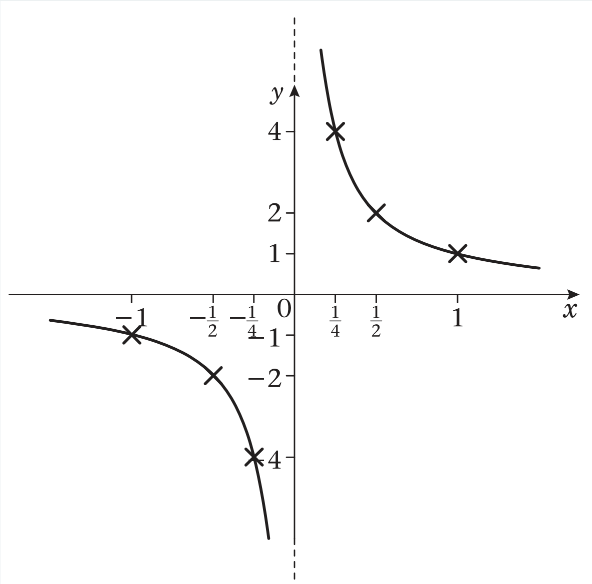
The curve does not cross the axes.

The curve tends towards the x-axis when x is large and positive or large and negative. The x-axis is a horizontal asymptote.

The curve tends towards the y-axis when y is large and positive or large and negative. The y-axis is a vertical asymptote.

$y = \frac{1}{x}$

x	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1
$y = \frac{1}{x}$	-1	-2	-4	4	2	1



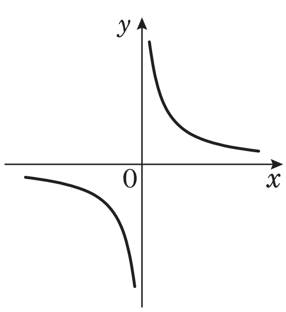
The curve does not cross the x-axis or y-axis. You need to plot some points.

You can draw a dashed line to indicate an asymptote. (In this case the asymptotes are the axes, but see Example 11.)

■ **The curves with equations $y = \frac{k}{x}$ fall into two categories:**

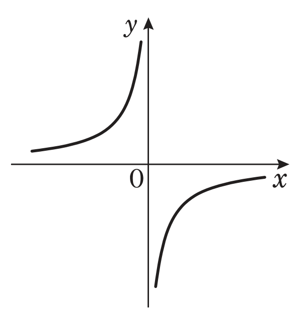
Type 1

$y = \frac{k}{x},\ k > 0$



Type 2

$y = \frac{k}{x},\ k < 0$

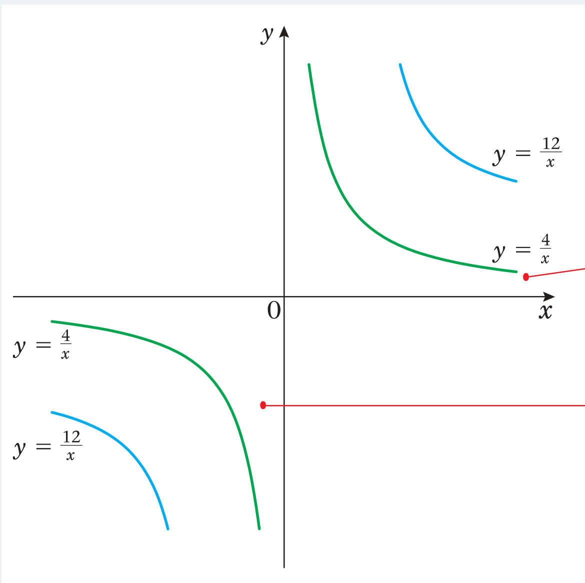


Example 7

Sketch on the same diagram:

a $y = \frac{4}{x}$ and $y = \frac{12}{x}$ **b** $y = -\frac{1}{x}$ and $y = -\frac{3}{x}$

a



The shape of these curves will be Type 1.

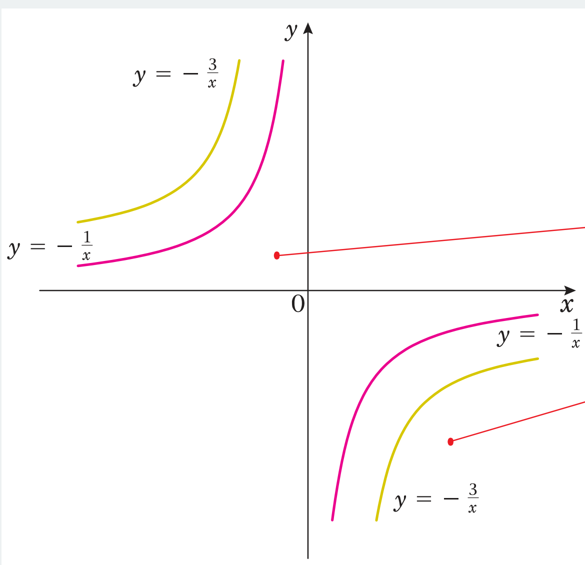
In this quadrant, $x > 0$

so for any values of x: $\frac{12}{x} > \frac{4}{x}$

In this quadrant, $x < 0$

so for any values of x: $\frac{12}{x} < \frac{4}{x}$

b



The shape of these curves will be Type 2.

In this quadrant, $x < 0$

so for any values of x: $\frac{-3}{x} > \frac{-1}{x}$

In this quadrant, $x > 0$

so for any values of x: $\frac{-3}{x} < \frac{-1}{x}$

Exercise 4C

Use a separate diagram to sketch each pair of graphs.

1 $y = \frac{2}{x}$ and $y = \frac{4}{x}$

2 $y = \frac{2}{x}$ and $y = -\frac{2}{x}$

3 $y = -\frac{4}{x}$ and $y = -\frac{2}{x}$

4 $y = \frac{3}{x}$ and $y = \frac{8}{x}$

5 $y = -\frac{3}{x}$ and $y = -\frac{8}{x}$

4.4 You can sketch curves of functions to show points of intersection and solutions to equations.

Example 8

a On the same diagram sketch the curves with equations $y = x(x - 3)$ and $y = x^2(1 - x)$.

b Find the coordinates of the point of intersection.

a $y = x(x - 3)$

$0 = x(x - 3)$

Put $y = 0$ and solve for x.

So $x = 0$ or $x = 3$.

So the curve crosses the x-axis at $(0, 0)$ and $(3, 0)$.

$y = x^2(1 - x)$

$0 = x^2(1 - x)$

Put $y = 0$ and solve for x to find where the curve crosses the x-axis.

So $x = 0$ or $x = 1$.

