After completing this chapter you should be able to

1. understand the link between the equation of a line, and its gradient and intercept
2. calculate the gradient of a line joining a pair of points
3. find the equation of a line in either the form $y = mx + c$ or alternatively $ax + by = c$
4. find the equation of a line passing through a pair of points
5. determine the point where a pair of straight lines intersect
6. know and use the rule concerning perpendicular gradients.

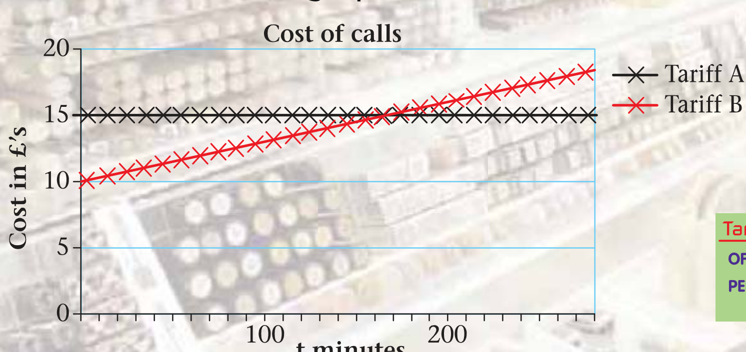
Understanding this chapter will help you find the equation of a tangent and normal to a curve in Chapter 7.

5 Coordinate geometry in the (x, y) plane

Did you know?

...that many bills (including mobile phones) are linear and will produce straight lines when they are graphed?

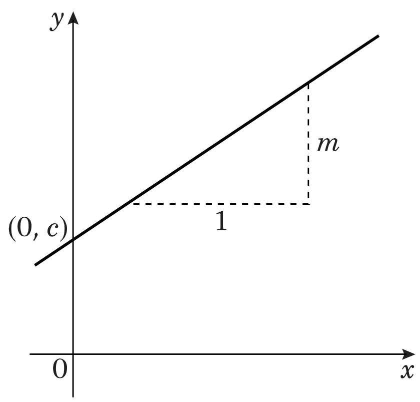
The problem below can easily be answered by solving where pair of straight lines intersect.
If C = the cost of the calls in £s and t = the time in minutes, then the graphs are



Can you work out when it would be cheaper to use Tariff A? You may wish to work out if your mobile phone contract is the most suitable one for the number of calls you make.

5.1 You can write the equation of a straight line in the form $y = mx + c$ or $ax + by + c = 0$.

- **In the general form $y = mx + c$, m is the gradient and $(0, c)$ is the intercept on the y-axis.**



- **In the general form $ax + by + c = 0$, a, b and c are integers.**

Example 1

Write down the gradient and intercept on the y-axis of these lines:

a $y = -3x + 2$

b $4x - 2y + 5 = 0$

a $y = -3x + 2$

The gradient $= -3$ and the intercept on the y-axis $= (0, 2)$.

Compare $y = -3x + 2$ with $y = mx + c$.
From this, $m = -3$ and $c = 2$.

b $4x - 2y + 5 = 0$

$4x + 5 = 2y$

So $2y = 4x + 5$

$y = 2x + \frac{5}{2}$

The gradient $= 2$ and the intercept on the y-axis $= (0, \frac{5}{2})$.

Rearrange the equation into the form $y = mx + c$.

Add $2y$ to each side.

Put the term in y at the front of the equation.

Divide each term by 2, so that:

$2y \div 2 = y$
$4 \div 2 = 2$
$5 \div 2 = \frac{5}{2}$. (Do not write this as 2.5)

Compare $y = 2x + \frac{5}{2}$ to $y = mx + c$.
From this, $m = 2$ and $c = \frac{5}{2}$.

Example 2

Write these lines in the form $ax + by + c = 0$:

a $y = 4x + 3$ **b** $y = -\frac{1}{2}x + 5$

a
$y = 4x + 3$
$0 = 4x + 3 - y$ — Rearrange the equation into the form $ax + by + c = 0$. Subtract y from each side.
So $4x - y + 3 = 0$

b
$y = -\frac{1}{2}x + 5$ — Collect all the terms on one side of the equation.
$\frac{1}{2}x + y = 5$ — Add $\frac{1}{2}x$ to each side.
$\frac{1}{2}x + y - 5 = 0$ — Subtract 5 from each side.
So $x + 2y - 10 = 0$ — Multiply each term by 2 to clear the fraction.

Example 3

A line is parallel to the line $y = \frac{1}{2}x - 5$ and its intercept on the y-axis is (0, 1). Write down the equation of the line.

$y = \frac{1}{2}x + 1$

Remember that parallel lines have the same gradient.

Compare $y = \frac{1}{2}x - 5$ with $y = mx + c$, so $m = \frac{1}{2}$.

The gradient of the required line $= \frac{1}{2}$.

The intercept on the y-axis is (0, 1), so $c = 1$.

Example 4

A line is parallel to the line $6x + 3y - 2 = 0$ and it passes through the point (0, 3). Work out the equation of the line.

$6x + 3y - 2 = 0$ — Rearrange the equation into the form $y = mx + c$ to find m.
$3y - 2 = -6x$ — Subtract $6x$ from each side.
$3y = -6x + 2$ — Add 2 to each side.
$y = -2x + \frac{2}{3}$ — Divide each term by 3, so that
$3y \div 3 = y$
$-6x \div 3 = -2x$
$2 \div 3 = \frac{2}{3}$. (Do not write this as a decimal.)

