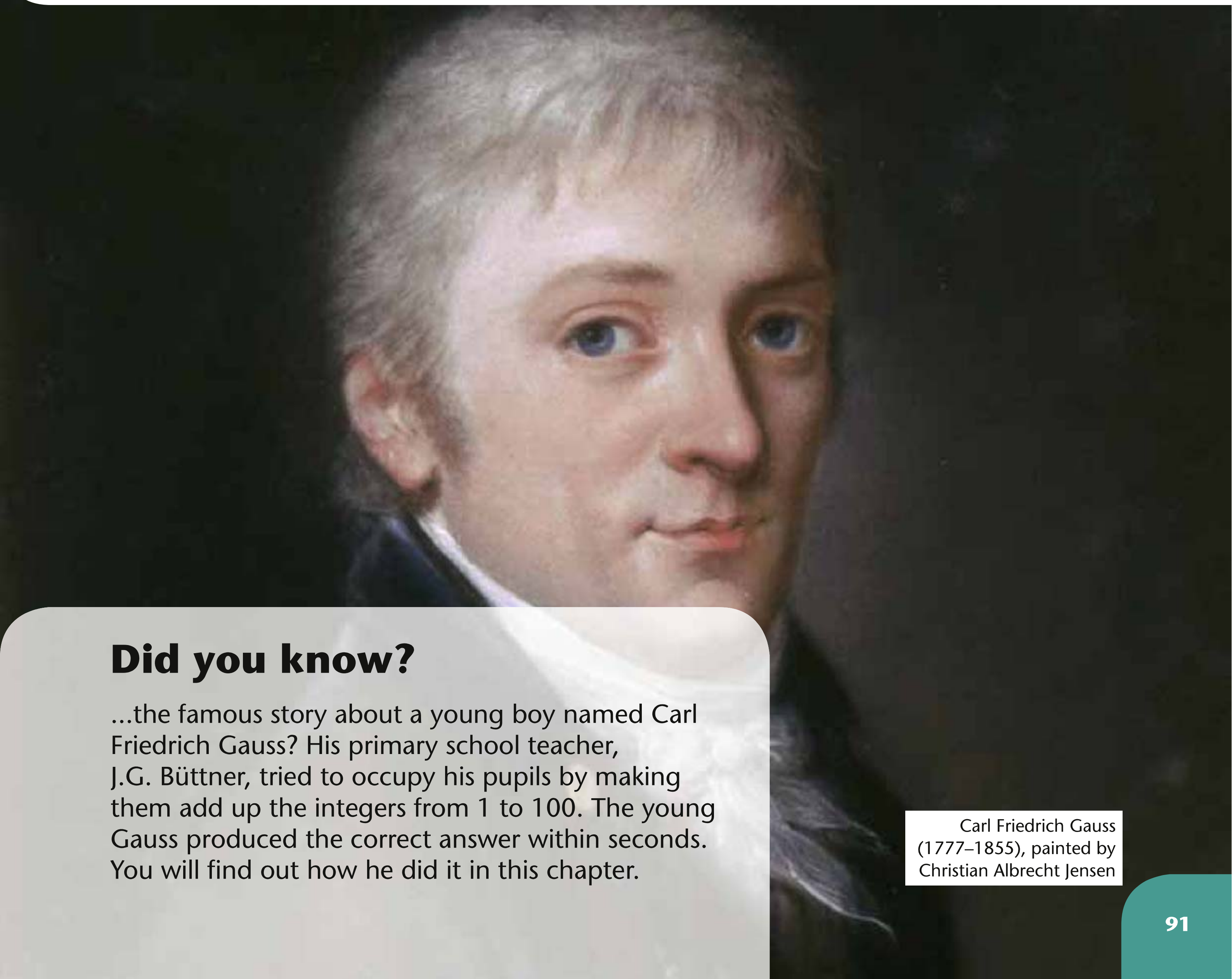
After completing this chapter you should be able to

1. generate a sequence from the *n*th term, or from a recurrence relationship
2. know how to find the *n*th term of an arithmetic sequence, U_n
3. know how to find the sum to *n* terms of an arithmetic series, S_n
4. solve problems on arithmetic series using the formulae for U_n and S_n
5. know the meaning of the symbol $\sum$.

6 Sequences and series



Did you know?