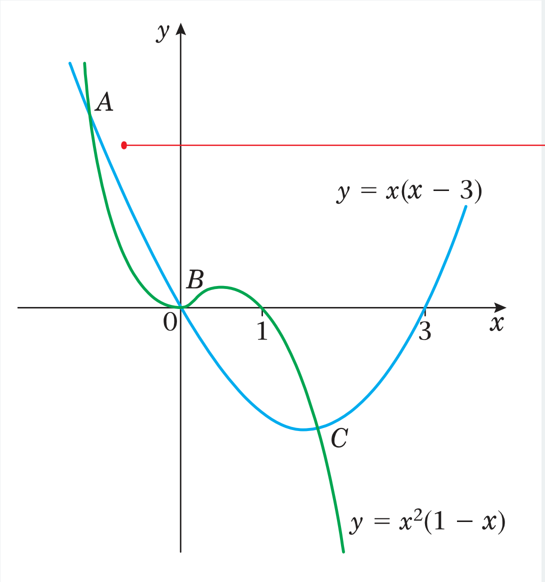
So the curve crosses the x-axis at $(0, 0)$ or $(1, 0)$.

The curve crosses the y-axis at $(0, 0)$.

$x \to \infty,\ y \to -\infty$

$x \to -\infty,\ y \to +\infty$

Check what happens to y for large positive and negative values of x.



A cubic curve is always steeper than a quadratic curve, so it will cross over somewhere on this side of the y-axis.

b From the graph there are three points where the curves cross, labelled A, B and C. The x-coordinates are given by the solutions to the equation.

$$x(x-3) = x^2(1-x)$$

$$x^2 - 3x = x^2 - x^3$$ — Multiply out brackets (see Section 1.3).

$$x^3 - 3x = 0$$ — Collect terms on one side.

$$x(x^2 - 3) = 0$$ — Factorise.

$$x(x-\sqrt{3})(x+\sqrt{3}) = 0$$ — Factorise using a difference of 2 squares.

So $x = -\sqrt{3}, 0, \sqrt{3}$

You can use the equation $y = x^2(1-x)$ to find the y-coordinates.

So the point where x is negative is $A\,(-\sqrt{3}, 3[1+\sqrt{3}])$, B is $(0, 0)$ and C is the point $(\sqrt{3}, 3[1-\sqrt{3}])$.

Example 9

a On the same diagram sketch the curves with equations $y = x^2(x-1)$ and $y = \dfrac{2}{x}$.

b Explain how your sketch shows that there are two solutions to the equation $x^2(x-1) - \dfrac{2}{x} = 0$.

a

$$y = x^2(x-1)$$

$$0 = x^2(x-1)$$ — Put $y = 0$ and solve for x.

So $x = 0$ or $x = 1$.

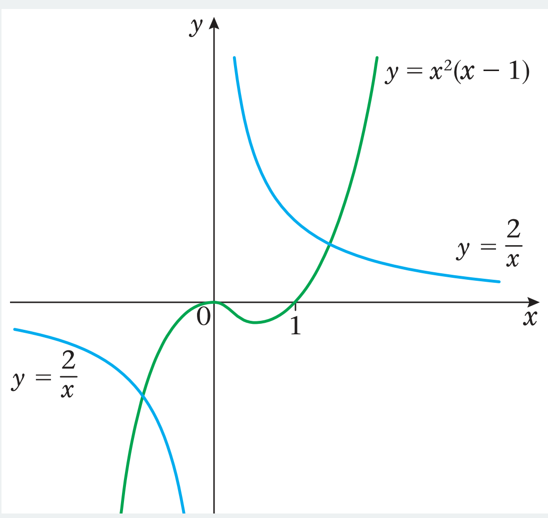
So the curve crosses the x-axis at $(0, 0)$ and $(1, 0)$.

The curve crosses the y-axis at (0, 0).

$x \to \infty,\ y \to \infty$

$x \to -\infty,\ y \to -\infty$

Check what happens to y for large positive and negative values of x.



b From the sketch there are only two points of intersection of the curves.

This means there are only two values of x where

$$x^2(x-1) = \frac{2}{x}$$

or $$x^2(x-1) - \frac{2}{x} = 0$$

So this equation has two solutions.

You would not be expected to solve this equation in C1.

Exercise 4D

1 In each case:

i sketch the two curves on the same axes

ii state the number of points of intersection

iii write down a suitable equation which would give the x-coordinates of these points. (You are not required to solve this equation.)

a $y = x^2, y = x(x^2 - 1)$

b $y = x(x + 2), y = -\frac{3}{x}$

c $y = x^2, y = (x + 1)(x - 1)^2$

d $y = x^2(1 - x), y = -\frac{2}{x}$

e $y = x(x - 4), y = \frac{1}{x}$

f $y = x(x - 4), y = -\frac{1}{x}$

g $y = x(x - 4), y = (x - 2)^3$

h $y = -x^3, y = -\frac{2}{x}$

i $y = -x^3, y = x^2$

j $y = -x^3, y = -x(x + 2)$

Hint: In question 1f, check the point $x = 2$ in both curves.

2 **a** On the same axes sketch the curves given by $y = x^2(x - 4)$ and $y = x(4 - x)$.

b Find the coordinates of the points of intersection.

3 **a** On the same axes sketch the curves given by $y = x(2x + 5)$ and $y = x(1 + x)^2$

b Find the coordinates of the points of intersection.

4 **a** On the same axes sketch the curves given by $y = (x - 1)^3$ and $y = (x - 1)(1 + x)$.

b Find the coordinates of the points of intersection.

5 **a** On the same axes sketch the curves given by $y = x^2$ and $y = -\frac{27}{x}$.

b Find the coordinates of the point of intersection.

6 **a** On the same axes sketch the curves given by $y = x^2 - 2x$ and $y = x(x - 2)(x - 3)$.

b Find the coordinates of the point of intersection.

7 **a** On the same axes sketch the curves given by $y = x^2(x - 3)$ and $y = \frac{2}{x}$.

b Explain how your sketch shows that there are only two solutions to the equation $x^3(x - 3) = 2$.

8 **a** On the same axes sketch the curves given by $y = (x + 1)^3$ and $y = 3x(x - 1)$.

b Explain how your sketch shows that there is only one solution to the equation $x^3 + 6x + 1 = 0$.

9 **a** On the same axes sketch the curves given by $y = \frac{1}{x}$ and $y = -x(x - 1)^2$.

b Explain how your sketch shows that there are no solutions to the equation $1 + x^2(x - 1)^2 = 0$.

10 **a** On the same axes sketch the curves given by $y = 1 - 4x^2$ and $y = x(x - 2)^2$.

b State, with a reason, the number of solutions to the equation $x^3 + 4x - 1 = 0$.