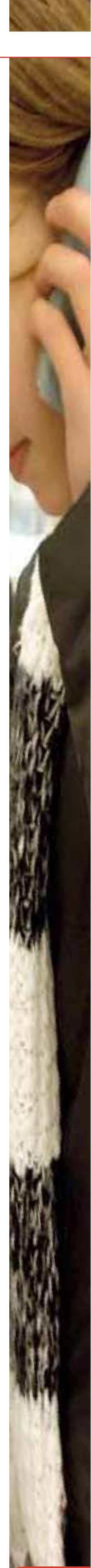
The gradient of this line is -2.

Compare $y = -2x + \frac{2}{3}$ with $y = mx + c$, so $m = -2$.

Parallel lines have the same gradient, so the gradient of the required line $= -2$.

The equation of the line is $y = -2x + 3$.

(0, 3) is the intercept on the y-axis, so $c = 3$.



Example 5

The line $y = 4x - 8$ meets the x-axis at the point P. Work out the coordinates of P.

$y = 4x - 8$

Substituting,

$4x - 8 = 0$

$4x = 8$ — Add 8 to each side.

$x = 2$ — Divide each side by 4.

So $P(2, 0)$.

The line meets the x-axis when $y = 0$, so substitute $y = 0$ into $y = 4x - 8$.

Rearrange the equation for x.

Add 8 to each side.

Divide each side by 4.

Always write down the coordinates of the point.

Exercise 5A

1 Work out the gradients of these lines:

a $y = -2x + 5$ **b** $y = -x + 7$ **c** $y = 4 + 3x$

d $y = \frac{1}{3}x - 2$ **e** $y = -\frac{2}{3}x$ **f** $y = \frac{5}{4}x + \frac{2}{3}$

g $2x - 4y + 5 = 0$ **h** $10x - 5y + 1 = 0$ **i** $-x + 2y - 4 = 0$

j $-3x + 6y + 7 = 0$ **k** $4x + 2y - 9 = 0$ **l** $9x + 6y + 2 = 0$

2 These lines intercept the y-axis at $(0, c)$. Work out the value of c in each case.

a $y = -x + 4$ **b** $y = 2x - 5$ **c** $y = \frac{1}{2}x - \frac{2}{3}$

d $y = -3x$ **e** $y = \frac{6}{7}x + \frac{7}{5}$ **f** $y = 2 - 7x$

g $3x - 4y + 8 = 0$ **h** $4x - 5y - 10 = 0$ **i** $-2x + y - 9 = 0$

j $7x + 4y + 12 = 0$ **k** $7x - 2y + 3 = 0$ **l** $-5x + 4y + 2 = 0$

3 Write these lines in the form $ax + by + c = 0$.

a $y = 4x + 3$ **b** $y = 3x - 2$ **c** $y = -6x + 7$

d $y = \frac{4}{5}x - 6$ **e** $y = \frac{5}{3}x + 2$ **f** $y = \frac{7}{3}x$

g $y = 2x - \frac{4}{7}$ **h** $y = -3x + \frac{2}{9}$ **i** $y = -6x - \frac{2}{3}$

j $y = -\frac{1}{3}x + \frac{1}{2}$ **k** $y = \frac{2}{3}x + \frac{5}{6}$ **l** $y = \frac{3}{5}x + \frac{1}{2}$

4 A line is parallel to the line $y = 5x + 8$ and its intercept on the y-axis is $(0, 3)$. Write down the equation of the line.

5 A line is parallel to the line $y = -\frac{2}{5}x + 1$ and its intercept on the y-axis is $(0, -4)$. Work out the equation of the line. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

6 A line is parallel to the line $3x + 6y + 11 = 0$ and its intercept on the y-axis is $(0, 7)$. Write down the equation of the line.

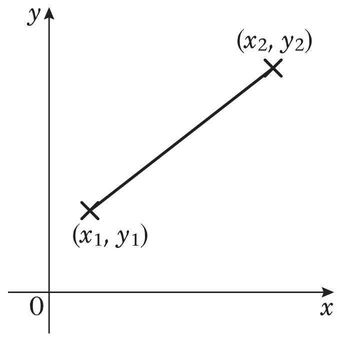
7 A line is parallel to the line $2x - 3y - 1 = 0$ and it passes through the point $(0, 0)$. Write down the equation of the line.

8 The line $y = 6x - 18$ meets the x-axis at the point P. Work out the coordinates of P.

9 The line $3x + 2y - 5 = 0$ meets the x-axis at the point R. Work out the coordinates of R.

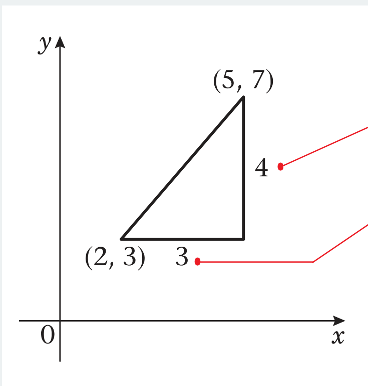
10 The line $5x - 4y + 20 = 0$ meets the y-axis at the point A and the x-axis at the point B. Work out the coordinates of the points A and B.

5.2 **You can work out the gradient m of the line joining the point with coordinates (x_1, y_1) to the point with coordinates (x_2, y_2) by using the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.**



Example 6

Work out the gradient of the line joining the points (2, 3) and (5, 7).