...the famous story about a young boy named Carl Friedrich Gauss? His primary school teacher, J.G. Büttner, tried to occupy his pupils by making them add up the integers from 1 to 100. The young Gauss produced the correct answer within seconds. You will find out how he did it in this chapter.

Carl Friedrich Gauss (1777–1855), painted by Christian Albrecht Jensen

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6.1 **A series of numbers following a set rule is called a sequence.**
3, 7, 11, 15, 19, … is an example of a sequence.

- **Each number in a sequence is called a term.**

Example 1

Work out:

i the next three terms in each of the following sequences and **ii** the rule to find the next term.

a 14, 11, 8, 5, …

b 1, 2, 4, 8, …

c 1, 3, 7, 15, …

Look for the rule that takes you from one term to the next.

a 14, 11, 8, 5, …

To go from one term to the next you subtract 3.

i The next three terms are 2, −1 and -4.

ii

Term no.	1	2	3	4	5
Term	14	11	8	5	2

The rule to find the next term is 'subtract 3 from the previous term'.

b 1, 2, 4, 8, …

To go from one term to the next you multiply by 2.

i The next three terms are 16, 32 and 64.

ii

Term no.	1	2	3	4	5
Term	1	2	4	8	16

The rule to find the next term is 'multiply the previous term by 2'.

c 1, 3, 7, 15, …

To go from one term to the next you multiply by 2, then add 1.

i The next three terms are 31, 63 and 127.

ii

Term no.	1	2	3	4	5
Term	1	3	7	15	31

The rule to find the next term is 'multiply the previous term by 2 then add 1'.

Exercise 6A

Work out the next three terms of the following sequences. State the rule to find the next term in each case:

1 4, 9, 14, 19, …

2 2, −2, 2, −2, …

3 30, 27, 24, 21, …

4 2, 6, 18, 54, …

5 4, −2, 1, $-\frac{1}{2}$, …

6 1, 2, 5, 14, …

7 1, 1, 2, 3, 5, …

8 1, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{4}{7}$, …

9 4, 3, 2.5, 2.25, 2.125, …

10 0, 3, 8, 15, …

Hints: Question 6 – Look for two operations. Question 8 – Treat numerator and denominator separately.

6.2 When you know a formula for the *n*th term of a sequence (e.g. $U_n = 3n - 1$) you can use this to find any term in the sequence.

■ **The *n*th term of a sequence is sometimes called the general term.**

Example 2

The *n*th term of a sequence is given by $U_n = 3n - 1$.
Work out:

a The first term. **b** The third term. **c** The nineteenth term.

a $U_1 = 3 \times 1 - 1$
$= 2$

Substitute $n = 1$

b $U_3 = 3 \times 3 - 1$
$= 8$

Substitute $n = 3$

c $U_{19} = 3 \times 19 - 1$
$= 56$

Substitute $n = 19$

Example 3

The *n*th term of a sequence is given by $U_n = \frac{n^2}{(n + 1)}$.

Work out:

a The first three terms. **b** The 49th term.

a $U_1 = \frac{1 \times 1}{1 + 1} = \frac{1}{2}$ (Substitute $n = 1$)

$U_2 = \frac{2 \times 2}{2 + 1} = \frac{4}{3}$ (Substitute $n = 2$)

$U_3 = \frac{3 \times 3}{3 + 1} = \frac{9}{4}$ (Substitute $n = 3$)

Use $U_n = \frac{n^2}{n + 1}$ with $n = 1$, 2 and 3.

b $U_n = \frac{49 \times 49}{49 + 1}$ (Substitute $n = 49$)

$= \frac{2401}{50}$

Use $U_n = \frac{n^2}{n + 1}$ with $n = 49$.

Example 4

Find the value of n for which U_n has the given value:

a $U_n = 5n - 2$, $U_n = 153$

b $U_n = n^2 + 5$, $U_n = 149$

c $U_n = n^2 - 7n + 12$, $U_n = 72$

a $153 = 5n - 2$	Here $U_n = 153$, so we substitute and solve the equation for n.
$155 = 5n$	Add 2 to both sides.
$n = 31$	Divide by 5.
b $149 = n^2 + 5$	Here $U_n = 149$.
$144 = n^2$	Take 5 from both sides.
$n = \pm 12$	Find square root.
$n = 12$	n can only be positive so $n = 12$.
c $72 = n^2 - 7n + 12$	Here $U_n = 72$.
$0 = n^2 - 7n - 60$	
$0 = (n - 12)(n + 5)$	Solve the quadratic equation by factorisation.
$n = 12$ or $n = -5$	
$n = 12$	n is positive so accept only $n = 12$.