11 **a** On the same axes sketch the curve $y = x^3 - 3x^2 - 4x$ and the line $y = 6x$.

b Find the coordinates of the points of intersection.

12 **a** On the same axes sketch the curve $y = (x^2 - 1)(x - 2)$ and the line $y = 14x + 2$.

b Find the coordinates of the points of intersection.

13 **a** On the same axes sketch the curves with equations $y = (x - 2)(x + 2)^2$ and $y = -x^2 - 8$.

b Find the coordinates of the points of intersection.

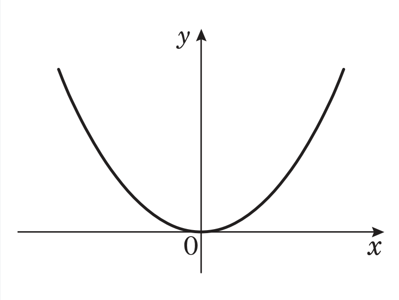
4.5 **You can transform the curve of a function f(x) by simple translations of the form:**
- **f($x + a$) is a horizontal translation of $-a$**
- **f(x) + a is a vertical translation of $+a$.**

Example 10

Sketch the curves for:

a $f(x) = x^2$ **b** $g(x) = (x - 2)^2$ **c** $h(x) = x^2 + 2$

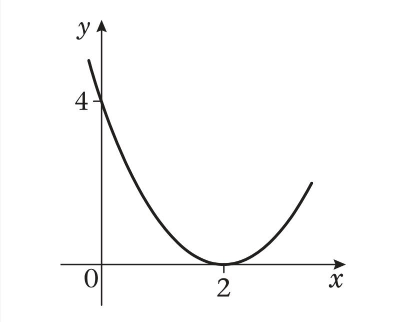
a $f(x) = x^2$



b $g(x) = (x - 2)^2$

So $g(x) = f(x - 2)$

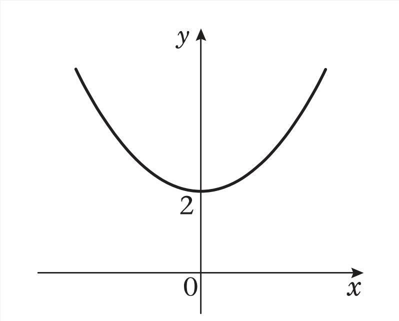
Here $a = -2$ so $g(x)$ is a horizontal translation of $-(-2) = +2$ along the x-axis.



c $h(x) = x^2 + 2$

So $h(x) = f(x) + 2$

Here $a = +2$ so $h(x)$ is a vertical translation of $+2$ along the y-axis.



Example 11

a Given that **i** $f(x) = x^3$

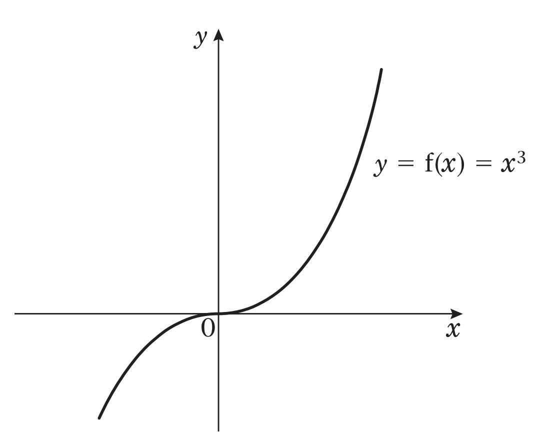
ii $g(x) = x(x - 2)$,

sketch the curves with equation $y = f(x + 1)$ and $g(x + 1)$ and mark on your sketch the points where the curves cross the axes.

b Given that $h(x) = \frac{1}{x}$, sketch the curve with equation $y = h(x) + 1$ and state the equations of any asymptotes and intersections with the axes.

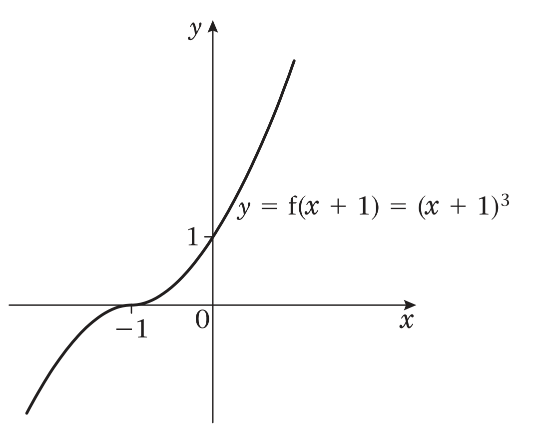
a

i The graph of $f(x) = x^3$ is



First sketch $f(x)$.

So the graph of $y = f(x + 1)$ is



Here $a = +1$ so it is a horizontal translation of -1 along the x-axis.

In this case the new equations can easily be found as $y = (x + 1)^3$ and this may help with the sketch.

ii $g(x) = x(x - 2)$

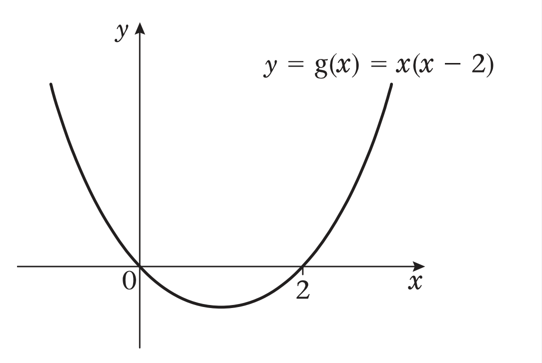
The curve is $y = x(x - 2)$

$0 = x(x - 2)$

Put $y = 0$ to find where the curve crosses the x-axis.

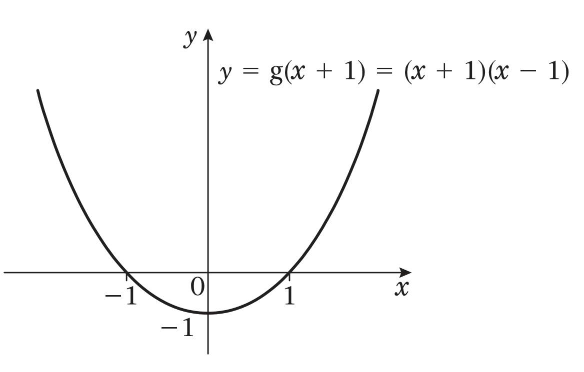
So $x = 0$ or $x = 2$

So the curve crosses the x-axis at $(0, 0)$ and $(2, 0)$.



First sketch $g(x)$.