The gradient of the line is $\frac{4}{3}$.

Draw a sketch.
$7 - 3 = 4$
$5 - 2 = 3$
Remember the gradient of a line
$= \dfrac{\text{difference in } y\text{-coordinates}}{\text{difference in } x\text{-coordinates}}$,
so $m = \dfrac{7-3}{5-2}$.
This is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ with $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (5, 7)$.

Example 7

Work out the gradient of the line joining these pairs of points:

a $(-2, 7)$ and $(4, 5)$ **b** $(2d, -5d)$ and $(6d, 3d)$

a $m = \dfrac{5-7}{4-(-2)}$

$= \dfrac{-2}{6}$

$= -\dfrac{1}{3}$

The gradient of the line is $-\dfrac{1}{3}$.

Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Here $(x_1, y_1) = (-2, 7)$ and $(x_2, y_2) = (4, 5)$.

$-(-2) = +2$, so $4 + 2 = 6$

Remember to simplify the fraction when possible, so divide by 2.

$\dfrac{-1}{3}$ is the same as $-\dfrac{1}{3}$.

b $m = \dfrac{3d - (-5d)}{6d - 2d}$ — Here $(x_1, y_1) = (2d, -5d)$ and $(x_2, y_2) = (6d, 3d)$.

$= \dfrac{8d}{4d}$ — $-(-5d) = +5d$, so $3d + 5d = 8d$.

$= 2$ — $8d \div 4d = 2$.

The gradient of the line is 2.

Example 8

The line joining (2, −5) to (4, *a*) has gradient −1. Work out the value of *a*.

$\dfrac{a - (-5)}{4 - 2} = -1$ — Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Here $m = -1$, $(x_1, y_1) = (2, -5)$ and $(x_2, y_2) = (4, a)$.

So $\dfrac{a + 5}{2} = -1$ — $a - (-5) = a + 5$

$a + 5 = -2$ — Multiply each side of the equation by 2 to clear the fraction.

$a = -7$ — Subtract 5 from each side of the equation.

Exercise 5B

1 Work out the gradient of the line joining these pairs of points:

a (4, 2), (6, 3) **b** (−1, 3), (5, 4)

c (−4, 5), (1, 2) **d** (2, −3), (6, 5)

e (−3, 4), (7, −6) **f** (−12, 3), (−2, 8)

g (−2, −4), (10, 2) **h** $(\frac{1}{2}, 2)$, $(\frac{3}{4}, 4)$

i $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, \frac{2}{3})$ **j** (−2.4, 9.6), (0, 0)

k (1.3, −2.2), (8.8, −4.7) **l** (0, 5*a*), (10*a*, 0)

m (3*b*, −2*b*), (7*b*, 2*b*) **n** (p, p^2), (q, q^2)

2 The line joining (3, −5) to (6, *a*) has gradient 4. Work out the value of *a*.

3 The line joining (5, *b*) to (8, 3) has gradient −3. Work out the value of *b*.

4 The line joining (*c*, 4) to (7, 6) has gradient $\frac{3}{4}$. Work out the value of *c*.

5 The line joining (−1, 2*d*) to (1, 4) has gradient $-\frac{1}{4}$. Work out the value of *d*.

6 The line joining (−3, −2) to (2*e*, 5) has gradient 2. Work out the value of *e*.

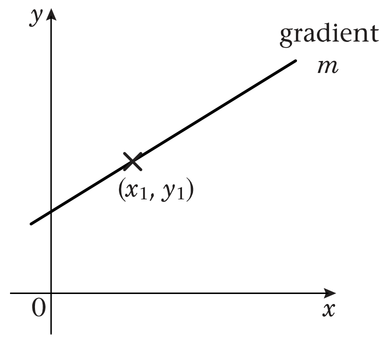
7 The line joining (7, 2) to (*f*, 3*f*) has gradient 4. Work out the value of *f*.

8 The line joining $(3, -4)$ to $(-g, 2g)$ has gradient -3. Work out the value of g.

9 Show that the points $A(2, 3)$, $B(4, 4)$, $C(10, 7)$ can be joined by a straight line.
(Hint: Find the gradient of the lines joining the points: **i** A and B and **ii** A and C.)

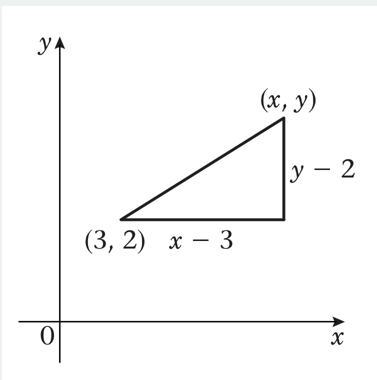
10 Show that the points $(-2a, 5a)$, $(0, 4a)$, $(6a, a)$ are collinear (i.e. on the same straight line).

5.3 **You can find the equation of a line with gradient m that passes through the point with coordinates (x_1, y_1) by using the formula $y - y_1 = m(x - x_1)$.**



Example 9

Find the equation of the line with gradient 5 that passes through the point (3, 2).



(x, y) is *any* point on the line.

The gradient $= 5$, so $\dfrac{y-2}{x-3} = 5$.