Example 5

A sequence is generated by the formula $U_n = an + b$ where a and b are constants to be found. Given that $U_3 = 5$ and $U_8 = 20$, find the values of the constants a and b.

We know $U_3 = 5$, so $3a + b = 5$. ①	Substitute $n = 3$ and $U_3 = 5$ in $U_n = an + b$.
We know $U_8 = 20$, so $8a + b = 20$. ②	Substitute $n = 8$ and $U_8 = 20$ in $U_n = an + b$.
② − ① gives:	Solve simultaneously.
$5a = 15$	
$a = 3$	
Substitute $a = 3$ in ①:	
$9 + b = 5$	
$b = -4$	
Constants are $a = 3$ and $b = -4$.	

Exercise 6B

1 Find the U_1, U_2, U_3 and U_{10} of the following sequences, where:

a $U_n = 3n + 2$
b $U_n = 10 - 3n$
c $U_n = n^2 + 5$
d $U_n = (n - 3)^2$
e $U_n = (-2)^n$
f $U_n = \dfrac{n}{n + 2}$
g $U_n = (-1)^n \dfrac{n}{n + 2}$
h $U_n = (n - 2)^3$

2 Find the value of n for which U_n has the given value:

a $U_n = 2n - 4$, $U_n = 24$
b $U_n = (n - 4)^2$, $U_n = 25$
c $U_n = n^2 - 9$, $U_n = 112$
d $U_n = \dfrac{2n + 1}{n - 3}$, $U_n = \dfrac{19}{6}$
e $U_n = n^2 + 5n - 6$, $U_n = 60$
f $U_n = n^2 - 4n + 11$, $U_n = 56$
g $U_n = n^2 + 4n - 5$, $U_n = 91$
h $U_n = (-1)^n \dfrac{n}{n + 4}$, $U_n = \dfrac{7}{9}$
i $U_n = \dfrac{n^3 + 3}{5}$, $U_n = 13.4$
j $U_n = \dfrac{n^3}{5} + 3$, $U_n = 28$

3 Prove that the $(2n + 1)$th term of the sequence $U_n = n^2 - 1$ is a multiple of 4.

4 Prove that the terms of the sequence $U_n = n^2 - 10n + 27$ are all positive. For what value of n is U_n smallest?

Hint: Question 4 – Complete the square.

5 A sequence is generated according to the formula $U_n = an + b$, where a and b are constants. Given that $U_3 = 14$ and $U_5 = 38$, find the values of a and b.

6 A sequence is generated according to the formula $U_n = an^2 + bn + c$, where a, b and c are constants. If $U_1 = 4$, $U_2 = 10$ and $U_3 = 18$, find the values of a, b and c.

7 A sequence is generated from the formula $U_n = pn^3 + q$, where p and q are constants. Given that $U_1 = 6$ and $U_3 = 19$, find the values of the constants p and q.

6.3 **When you know the rule to get from one term to the next, you can use this information to produce a recurrence relationship (or recurrence formula).**

Look at the following sequence of numbers:

5, 8, 11, 14, 17, …

We can describe this by the rule 'add 3 to the previous term'.
We can see that:

$U_2 = U_1 + 3$
$U_3 = U_2 + 3$
$U_4 = U_3 + 3$
etc.

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This sequence can also be described by the recurrence formula:

$$U_{k+1} = U_k + 3 \quad (k \geqslant 1)$$

It works for all values of k bigger than or equal to 1.

The $(k + 1)$th term in the sequence.

The kth term in the sequence.

You must always state the first term of the sequence, as many different sequences have the same recurrence relationship. For example, the sequences

4, 7, 10, 13, 16, …

and

5, 8, 11, 14, 17, …

could both be described by the recurrence formula $U_{k+1} = U_k + 3$, but we can distinguish between them by stating

$U_{k+1} = U_k + 3$, $k \geqslant 1$ with $U_1 = 4$ in the first example

but

$U_{k+1} = U_k + 3$, $k \geqslant 1$ and $U_1 = 5$ in the second example.