So the graph of $y = g(x + 1)$ is

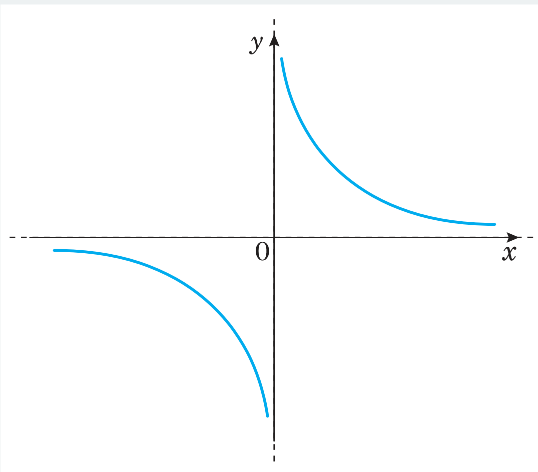


$a = +1$ so it is a horizontal translation of -1 along the x-axis.

You find the equation for $g(x + 1)$ by replacing x by $(x + 1)$ in the original equation. So $y = g(x + 1) = (x + 1)(x + 1 - 2) = (x+ 1)(x - 1)$.

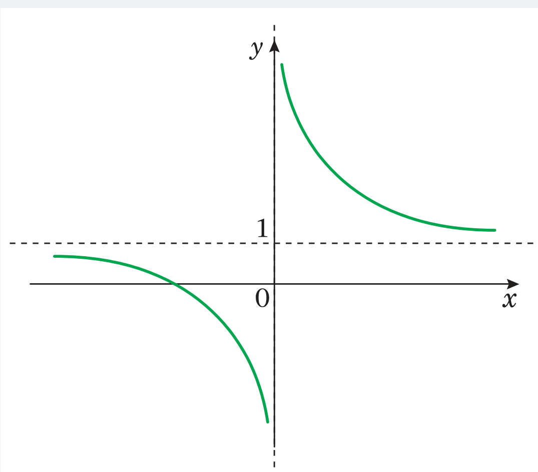
You can see this matches your sketch. The intersection with the y-axis is now at $(0, -1)$.

b The graph of $h(x) = \frac{1}{x}$ is



First sketch $h(x)$.

So the graph of $y = h(x) + 1$ is



Here $a = +1$ so it is a vertical translation of $+1$ along the y-axis.

The curve crosses the x-axis once.

$$y = \mathrm{h}(x) + 1 = \frac{1}{x} + 1$$

$$0 = \frac{1}{x} + 1$$

$$-1 = \frac{1}{x}$$

$$x = -1$$

So the curve intersects the x-axis at $(-1, 0)$.

The horizontal asymptote is $y = 1$.

The vertical asymptote is $x = 0$.

Put $y = 0$ to find where the curve crosses the x-axis.

Exercise 4E

1 Apply the following transformations to the curves with equations $y = \mathrm{f}(x)$ where:

i $\mathrm{f}(x) = x^2$ **ii** $\mathrm{f}(x) = x^3$ **iii** $\mathrm{f}(x) = \frac{1}{x}$

In each case state the coordinates of points where the curves cross the axes and in **iii** state the equations of any asymptotes.

a $\mathrm{f}(x + 2)$ **b** $\mathrm{f}(x) + 2$ **c** $\mathrm{f}(x - 1)$

d $\mathrm{f}(x) - 1$ **e** $\mathrm{f}(x) - 3$ **f** $\mathrm{f}(x - 3)$

2 **a** Sketch the curve $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = (x - 1)(x + 2)$.

b On separate diagrams sketch the graphs of **i** $y = \mathrm{f}(x + 2)$ **ii** $y = \mathrm{f}(x) + 2$.

c Find the equations of the curves $y = \mathrm{f}(x + 2)$ and $y = \mathrm{f}(x) + 2$, in terms of x, and use these equations to find the coordinates of the points where your graphs in part **b** cross the y-axis.

3 **a** Sketch the graph of $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = x^2(1 - x)$.

b Sketch the curve with equation $y = \mathrm{f}(x + 1)$.

c By finding the equation $\mathrm{f}(x + 1)$ in terms of x, find the coordinates of the point in part **b** where the curve crosses the y-axis.

4 **a** Sketch the graph of $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = x(x - 2)^2$.

b Sketch the curves with equations $y = \mathrm{f}(x) + 2$ and $y = \mathrm{f}(x + 2)$.

c Find the coordinates of the points where the graph of $y = \mathrm{f}(x + 2)$ crosses the axes.

5 **a** Sketch the graph of $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = x(x - 4)$.

b Sketch the curves with equations $y = \mathrm{f}(x + 2)$ and $y = \mathrm{f}(x) + 4$.

c Find the equations of the curves in part **b** in terms of x and hence find the coordinates of the points where the curves cross the axes.

4.6 **You can transform the curve of a function f(x) by simple stretches of these forms:**

- **f(ax) is a horizontal stretch of scale factor $\frac{1}{a}$, so you multiply the x-coordinates by $\frac{1}{a}$ and leave the y-coordinates unchanged.**
- **af(x) is a vertical stretch of scale factor a, so you multiply the y-coordinates by a and leave the x-coordinates unchanged.**

Example 12

Given that $f(x) = 9 - x^2$, sketch the curves with equations:

a $y = f(2x)$ **b** $y = 2f(x)$

a $f(x) = 9 - x^2$

So $f(x) = (3 - x)(3 + x)$

You can factorise the expression.

The curve is $y = (3 - x)(3 + x)$

$0 = (3 - x)(3 + x)$

So $x = 3$ or $x = -3$

Put $y = 0$ to find where the curve crosses the x-axis.

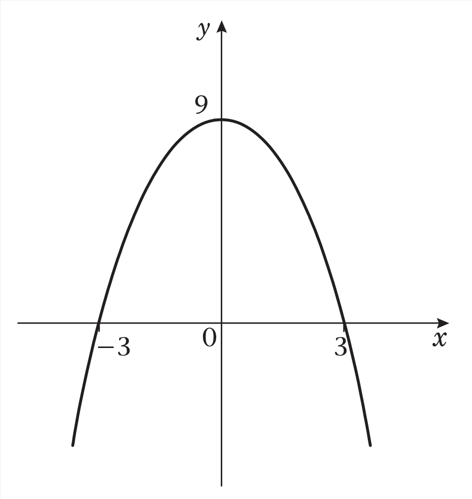
So the curve crosses the x-axis at $(3, 0)$ and $(-3, 0)$.

When $x = 0$, $y = 3 \times 3 = 9$

So the curve crosses the y-axis at $(0, 9)$.