Multiply each side of the equation by $x - 3$ to clear the fraction, so that:

$$\frac{y-2}{x-3} \times \frac{x-3}{1} = y - 2$$

$$5 \times (x - 3) = 5(x - 3)$$

$y - 2 = 5(x - 3)$ — This is in the form $y - y_1 = m(x - x_1)$. Here $m = 5$ and $(x_1, y_1) = (3, 2)$.

$y - 2 = 5x - 15$ — Expand the brackets.

$y = 5x - 13$ — Add 2 to each side.

Example 10

Find the equation of the line with gradient $-\frac{1}{2}$ that passes through the point $(4, -6)$.

$y - (-6) = -\frac{1}{2}(x - 4)$ — Use $y - y_1 = m(x - x_1)$. Here $m = -\frac{1}{2}$ and $(x_1, y_1) = (4, -6)$.

So $y + 6 = -\frac{1}{2}(x - 4)$

$y + 6 = -\frac{1}{2}x + 2$ — Expand the brackets. Remember $-\frac{1}{2} \times -4 = +2$.

$y = -\frac{1}{2}x - 4$ — Subtract 6 from each side.

Example 11

The line $y = 3x - 9$ meets the x-axis at the point A. Find the equation of the line with gradient $\frac{2}{3}$ that passes through the point A. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

Working	Explanation
$y = 3x - 9$	
$3x - 9 = 0$	The line meets the x-axis when $y = 0$, so substitute $y = 0$ into $y = 3x - 9$.
$3x = 9$	Rearrange the equation to find x.
$x = 3$	
So $A(3, 0)$.	Always write down the coordinates of the point.
$y - 0 = \frac{2}{3}(x - 3)$	Use $y - y_1 = m(x - x_1)$. Here $m = \frac{2}{3}$ and $(x_1, y_1) = (3, 0)$.
$y = \frac{2}{3}(x - 3)$	Rearrange the equation into the form $ax + by + c = 0$.
$3y = 2(x - 3)$	Multiply by 3 to clear the fraction.
$3y = 2x - 6$	Expand the brackets.
$-2x + 3y = -6$	Subtract $2x$ from each side.
$-2x + 3y + 6 = 0$	Add 6 to each side.

Exercise 5C

1 Find the equation of the line with gradient m that passes through the point (x_1, y_1) when:

a $m = 2$ and $(x_1, y_1) = (2, 5)$
b $m = 3$ and $(x_1, y_1) = (-2, 1)$
c $m = -1$ and $(x_1, y_1) = (3, -6)$
d $m = -4$ and $(x_1, y_1) = (-2, -3)$
e $m = \frac{1}{2}$ and $(x_1, y_1) = (-4, 10)$
f $m = -\frac{2}{3}$ and $(x_1, y_1) = (-6, -1)$
g $m = 2$ and $(x_1, y_1) = (a, 2a)$
h $m = -\frac{1}{2}$ and $(x_1, y_1) = (-2b, 3b)$

2 The line $y = 4x - 8$ meets the x-axis at the point A. Find the equation of the line with gradient 3 that passes through the point A.

3 The line $y = -2x + 8$ meets the y-axis at the point B. Find the equation of the line with gradient 2 that passes through the point B.

4 The line $y = \frac{1}{2}x + 6$ meets the x-axis at the point C. Find the equation of the line with gradient $\frac{2}{3}$ that passes through the point C. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

5 The line $y = \frac{1}{4}x + 2$ meets the y-axis at the point B. The point C has coordinates $(-5, 3)$. Find the gradient of the line joining the points B and C.

6 The lines $y = x$ and $y = 2x - 5$ intersect at the point A. Find the equation of the line with gradient $\frac{2}{5}$ that passes through the point A. (Hint: Solve $y = x$ and $y = 2x - 5$ simultaneously.)

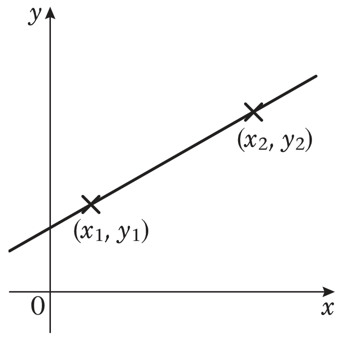
7 The lines $y = 4x - 10$ and $y = x - 1$ intersect at the point T. Find the equation of the line with gradient $-\frac{2}{3}$ that passes through the point T. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

8 The line p has gradient $\frac{2}{3}$ and passes through the point $(6, -12)$. The line q has gradient -1 and passes through the point $(5, 5)$. The line p meets the y-axis at A and the line q meets the x-axis at B. Work out the gradient of the line joining the points A and B.

9 The line $y = -2x + 6$ meets the x-axis at the point P. The line $y = \frac{3}{2}x - 4$ meets the y-axis at the point Q. Find the equation of the line joining the points P and Q. (Hint: First work out the gradient of the line joining the points P and Q.)

10 The line $y = 3x - 5$ meets the x-axis at the point M. The line $y = -\frac{2}{3}x + \frac{2}{3}$ meets the y-axis at the point N. Find the equation of the line joining the points M and N. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

5.4 **You can find the equation of the line that passes through the points with coordinates (x_1, y_1) and (x_2, y_2) by using the formula $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$.**



Example 12

Work out the gradient of the line that passes through the points $(5, 7)$ and $(3, -1)$ and hence find the equation of the line.

$m = \dfrac{(-1) - 7}{3 - 5}$ — Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Here $(x_1, y_1) = (5, 7)$ and $(x_2, y_2) = (3, -1)$.