■ **A sequence can be expressed by a recurrence relationship. For example, the sequence 5, 9, 13, 17, … can be formed from $U_{n+1} = U_n + 4$, $U_1 = 5$ (U_1 must be given).**

Example 6

Find the first four terms of the following sequences:

a $U_{n+1} = U_n + 4$, $U_1 = 7$ **b** $U_{n+1} = U_n + 4$, $U_1 = 5$ **c** $U_{n+2} = 3U_{n+1} - U_n$, $U_1 = 4$ and $U_2 = 2$

a $U_{n+1} = U_n + 4$, $U_1 = 7$

Substituting $n = 1$, $U_2 = U_1 + 4 = 7 + 4 = 11$.

Substituting $n = 2$, $U_3 = U_2 + 4 = 11 + 4 = 15$.

Substituting $n = 3$, $U_4 = U_3 + 4 = 15 + 4 = 19$.

Sequence is 7, 11, 15, 19, …

Substitute $n = 1$, 2 and 3. As you are given U_1 you have the first term.

b $U_{n+1} = U_n + 4$, $U_1 = 5$

This is the same recurrence formula. It produces a different sequence because U_1 is different.

Substituting $n = 1$, $U_2 = U_1 + 4 = 5 + 4 = 9$.

Substituting $n = 2$, $U_3 = U_2 + 4 = 9 + 4 = 13$.

Substituting $n = 3$, $U_4 = U_3 + 4 = 13 + 4 = 17$.

Sequence is 5, 9, 13, 17, …

c $U_{n+2} = 3U_{n+1} - U_n$, $U_1 = 4$, $U_2 = 2$.

This formula links up three terms. Simply substitute in the values of n to see how the relationship works.

Substituting $n = 1$, $U_3 = 3U_2 - U_1 = 3 \times 2 - 4 = 2$.

Substituting $n = 2$, $U_4 = 3U_3 - U_2 = 3 \times 2 - 2 = 4$.

Sequence is 4, 2, 2, 4, …

Example 7

A sequence of terms $\{U_n\}$, $n \geqslant 1$ is defined by the recurrence relation $U_{n+2} = mU_{n+1} + U_n$ where m is a constant. Given also that $U_1 = 2$ and $U_2 = 5$:

a find an expression in terms of m for U_3

b find an expression in terms of m for U_4.

Given the value of $U_4 = 21$:

c find the possible values of m.

a $U_3 = mU_2 + U_1$ — Substitute $n = 1$.

$= 5m + 2$ — Substitute $U_1 = 2$ and $U_2 = 5$.

b $U_4 = mU_3 + U_2$ — Substitute $n = 2$.

$= m(5m + 2) + 5$ — Substitute $U_3 = 5m + 2$ and $U_2 = 5$.

$= 5m^2 + 2m + 5$ — Simplify.

c $U_4 = 21$ — Set $U_4 = 21$.

$5m^2 + 2m + 5 = 21$

$5m^2 + 2m - 16 = 0$ — Subtract 21 from both sides.

$(5m - 8)(m + 2) = 0$ — Factorise (if possible).

$m = 1.6$ or -2.

Exercise 6C

1 Find the first four terms of the following recurrence relationships:

a $U_{n+1} = U_n + 3$, $U_1 = 1$

b $U_{n+1} = U_n - 5$, $U_1 = 9$

c $U_{n+1} = 2U_n$, $U_1 = 3$

d $U_{n+1} = 2U_n + 1$, $U_1 = 2$

e $U_{n+1} = \dfrac{U_n}{2}$, $U_1 = 10$

f $U_{n+1} = (U_n)^2 - 1$, $U_1 = 2$

g $U_{n+2} = 2U_{n+1} + U_n$, $U_1 = 3$, $U_2 = 5$

2 Suggest possible recurrence relationships for the following sequences (remember to state the first term):

a 3, 5, 7, 9, …

b 20, 17, 14, 11, …

c 1, 2, 4, 8, …

d 100, 25, 6.25, 1.5625, …

e 1, −1, 1, −1, 1, …

f 3, 7, 15, 31, …

g 0, 1, 2, 5, 26, …

h 26, 14, 8, 5, 3.5, …

i 1, 1, 2, 3, 5, 8, 13, …

j 4, 10, 18, 38, 74, …

3 By writing down the first four terms or otherwise, find the recurrence formula that defines the following sequences:

a $U_n = 2n - 1$ **b** $U_n = 3n + 2$

c $U_n = n + 2$ **d** $U_n = \dfrac{n + 1}{2}$

e $U_n = n^2$ **f** $U_n = (-1)^n n$

4 A sequence of terms $\{U_n\}$ is defined $n \geqslant 1$ by the recurrence relation $U_{n+1} = kU_n + 2$, where k is a constant. Given that $U_1 = 3$:

a find an expression in terms of k for U_2

b hence find an expression for U_3.

Given that $U_3 = 42$:

c find possible values of k.

5 A sequence of terms $\{U_k\}$ is defined $k \geqslant 1$ by the recurrence relation $U_{k+2} = U_{k+1} - pU_k$, where p is a constant. Given that $U_1 = 2$ and $U_2 = 4$:

a find an expression in terms of p for U_3

b hence find an expression in terms of p for U_4.

Given also that U_4 is twice the value of U_3:

c find the value of p.

6.4 A sequence that increases by a constant amount each time is called an arithmetic sequence.

The following are examples of arithmetic sequences:

3, 7, 11, 15, 19, … (because you add 4 each time)
2, 7, 12, 17, 22, … (because you add 5 each time)
17, 14, 11, 8, … (because you add −3 each time)
a, $a + d$, $a + 2d$, $a + 3d$, … (because you add d each time)

■ **A recurrence relationship of the form**

$$U_{k+1} = U_k + n,\ k \geqslant 1\ \ n \in \mathbb{Z}$$

is called an arithmetic sequence.

Example 8

Find the **a** 10th, **b** nth and **c** 50th terms of the arithmetic sequence 3, 7, 11, 15, 19, …

Sequence is 3, 7, 11, 15, …
First term = 3
Second term = 3 + 4
Third term = 3 + 4 + 4
Fourth term = 3 + 4 + 4 + 4

The sequence is going up in fours.
It is starting at 3.
The first term is 3 + 0 × 4.
The second term is 3 + 1 × 4.
The third term is 3 + 2 lots of 4.
The fourth term is 3 + 3 lots of 4.

a 10th term is

$3 + 9 \times 4 = 3 + 36 = 39$ — 10th term = first term + 9 fours.

b nth term is

$3 + (n - 1) \times 4 = 4n - 1$ — nth term = first term + $(n - 1)$ fours.

c 50th term is

$3 + (50 - 1) \times 4 = 3 + 196 = 199$ — 50th term = first term + 49 fours.

Example 9

A 6 metre high tree is planted in a garden. If it grows 1.5 metres a year:

a How high will it be after it has been in the garden for 8 years?

b After how many years will it be 24 metres high?

a $6 + 8 \times 1.5$ — It starts at 6 m. It has 8 years' growth at 1.5 m a year.

$= 6 + 12$

$= 18$ metres

b $24 - 6 = 18$ metres — Find out how much it has grown in total.

So number of years $= \frac{18}{1.5}$ — It grows at 1.5 metres a year.

$= 12$ years

Example 10

Find the number of terms in the arithmetic sequence 7, 11, 15, …, 143:

The sequence goes up in fours. — Work out how to get from one term to the next.

It goes from 7 to 143, a difference of 136. — Work out the difference between largest and smallest numbers.

136 in fours is $\frac{136}{4} = 34$ jumps.

7, 11, 15, …, …, 143

There is one more term than the number of jumps, so 34 jumps means 35 terms.

Exercise 6D

1 Which of the following sequences are arithmetic?

a 3, 5, 7, 9, 11, …
b 10, 7, 4, 1, …
c $y, 2y, 3y, 4y, \ldots$
d 1, 4, 9, 16, 25, …
e 16, 8, 4, 2, 1, …
f 1, −1, 1, −1, 1, …
g $y, y^2, y^3, y^4, \ldots$
h $U_{n+1} = U_n + 2,\ U_1 = 3$
i $U_{n+1} = 3U_n - 2,\ U_1 = 4$
j $U_{n+1} = (U_n)^2,\ U_1 = 2$
k $U_n = n(n+1)$
l $U_n = 2n + 3$

2 Find the 10th and nth terms in the following arithmetic progressions:

a 5, 7, 9, 11, …
b 5, 8, 11, 14, …
c 24, 21, 18, 15, …
d −1, 3, 7, 11, …
e $x, 2x, 3x, 4x, \ldots$
f $a, a+d, a+2d, a+3d, \ldots$

3 An investor puts £4000 in an account. Every month thereafter she deposits another £200. How much money in total will she have invested at the start of **a** the 10th month and **b** the mth month? (Note that at the start of the 6th month she will have made only 5 deposits of £200.)