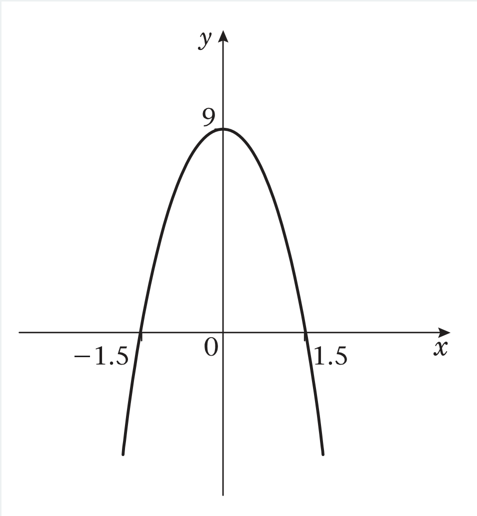
Put $x = 0$ to find where the curve crosses the y-axis.

The curve $y = f(x)$ is



First sketch $y = f(x)$.

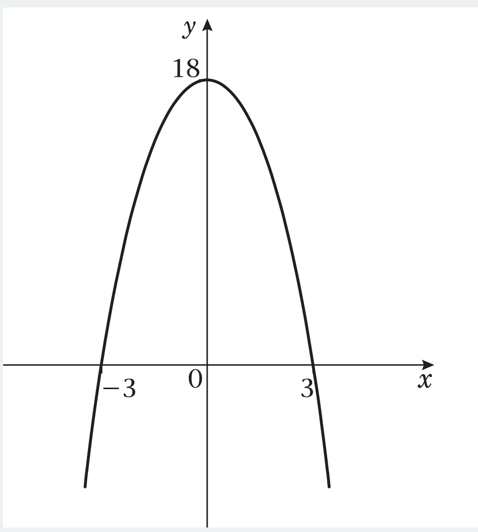
$y = \mathrm{f}(2x)$ so the curve is



$y = \mathrm{f}(ax)$ where $a = 2$ so it is a horizontal stretch with scale factor $\frac{1}{2}$.

Check: The curve is $y = \mathrm{f}(2x)$.
So $y = (3 - 2x)(3 + 2x)$.
When $y = 0$, $x = -1.5$ or $x = 1.5$.
So the curve crosses the x-axis at $(-1.5, 0)$ and $(1.5, 0)$.
When $x = 0$, $y = 9$.
So the curve crosses the y-axis at $(0, 9)$.

b $y = 2\mathrm{f}(x)$
So the curve is



$y = a\mathrm{f}(x)$ where $a = 2$ so it is a vertical stretch with scale factor 2.

Check: The curve is $y = 2\mathrm{f}(x)$.
So $y = 2(3 - x)(3 + x)$.
When $y = 0$, $x = 3$ or $x = -3$.
So the curve crosses the x-axis at $(-3, 0)$ and $(3, 0)$.
When $x = 0$, $y = 2 \times 9 = 18$.
So the curve crosses the y-axis at $(0, 18)$.

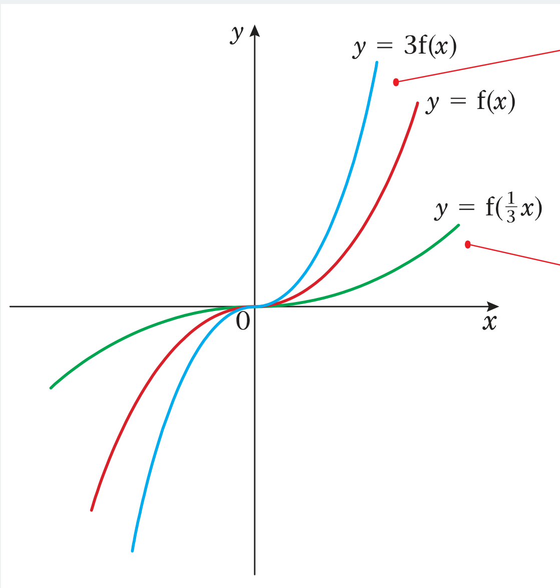
Example 13

a On the same axes sketch the graphs of $y = f(x)$, $y = 3f(x)$ and $y = f(\frac{1}{3}x)$ where:

i $f(x) = x^3$ **ii** $f(x) = \dfrac{1}{x}$

b On the same axes sketch the graphs of $y = f(x)$, $y = -f(x)$ and $y = f(-x)$ where $f(x) = x(x + 2)$.

a i $f(x) = x^3$

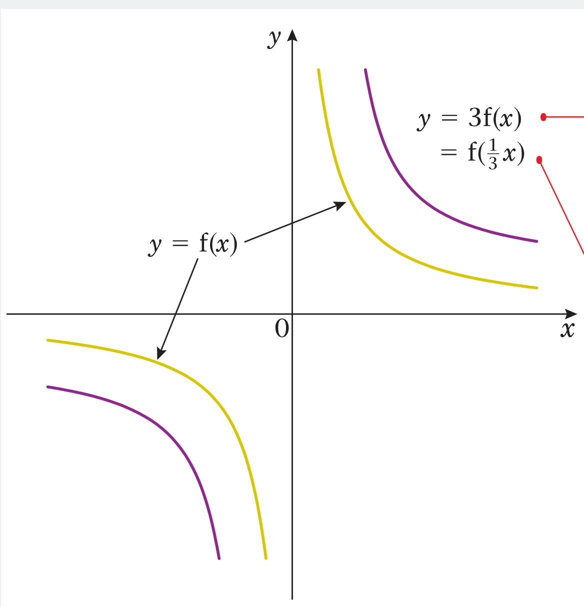


$y = 3f(x)$ is equivalent to $y = 3x^3$ and this will be steeper than $y = x^3$. It is a vertical stretch of $f(x)$ with scale factor 3.

This is equivalent to $y = \dfrac{x^3}{27}$ and this will be more shallow than $y = x^3$.

It is a horizontal stretch of $f(x)$ with scale factor 3.

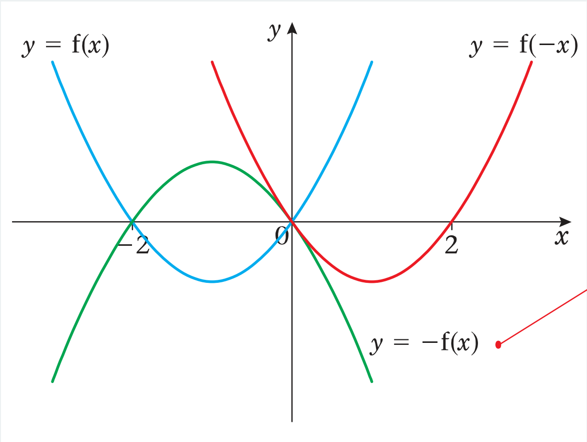
ii $f(x) = \dfrac{1}{x}$



$y = \dfrac{3}{x}$ will be above $y = \dfrac{1}{x}$.