$= \dfrac{-8}{-2}$

So $m = 4$. — $-8 \div -2 = +4$

$y - 7 = 4(x - 5)$ — Use $y - y_1 = m(x - x_1)$. Here $m = 4$ and $(x_1, y_1) = (5, 7)$.

$y - 7 = 4x - 20$ — Expand the brackets.

$y = 4x - 13$ — Simplify into the form $y = mx + c$. Add 7 to each side.

Example 13

Use $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ to find the equation of the line that passes through the points $(5, 7)$ and $(3, -1)$.

$$\frac{y-(-1)}{7-(-1)} = \frac{x-3}{5-3}$$

Use $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$.

Here $(x_1, y_1) = (3, -1)$ and $(x_2, y_2) = (5, 7)$.

(x_1, y_1) and (x_2, y_2) have been chosen to make the denominators positive.

So $$\frac{y+1}{8} = \frac{x-3}{2}$$

$$y + 1 = 4(x-3)$$

Multiply each side by 8 to clear the fraction, so that:

$$8 \times \frac{y+1}{8} = y + 1$$

$$8 \times \frac{x-3}{2} = 4(x-3)$$

$$y + 1 = 4x - 12$$

Expand the brackets.

$$y = 4x - 13$$

Subtract 1 from each side.

Example 14

The lines $y = 4x - 7$ and $2x + 3y - 21 = 0$ intersect at the point A. The point B has coordinates $(-2, 8)$. Find the equation of the line that passes through the points A and B. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

$y = 4x - 7,\ 2x + 3y - 21 = 0$

Solve the equations $y = 4x - 7$ and $2x + 3y - 21 = 0$ simultaneously to find the point A.

$2x + 3(4x - 7) - 21 = 0$

Substitute $y = 4x - 7$ into $2x + 3y - 21 = 0$ to eliminate y.

$2x + 12x - 21 - 21 = 0$

Expand the brackets.

$14x - 42 = 0$

Collect like terms.

$14x = 42$

Add 42 to each side.

$x = 3$

Divide each term by 14.

Substituting,

$y = 4(3) - 7$

Substitute $x = 3$ into either equation to find y. $y = 4x - 7$ is easier.

$y = 5$

So $A(3, 5)$.

Write down the coordinates of A.

$A(3, 5)$ and $B(-2, 8)$

$$\frac{y-5}{8-5} = \frac{x-3}{-2-3}$$

Use $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$. Here $(x_1, y_1) = (3, 5)$ and $(x_2, y_2) = (-2, 8)$.

$$\frac{y-5}{3} = \frac{x-3}{-5}$$

Simplify the denominators.

$5(y - 5) = -3(x - 3)$

Clear the fraction. Multiply each side by 15 so that

$$15 \times \frac{y-5}{3} = 5(y-5)$$

$$15 \times \frac{x-3}{-5} = -3(x-3)$$

$5y - 25 = -3x + 9$

Expand the brackets.

$-3 \times -3 = +9$

$3x + 5y - 25 = 9$

Add $3x$ to each side.

$3x + 5y - 34 = 0$

Subtract 9 from each side.

Exercise 5D

1 Find the equation of the line that passes through these pairs of points:

a $(2, 4)$ and $(3, 8)$

b $(0, 2)$ and $(3, 5)$

c $(-2, 0)$ and $(2, 8)$

d $(5, -3)$ and $(7, 5)$

e $(3, -1)$ and $(7, 3)$

f $(-4, -1)$ and $(6, 4)$

g $(-1, -5)$ and $(-3, 3)$

h $(-4, -1)$ and $(-3, -9)$

i $(\frac{1}{3}, \frac{2}{5})$ and $(\frac{2}{3}, \frac{4}{5})$

j $(-\frac{3}{4}, \frac{1}{7})$ and $(\frac{1}{4}, \frac{3}{7})$

2 The line that passes through the points $(2, -5)$ and $(-7, 4)$ meets the x-axis at the point P. Work out the coordinates of the point P.

3 The line that passes through the points $(-3, -5)$ and $(4, 9)$ meets the y-axis at the point G. Work out the coordinates of the point G.

4 The line that passes through the points $(3, 2\frac{1}{2})$ and $(-1\frac{1}{2}, 4)$ meets the y-axis at the point J. Work out the coordinates of the point J.

5 The line $y = 2x - 10$ meets the x-axis at the point A. The line $y = -2x + 4$ meets the y-axis at the point B. Find the equation of the line joining the points A and B. (Hint: First work out the coordinates of the points A and B.)

6 The line $y = 4x + 5$ meets the y-axis at the point C. The line $y = -3x - 15$ meets the x-axis at the point D. Find the equation of the line joining the points C and D. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

7 The lines $y = x - 5$ and $y = 3x - 13$ intersect at the point S. The point T has coordinates $(-4, 2)$. Find the equation of the line that passes through the points S and T.

8 The lines $y = -2x + 1$ and $y = x + 7$ intersect at the point L. The point M has coordinates $(-3, 1)$. Find the equation of the line that passes through the points L and M.

9 The vertices of the triangle ABC have coordinates $A(3, 5)$, $B(-2, 0)$ and $C(4, -1)$. Find the equations of the sides of the triangle.

10 The line V passes through the points $(-5, 3)$ and $(7, -3)$ and the line W passes through the points $(2, -4)$ and $(4, 2)$. The lines V and W intersect at the point A. Work out the coordinates of the point A.