4 Calculate the number of terms in the following arithmetic sequences:

a 3, 7, 11, …, 83, 87
b 5, 8, 11, …, 119, 122
c 90, 88, 86, …, 16, 14
d 4, 9, 14, …, 224, 229
e $x, 3x, 5x, \ldots, 35x$
f $a, a+d, a+2d, \ldots, a+(n-1)d$

6.5 Arithmetic series are formed by adding together the terms of an arithmetic sequence, $U_1 + U_2 + U_3 + \ldots + U_n$.

In an arithmetic series the next term is found by adding (or subtracting) a constant number. This number is called the common difference d.
The first term is represented by a.

- **Therefore all arithmetic series can be put in the form**

$$\underbrace{a}_{\text{1st term}} + \underbrace{(a+d)}_{\text{2nd term}} + \underbrace{(a+2d)}_{\text{3rd term}} + \underbrace{(a+3d)}_{\text{4th term}} + \underbrace{(a+4d)}_{\text{5th term}} + \underbrace{(a+5d)}_{\text{6th term}}$$

Look at the relationship between the number of the term and the coefficient of d. You should be able to see that the coefficient of d is one less than the number of the term.

We can use this fact to produce a formula for the nth term of an arithmetic series.

- **The nth term of an arithmetic series is $a + (n-1)d$, where a is the first term and d is the common difference.**

Example 11

Find **i** the 20th and **ii** the 50th terms of the following series:

a $4 + 7 + 10 + 13 + \dots$ **b** $100 + 93 + 86 + 79 + \dots$

a $4 + 7 + 10 + 13 + \dots$

In this series $a = 4$ and $d = 3$

i 20th term

$= 4 + (20 - 1) \times 3$

$= 4 + 19 \times 3$

$= 61$

ii 50th term

$= 4 + (50 - 1) \times 3$

$= 4 + 49 \times 3$

$= 151$

b $100 + 93 + 86 + 79 + \dots$

In this series $a = 100$ and $d = -7$

i 20th term

$= 100 + (20 - 1) \times -7$

$= 100 + 19 \times -7$

$= -33$

ii 50th term

$= 100 + (50 - 1) \times -7$

$= 100 + 49 \times -7$

$= -243$

First calculate the values of a and d. (In this case d is $7 - 4$.)

Use the formula $a + (n - 1)d$, with $n = 20$ for the 20th term and $n = 50$ for the 50th term.

d is negative this time.
$d = (93 - 100) = -7$.
To calculate d you can use $U_2 - U_1$ or $U_3 - U_2$ or $U_4 - U_3$, etc.

Example 12

For the arithmetic series $5 + 9 + 13 + 17 + 21 + \dots + 805$:

a find the number of terms **b** which term of the series would be 129?

Series is $5 + 9 + 13 + 17 + 21 + \dots + 805$.

In this series $a = 5$ and $d = 4$.

a Using nth term $= a + (n - 1)d$

$805 = 5 + (n - 1) \times 4$

$805 = 5 + 4n - 4$

$805 = 4n + 1$

$804 = 4n$

$n = 201$

There are 201 terms in this series.

A good starting point in all questions is to find the values of a and d.
Here $a = 5$ and $a + d = 9$, so $d = 4$.

The nth term is $a + (n - 1)d$.
So replace U_n with 805 and solve for n.

Subtract 1.

Divide by 4.

b Using nth term $= a + (n-1)d$

$129 = 5 + (n-1) \times 4$

$129 = 4n + 1$

$128 = 4n$

$n = 32$

The 32nd term is 129.

> This time the nth term is 129. So replace U_n with 129.
>
> Subtract 1.
>
> Divide by 4.