$f(\frac{1}{3}x) = \dfrac{1}{\frac{1}{3}x} = \dfrac{3}{x}$ so this curve will be the same as $y = \dfrac{3}{x}$.

b $f(x) = x(x + 2)$



$y = f(-x)$ is $y = (-x)(-x + 2)$ which is $y = x^2 - 2x$ or $y = x(x - 2)$ and this is a reflection of the original curve in the y-axis.

Alternatively multiply each x-coordinate by -1 and leave the y coordinates unchanged.

$y = -f(x)$ is $y = -x(x + 2)$ and this is a reflection of the original curve in the x-axis.

Alternatively simply remember each y-coordinate is multiplied by -1 and the x-coordinates remain unchanged.

Exercise 4F

1 Apply the following transformations to the curves with equations $y = f(x)$ where:

i $f(x) = x^2$ **ii** $f(x) = x^3$ **iii** $f(x) = \frac{1}{x}$

In each case show both $f(x)$ and the transformation on the same diagram.

a $f(2x)$ **b** $f(-x)$

c $f(\frac{1}{2}x)$ **d** $f(4x)$

e $f(\frac{1}{4}x)$ **f** $2f(x)$

g $-f(x)$ **h** $4f(x)$

i $\frac{1}{2}f(x)$ **j** $\frac{1}{4}f(x)$

2 **a** Sketch the curve with equation $y = f(x)$ where $f(x) = x^2 - 4$.

b Sketch the graphs of $y = f(4x)$, $y = 3f(x)$, $y = f(-x)$ and $y = -f(x)$.

3 **a** Sketch the curve with equation $y = f(x)$ where $f(x) = (x - 2)(x + 2)x$.

b Sketch the graphs of $y = f(\frac{1}{2}x)$, $y = f(2x)$ and $y = -f(x)$.

4 **a** Sketch the curve with equation $y = f(x)$ where $f(x) = x^2(x - 3)$.

b Sketch the curves with equations $y = f(2x)$, $y = -f(x)$ and $y = f(-x)$.

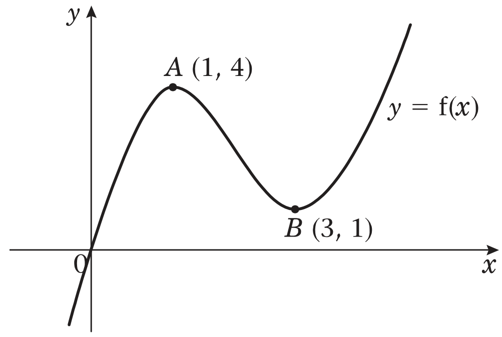
5 **a** Sketch the curve with equation $y = f(x)$ where $f(x) = (x - 2)(x - 1)(x + 2)$.

b Sketch the curves with equations $y = f(2x)$ and $f(\frac{1}{2}x)$.

4.7 You need to be able to perform simple transformations on a given sketch of a function.

Example 14

The following diagram shows a sketch of the curve f(x) which passes through the origin. The points $A(1, 4)$ and $B(3, 1)$ also lie on the curve.

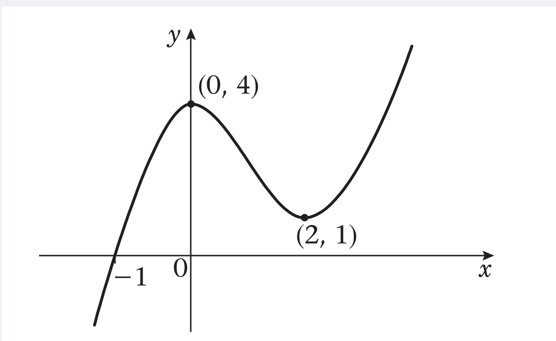


Sketch the following:

a $y = f(x + 1)$ **b** $y = f(x - 1)$ **c** $y = f(x) - 4$

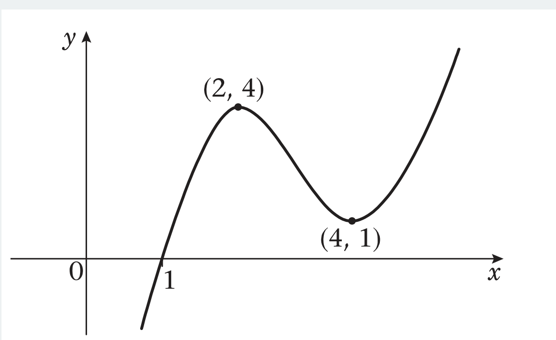
In each case you should show the coordinates of the images of the points O, A and B.

a f(x + 1)



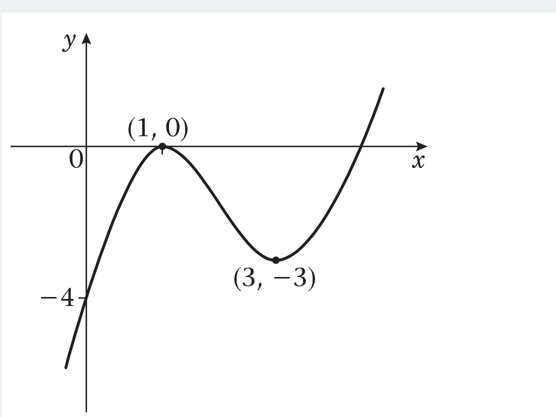
Move f(x) 1 unit to the left.

b f(x − 1)



This means move f(x) 1 unit to the right.

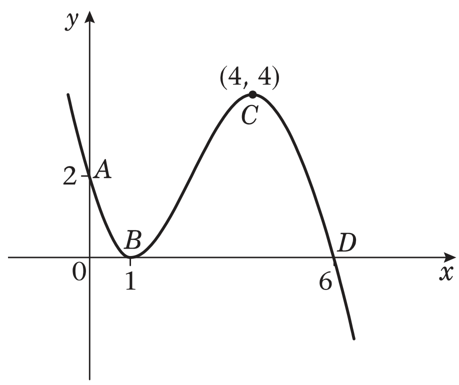
c f(x) − 4



Move f(x) down 4 units.

Exercise 4G

1 The following diagram shows a sketch of the curve with equation $y = f(x)$. The points $A(0, 2)$, $B(1, 0)$, $C(4, 4)$ and $D(6, 0)$ lie on the curve.