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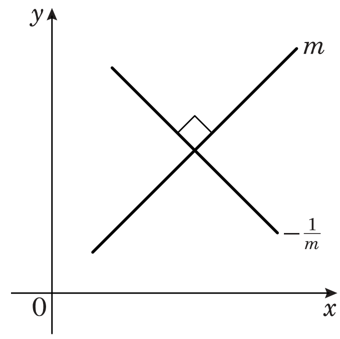
5.5 You can work out the gradient of a line that is perpendicular to the line $y = mx + c$.



- **If a line has a gradient of *m*, a line perpendicular to it has a gradient of $-\frac{1}{m}$.**
- **If two lines are perpendicular, the product of their gradients is −1.**

Example 15

Work out the gradient of the line that is perpendicular to the lines with these gradients:

a 3 **b** $\frac{1}{2}$ **c** $-\frac{2}{5}$

a $m = 3$

So the gradient of the perpendicular line is $-\frac{1}{3}$.

Use $-\frac{1}{m}$ with $m = 3$.

b $m = \frac{1}{2}$

So the gradient of the perpendicular line is

$$-\frac{1}{\left(\frac{1}{2}\right)}$$

$$= -\frac{2}{1}$$

$$= -2$$

Use $-\frac{1}{m}$ with $m = \frac{1}{2}$.

Remember $\frac{1}{\left(\frac{a}{b}\right)} = \frac{b}{a}$, so $\frac{1}{\left(\frac{1}{2}\right)} = \frac{2}{1}$.

c $m = -\frac{2}{5}$

So the gradient of the perpendicular line is

$$-\frac{1}{\left(-\frac{2}{5}\right)}$$

$$= -\left(-\frac{5}{2}\right)$$

$$= \frac{5}{2}$$

Use $-\frac{1}{m}$ with $m = -\frac{2}{5}$.

Here $\frac{1}{\left(\frac{2}{5}\right)} = \frac{5}{2}$, so $\frac{1}{\left(-\frac{2}{5}\right)} = -\frac{5}{2}$.

$-1 \times -\frac{5}{2} = +\frac{5}{2}$

Example 16

Show that the line $y = 3x + 4$ is perpendicular to the line $x + 3y - 3 = 0$.

$y = 3x + 4$

The gradient of this line is 3.

Compare $y = 3x + 4$ with $y = mx + c$, so $m = 3$.

$x + 3y - 3 = 0$

Rearrange the equation into the form $y = mx + c$ to find m.

$3y - 3 = -x$

Subtract x from each side.

$3y = -x + 3$

Add 3 to each side.

$y = -\frac{1}{3}x + 1$

Divide each term by 3.

$-x \div 3 = \frac{-x}{3} = -\frac{1}{3}x$.

The gradient of this line is $-\frac{1}{3}$.

Compare $y = -\frac{1}{3}x + 1$ with $y = mx + c$, so $m = -\frac{1}{3}$.

$3 \times -\frac{1}{3} = -1$

Multiply the gradients of the lines.

The lines are perpendicular because the product of their gradients is -1.

Example 17

Work out whether these pairs of lines are parallel, perpendicular or neither:

a $y = -2x + 9$
$y = -2x - 3$

b $3x - y - 2 = 0$
$x + 3y - 6 = 0$

c $y = \frac{1}{2}x$
$2x - y + 4 = 0$

a $y = -2x + 9$

The gradient of this line is -2.

Compare $y = -2x + 9$ with $y = mx + c$, so $m = -2$.

$y = -2x - 3$

Compare $y = -2x - 3$ with $y = mx + c$, so $m = -2$.

The gradient of this line is -2.

So the lines are parallel, since the gradients are equal.

Remember that parallel lines have the same gradient.

b $3x - y - 2 = 0$

Rearrange the equation into the form $y = mx + c$.

$3x - 2 = y$

Add y to each side.

So $y = 3x - 2$

The gradient of this line is 3.

Compare $y = 3x - 2$ with $y = mx + c$, so $m = 3$.

$x + 3y - 6 = 0$

$3y - 6 = -x$

Subtract x from each side.

$3y = -x + 6$

Add 6 to each side.

$y = -\frac{1}{3}x + 2$

Divide each term by 3.

The gradient of this line is $-\frac{1}{3}$.

Compare $y = -\frac{1}{3}x + 2$ with $y = mx + c$, so $m = -\frac{1}{3}$.

So the lines are perpendicular as $3 \times \frac{1}{3} = -1$.

c $y = \frac{1}{2}x$ — Compare $y = \frac{1}{2}x$ with $y = mx + c$, so $m = \frac{1}{2}$.

The gradient of this line is $\frac{1}{2}$.

$2x - y + 4 = 0$ — Rearrange the equation into the form $y = mx + c$ to find m.

$2x + 4 = y$ — Add y to each side.

So $y = 2x + 4$

The gradient of this line is 2. — Compare $y = 2x + 4$ with $y = mx + c$, so $m = 2$.

The lines are not parallel as they have different gradients.

The lines are not perpendicular as $\frac{1}{2} \times 2 = 1$.

Example 18

Find an equation of the line that passes through the point $(3, -1)$ and is perpendicular to the line $y = 2x - 4$.

$y = 2x - 4$ — Compare $y = 2x - 4$ with $y = mx + c$.

$m = 2$

So the gradient of the perpendicular line is $-\frac{1}{2}$. — Use the rule $-\frac{1}{m}$ with $m = 2$.