Example 13

Given that the 3rd term of an arithmetic series is 20 and the 7th term is 12:

a find the first term **b** find the 20th term.

(Note: These are very popular questions and involve setting up and solving simultaneous equations.)

a 3rd term $= 20$, so $a + 2d = 20$. ①

7th term $= 12$, so $a + 6d = 12$. ②

Taking ① from ②:

$4d = -8$

$d = -2$

The common difference is -2.

$a + 2 \times -2 = 20$

$a - 4 = 20$

$a = 24$

The first term is 24.

> Use nth term $= a + (n-1)d$, with $n = 3$ and $n = 7$.

> Substitute $d = -2$ back into equation ①.
>
> Add 4 to both sides.

b 20th term $= a + 19d$

$= 24 + 19 \times -2$

$= 24 - 38$

$= -14$

The 20th term is -14.

> Use nth term is $a + (n-1)d$ with $n = 20$.
>
> Substitute a $= 24$ and $d = -2$.

Exercise 6E

1 Find **i** the 20th and **ii** the nth terms of the following arithmetic series:

a $2 + 6 + 10 + 14 + 18 \ldots$

b $4 + 6 + 8 + 10 + 12 + \ldots$

c $80 + 77 + 74 + 71 + \ldots$

d $1 + 3 + 5 + 7 + 9 + \ldots$

e $30 + 27 + 24 + 21 + \ldots$

f $2 + 5 + 8 + 11 + \ldots$

g $p + 3p + 5p + 7p + \ldots$

h $5x + x + (-3x) + (-7x) + \ldots$

2 Find the number of terms in the following arithmetic series:

a $5 + 9 + 13 + 17 + \ldots + 121$ **b** $1 + 1.25 + 1.5 + 1.75 \ldots + 8$

c $-4 + -1 + 2 + 5 \ldots + 89$ **d** $70 + 61 + 52 + 43 \ldots + -200$

e $100 + 95 + 90 + \ldots + (-1000)$ **f** $x + 3x + 5x \ldots + 153x$

3 The first term of an arithmetic series is 14. If the fourth term is 32, find the common difference.

4 Given that the 3rd term of an arithmetic series is 30 and the 10th term is 9 find a and d. Hence find which term is the first one to become negative.

5 In an arithmetic series the 20th term is 14 and the 40th term is -6. Find the 10th term.

6 The first three terms of an arithmetic series are $5x$, 20 and $3x$. Find the value of x and hence the values of the three terms.

7 For which values of x would the expression -8, x^2 and $17x$ form the first three terms of an arithmetic series?

Hint: Question 6 – Find two expressions equal to the common difference and set them equal to each other.

6.6 You need to be able to find the sum of an arithmetic series.

The method of finding this sum is attributed to a famous mathematician called Carl Friedrich Gauss (1777–1855). He reputedly solved the following sum whilst in Junior School:

$$1 + 2 + 3 + 4 + 5 + \ldots + 99 + 100$$

Here is how he was able to work it out:

Let $S = 1 + 2 + 3 + 4 \ldots + 98 + 99 + 100$

Reversing the sum $S = 100 + 99 + 98 + 97 \ldots + 3 + 2 + 1$

Adding the two sums $2S = 101 + 101 + 101 + \ldots + 101 + 101 + 101$

$$2S = 100 \times 101$$
$$S = (100 \times 101) \div 2$$
$$S = 5050$$

In general:

$$S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 2)d) + (a + (n - 1)d)$$

Reversing the sum:

$$S_n = (a + (n - 1)d) + (a + (n - 2)d) + (a + (n - 3)d) + \ldots + (a + d) + a$$

Adding the two sums:

$$2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + \ldots + [2a + (n - 1)d]$$
$$2S_n = n[2a + (n - 1)d]$$
$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Hint: There are n lots of $2a + (n - 1)d$.

Prove for yourself that it could be $S_n = \frac{n}{2}(a + L)$ where $L = a + (n - 1)d$.

■ **The formula for the sum of an arithmetic series is**

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

or $$S_n = \frac{n}{2}(a + L)$$

where a is the first term, d is the common difference, n is the number of terms and L is the last term in the series.

You could be asked to prove these formulae.