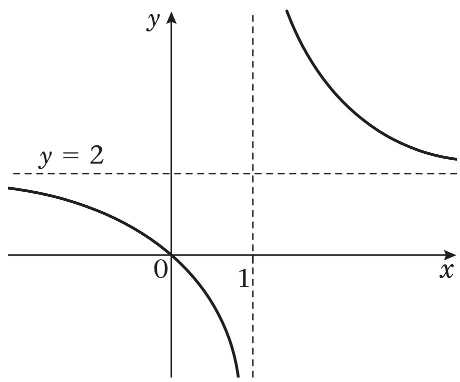
Sketch the following graphs and give the coordinates of the points A, B, C and D after each transformation:

a $f(x + 1)$ **b** $f(x) - 4$ **c** $f(x + 4)$

d $f(2x)$ **e** $3f(x)$ **f** $f(\frac{1}{2}x)$

g $\frac{1}{2}f(x)$ **h** $f(-x)$

2 The curve $y = f(x)$ passes through the origin and has horizontal asymptote $y = 2$ and vertical asymptote $x = 1$, as shown in the diagram.



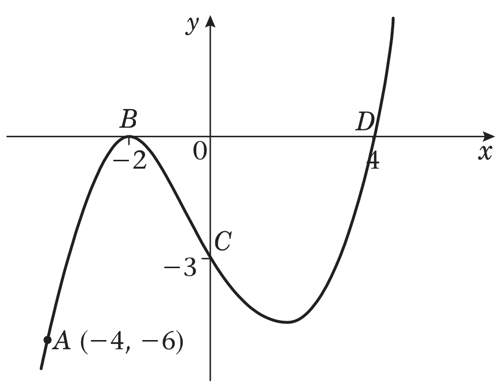
Sketch the following graphs and give the equations of any asymptotes and, for all graphs except **a**, give coordinates of intersections with the axes after each transformation.

a $f(x) + 2$ **b** $f(x + 1)$ **c** $2f(x)$

d $f(x) - 2$ **e** $f(2x)$ **f** $f(\frac{1}{2}x)$

g $\frac{1}{2}f(x)$ **h** $-f(x)$

3 The curve with equation $y = f(x)$ passes through the points $A(-4, -6)$, $B(-2, 0)$, $C(0, -3)$ and $D(4, 0)$ as shown in the diagram.



Sketch the following and give the coordinates of the points A, B, C and D after each transformation.

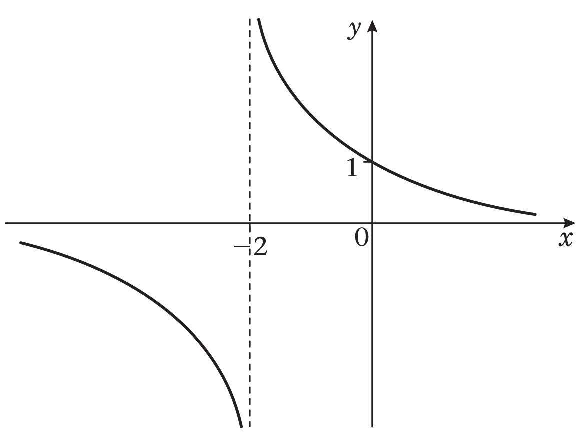
a $f(x - 2)$ **b** $f(x) + 6$ **c** $f(2x)$

d $f(x + 4)$ **e** $f(x) + 3$ **f** $3f(x)$

g $\frac{1}{3}f(x)$ **h** $f(\frac{1}{4}x)$ **i** $-f(x)$

j $f(-x)$

4 A sketch of the curve $y = f(x)$ is shown in the diagram. The curve has a vertical asymptote with equation $x = -2$ and a horizontal asymptote with equation $y = 0$. The curve crosses the y-axis at $(0, 1)$.



a Sketch, on separate diagrams, the graphs of:

i $2f(x)$ **ii** $f(2x)$ **iii** $f(x - 2)$

iv $f(x) - 1$ **v** $f(-x)$ **vi** $-f(x)$

In each case state the equations of any asymptotes and, if possible, points where the curve cuts the axes.

b Suggest a possible equation for $f(x)$.

Mixed exercise 4H

1 **a** On the same axes sketch the graphs of $y = x^2(x - 2)$ and $y = 2x - x^2$.

b By solving a suitable equation find the points of intersection of the two graphs.

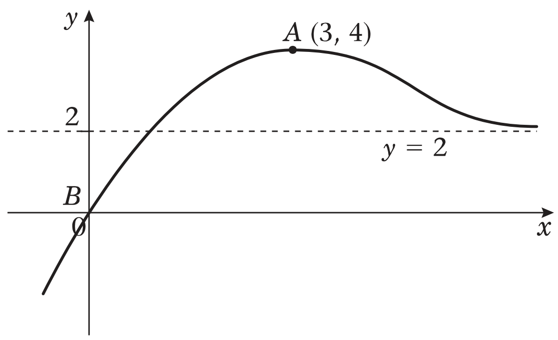
2 **a** On the same axes sketch the curves with equations $y = \frac{6}{x}$ and $y = 1 + x$.

b The curves intersect at the points A and B. Find the coordinates of A and B.

c The curve C with equation $y = x^2 + px + q$, where p and q are integers, passes through A and B. Find the values of p and q.

d Add C to your sketch.

3 The diagram shows a sketch of the curve $y = f(x)$. The point $B(0, 0)$ lies on the curve and the point $A(3, 4)$ is a maximum point. The line $y = 2$ is an asymptote.

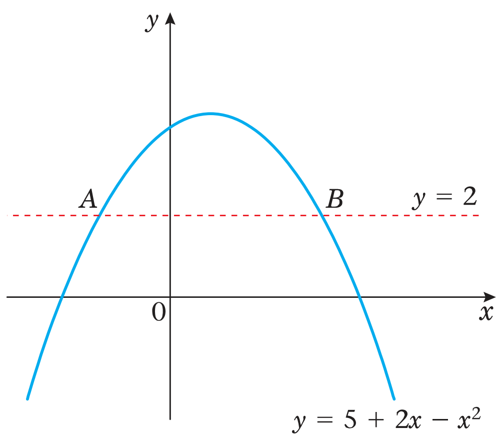


Sketch the following and in each case give the coordinates of the new positions of A and B and state the equation of the asymptote:

a $f(2x)$ **b** $\frac{1}{2}f(x)$ **c** $f(x) - 2$

d $f(x + 3)$ **e** $f(x - 3)$ **f** $f(x) + 1$

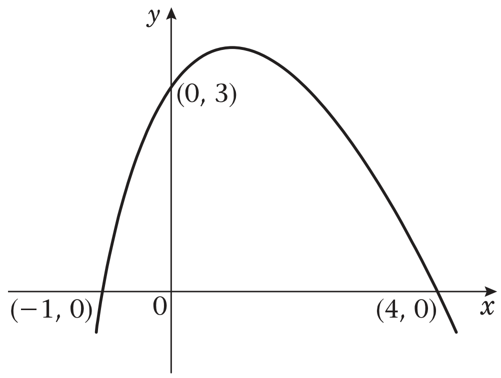
4 The diagram shows the curve with equation $y = 5 + 2x - x^2$ and the line with equation $y = 2$. The curve and the line intersect at the points A and B.