$y - (-1) = -\frac{1}{2}(x - 3)$ — Use $y - y_1 = m(x - x_1)$. Here $m = -\frac{1}{2}$ and $(x_1, y_1) = (3, -1)$.

$y + 1 = -\frac{1}{2}x + \frac{3}{2}$ — Expand the brackets. $-\frac{1}{2} \times -3 = \frac{3}{2}$

$y = -\frac{1}{2}x + \frac{1}{2}$ — Subtract 1 from each side, so that $\frac{3}{2} - 1 = \frac{1}{2}$.

Exercise 5E

1 Work out whether these pairs of lines are parallel, perpendicular or neither:

a $y = 4x + 2$, $y = -\frac{1}{4}x - 7$

b $y = \frac{2}{3}x - 1$, $y = \frac{2}{3}x - 11$

c $y = \frac{1}{5}x + 9$, $y = 5x + 9$

d $y = -3x + 2$, $y = \frac{1}{3}x - 7$

e $y = \frac{3}{5}x + 4$, $y = -\frac{5}{3}x - 1$

f $y = \frac{5}{7}x$, $y = \frac{5}{7}x - 3$

g $y = 5x - 3$, $5x - y + 4 = 0$

h $5x - y - 1 = 0$, $y = -\frac{1}{5}x$

i $y = -\frac{3}{2}x + 8$, $2x - 3y - 9 = 0$

j $4x - 5y + 1 = 0$, $8x - 10y - 2 = 0$

k $3x + 2y - 12 = 0$, $2x + 3y - 6 = 0$

l $5x - y + 2 = 0$, $2x + 10y - 4 = 0$

2 Find an equation of the line that passes through the point $(6, -2)$ and is perpendicular to the line $y = 3x + 5$.

3 Find an equation of the line that passes through the point $(-2, 7)$ and is parallel to the line $y = 4x + 1$. Write your answer in the form $ax + by + c = 0$.

4 Find an equation of the line:

a parallel to the line $y = -2x - 5$, passing through $(-\frac{1}{2}, \frac{3}{2})$

b parallel to the line $x - 2y - 1 = 0$, passing through $(0, 0)$

c perpendicular to the line $y = x - 4$, passing through $(-1, -2)$

d perpendicular to the line $2x + y - 9 = 0$, passing through $(4, -6)$.

5 Find an equation of the line:

a parallel to the line $y = 3x + 6$, passing through $(-2, 5)$

b perpendicular to the line $y = 3x + 6$, passing through $(-2, 5)$

c parallel to the line $4x - 6y + 7 = 0$, passing through $(3, 4)$

d perpendicular to the line $4x - 6y + 7 = 0$, passing through $(3, 4)$.

6 Find an equation of the line that passes through the point $(5, -5)$ and is perpendicular to the line $y = \frac{2}{3}x + 5$. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

7 Find an equation of the line that passes through the point $(-2, -3)$ and is perpendicular to the line $y = -\frac{4}{7}x + 5$. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

8 The line r passes through the points $(1, 4)$ and $(6, 8)$ and the line s passes through the points $(5, -3)$ and $(20, 9)$. Show that the lines r and s are parallel.

9 The line l passes through the points $(-3, 0)$ and $(3, -2)$ and the line n passes through the points $(1, 8)$ and $(-1, 2)$. Show that the lines l and n are perpendicular.

10 The vertices of a quadrilateral $ABCD$ has coordinates $A(-1, 5)$, $B(7, 1)$, $C(5, -3)$, $D(-3, 1)$. Show that the quadrilateral is a rectangle.

Mixed exercise 5F

1 The points A and B have coordinates $(-4, 6)$ and $(2, 8)$ respectively. A line p is drawn through B perpendicular to AB to meet the y-axis at the point C.

a Find an equation of the line p.

b Determine the coordinates of C.

E

2 The line l has equation $2x - y - 1 = 0$.
The line m passes through the point $A(0, 4)$ and is perpendicular to the line l.

a Find an equation of m and show that the lines l and m intersect at the point $P(2, 3)$.

The line n passes through the point $B(3, 0)$ and is parallel to the line m.

b Find an equation of n and hence find the coordinates of the point Q where the lines l and n intersect.

3 The line L_1 has gradient $\frac{1}{7}$ and passes through the point $A(2, 2)$. The line L_2 has gradient -1 and passes through the point $B(4, 8)$. The lines L_1 and L_2 intersect at the point C.

a Find an equation for L_1 and an equation for L_2.

b Determine the coordinates of C.

4 The straight line passing through the point $P(2, 1)$ and the point $Q(k, 11)$ has gradient $-\frac{5}{12}$.

a Find the equation of the line in terms of x and y only.

b Determine the value of k.

5 **a** Find an equation of the line l which passes through the points $A(1, 0)$ and $B(5, 6)$.

The line m with equation $2x + 3y = 15$ meets l at the point C.

b Determine the coordinates of the point C.

6 The line L passes through the points $A(1, 3)$ and $B(-19, -19)$.
Find an equation of L in the form $ax + by + c = 0$, where a, b and c are integers.

7 The straight line l_1 passes through the points A and B with coordinates $(2, 2)$ and $(6, 0)$ respectively.

a Find an equation of l_1.

The straight line l_2 passes through the point C with coordinates $(-9, 0)$ and has gradient $\frac{1}{4}$.

b Find an equation of l_2.

8 The straight line l_1 passes through the points A and B with coordinates $(0, -2)$ and $(6, 7)$ respectively.

a Find the equation of l_1 in the form $y = mx + c$.

The straight line l_2 with equation $x + y = 8$ cuts the y-axis at the point C. The lines l_1 and l_2 intersect at the point D.

b Calculate the coordinates of the point D.

c Calculate the area of $\triangle ACD$.

9 The points A and B have coordinates $(2, 16)$ and $(12, -4)$ respectively. A straight line l_1 passes through A and B.

a Find an equation for l_1 in the form $ax + by = c$.

The line l_2 passes through the point C with coordinates $(-1, 1)$ and has gradient $\frac{1}{3}$.

b Find an equation for l_2.

10 The points $A(-1, -2)$, $B(7, 2)$ and $C(k, 4)$, where k is a constant, are the vertices of $\triangle ABC$. Angle ABC is a right angle.

a Find the gradient of AB.

b Calculate the value of k.

c Find an equation of the straight line passing through B and C. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers. *E*

11 The straight line l passes through $A(1, 3\sqrt{3})$ and $B(2 + \sqrt{3}, 3 + 4\sqrt{3})$.

a Calculate the gradient of l giving your answer as a surd in its simplest form.

b Give the equation of l in the form $y = mx + c$, where constants m and c are surds given in their simplest form.

c Show that l meets the x-axis at the point $C(-2, 0)$. *E*

12 **a** Find an equation of the straight line passing through the points with coordinates $(-1, 5)$ and $(4, -2)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

The line crosses the x-axis at the point A and the y-axis at the point B, and O is the origin.

b Find the area of $\triangle OAB$. *E*

13 The points A and B have coordinates $(k, 1)$ and $(8, 2k - 1)$ respectively, where k is a constant. Given that the gradient of AB is $\frac{1}{3}$:

a show that $k = 2$

b find an equation for the line through A and B. *E*

14 The straight line l_1 has equation $4y + x = 0$.
The straight line l_2 has equation $y = 2x - 3$.

a On the same axes, sketch the graphs of l_1 and l_2. Show clearly the coordinates of all points at which the graphs meet the coordinate axes.

The lines l_1 and l_2 intersect at the point A.

b Calculate, as exact fractions, the coordinates of A.

c Find an equation of the line through A which is perpendicular to l_1. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers. *E*

15 The points A and B have coordinates $(4, 6)$ and $(12, 2)$ respectively.
The straight line l_1 passes through A and B.

a Find an equation for l_1 in the form $ax + by + c = 0$, where a, b and c are integers.

The straight line l_2 passes through the origin and has gradient -4.

b Write down an equation for l_2.

The lines l_1 and l_2 intersect at the point C.

c Find the coordinates of C. *E*

Summary of key points

1 • In the general form

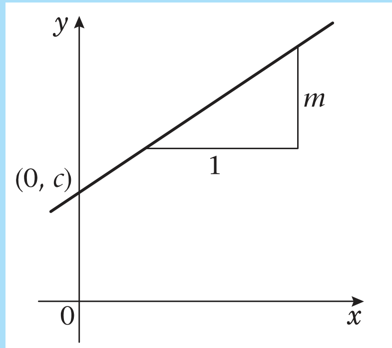
$$y = mx + c,$$

where m is the gradient and $(0, c)$ is the intercept on the y-axis.

• In the general form

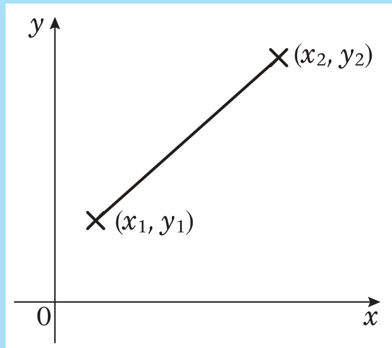
$$ax + by + c = 0,$$

where a, b and c are integers.



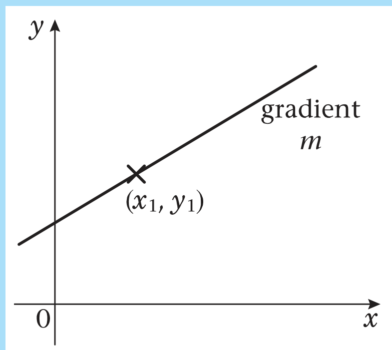
2 You can work out the gradient m of the line joining the point with coordinates (x_1, y_1) to the point with coordinates (x_2, y_2) by using the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



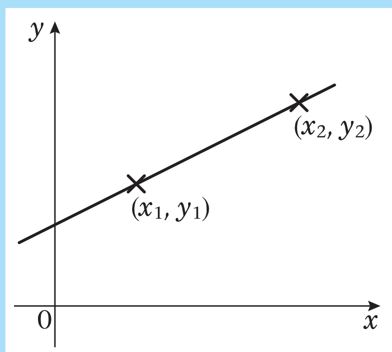
3 You can find the equation of a line with gradient m that passes through the point with coordinates (x_1, y_1) by using the formula

$$y - y_1 = m(x - x_1)$$



4 You can find the equation of the line that passes through the points with coordinates (x_1, y_1) and (x_2, y_2) by using the formula

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$



5 If a line has a gradient m, a line perpendicular to it has a gradient of $-\frac{1}{m}$.

6 If two lines are perpendicular, the product of their gradients is -1.