Find the x-coordinates of A and B.

E

5 The curve with equation $y = f(x)$ meets the coordinate axes at the points $(-1, 0)$, $(4, 0)$ and $(0, 3)$, as shown in the diagram.



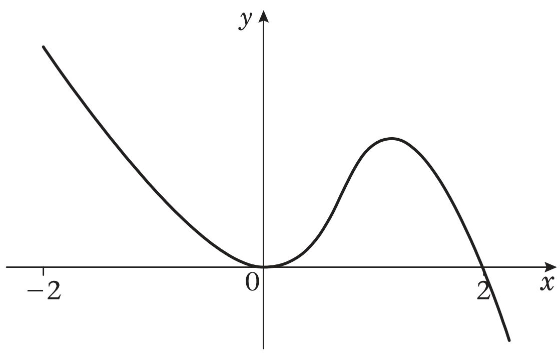
Using a separate diagram for each, sketch the curve with equation

a $y = f(x - 1)$ **b** $y = -f(x)$

On each sketch, write in the coordinates of the points at which the curve meets the coordinate axes.

E

6 The figure shows a sketch of the curve with equation $y = f(x)$.



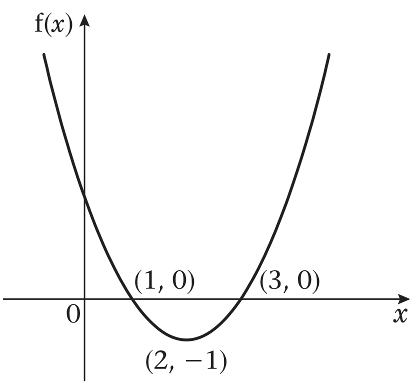
In separate diagrams show, for $-2 \leqslant x \leqslant 2$, sketches of the curves with equation:

a $y = f(-x)$ **b** $y = -f(x)$

Mark on each sketch the x-coordinate of any point, or points, where a curve touches or crosses the x-axis.

E

7 The diagram shows the graph of the quadratic function f. The graph meets the x-axis at $(1, 0)$ and $(3, 0)$ and the minimum point is $(2, -1)$.



a Find the equation of the graph in the form $y = f(x)$.

b On separate axes, sketch the graphs of

i $y = f(x + 2)$ **ii** $y = f(2x)$.

c On each graph write in the coordinates of the points at which the graph meets the x-axis and write in the coordinates of the minimum point.

E

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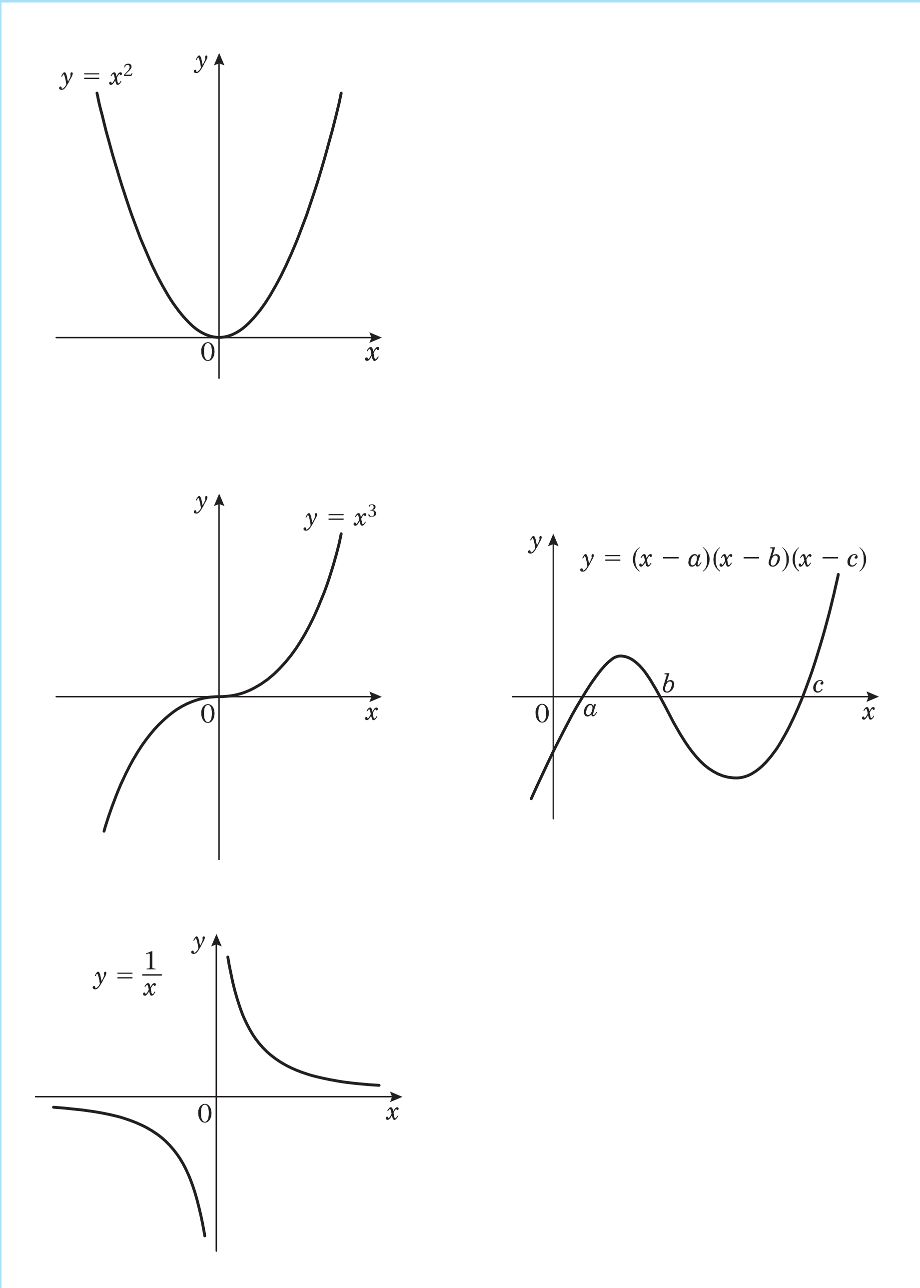
Summary of key points

1 You should know the shapes of the following basic curves.



2 Transformations:

$f(x + a)$ is a translation of $-a$ in the x-direction.

$f(x) + a$ is a translation of $+a$ in the y-direction.

$f(ax)$ is a stretch of $\frac{1}{a}$ in the x-direction $\left(\text{multiply } x\text{-coordinates by } \frac{1}{a}\right)$.

$af(x)$ is a stretch of a in the y-direction (multiply y-coordinates by a).