Example 14

Find the sum of the first 100 odd numbers.

$$\begin{aligned} S &= 1 + 3 + 5 + 7 + \dots \\ &= \frac{n}{2}[2a + (n - 1)d] \\ &= \frac{100}{2}[2 \times 1 + (100 - 1)2] \\ &= 50[2 + 198] \\ &= 50 \times 200 \\ &= 10\,000 \end{aligned}$$

This can be found simply using the formula

$$S = \frac{n}{2}[2a + (n - 1)d]$$

with $a = 1$, $d = 2$ and $n = 100$.

$$\begin{aligned} L &= a + (n - 1)d \\ &= 1 + 99 \times 2 \\ &= 199 \\ S &= \frac{n}{2}(a + L) \\ &= \frac{100}{2}(1 + 199) \\ &= 10\,000 \end{aligned}$$

Alternatively, find L and use

$$S = \frac{n}{2}(a + L)$$

This is a very useful formula and is well worth remembering.

Example 15

Find the greatest number of terms required for the sum of $4 + 9 + 14 + 19 + \dots$ to exceed 2000.

Always establish what you are given in a question. As you are adding on positive terms, it is easier to solve the equality $S_n = 2000$.

$4 + 9 + 14 + 19 + \ldots > 2000$

Using $S = \frac{n}{2}[2a + (n-1)d]$

$2000 = \frac{n}{2}[2 \times 4 + (n-1)5]$

$4000 = n(8 + 5n - 5)$

$4000 = n(5n + 3)$

$4000 = 5n^2 + 3n$

$0 = 5n^2 + 3n - 4000$

$n = \frac{-3 \pm \sqrt{(9 + 80\,000)}}{10}$

$= 27.9, -28.5$

28 terms are needed.

> Knowing $a = 4$, $d = 5$ and $S_n = 2000$, you need to find n.
>
> Substitute into $S = \frac{n}{2}[2a + (n-1)d]$.

> Solve using formula $n = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$.

> Accept positive answer and round up.

Example 16

Robert starts his new job on a salary of £15 000. He is promised rises of £1000 a year, at the end of every year, until he reaches his maximum salary of £25 000. Find his total earnings (since appointed) after **a** 8 years with the firm and **b** 14 years with the firm.

a Total earnings

$= £15\,000 + £16\,000 + \ldots$ (for 8 years)

$a = 15\,000$, $d = 1000$ and $n = 8$

$S = \frac{n}{2}[2a + (n-1)d]$

$S = \frac{8}{2}[30\,000 + 7 \times 1000]$

$= £148\,000$

> Note that it will take Robert 11 years to reach his maximum (his first year and 10 wage rises).

> Write down what you know.

> Use $S = \frac{n}{2}[2a + (n-1)d]$

b Total earnings

$= £15\,000 + £16\,000 + \ldots + £25\,000$

$+ £25\,000 + £25\,000 + £25\,000$

$a = 15\,000$, $d = 1000$ and

$n = 11$ for the first 11 years.

$S = \frac{n}{2}[2a + (n-1)d]$

$S = \frac{11}{2}[30\,000 + 10 \times 1000]$

$= £220\,000$

3 years at $£25\,000 = £75\,000$.

Total amount earned $= £295\,000$.

> This time there are 10 years of increases, taking him to the end of his 11th year, and 3 years of the same salary.

> Use $S = \frac{n}{2}[2a + (n-1)d]$ for the first 11 years.

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Example 17

Show that the sum of the first n natural numbers is $\frac{1}{2}n(n + 1)$.

$S = 1 + 2 + 3 + 4 + \ldots + n$

This is an arithmetic series with

$a = 1, d = 1, n = n.$

$$S = \frac{n}{2}[2a + (n - 1)d]$$

$$S = \frac{n}{2}[2 \times 1 + (n - 1) \times 1]$$

$$S = \frac{n}{2}(2 + n - 1)$$

$$S = \frac{n}{2}(n + 1)$$

$$= \frac{1}{2}n(n + 1)$$

Use $S = \frac{n}{2}[2a + (n - 1)d]$ with $a = 1$, $d = 1$ and $n = n$.

Exercise 6F

1 Find the sums of the following series:

a $3 + 7 + 11 + 14 + \ldots$ (20 terms)
b $2 + 6 + 10 + 14 + \ldots$ (15 terms)
c $30 + 27 + 24 + 21 + \ldots$ (40 terms)
d $5 + 1 + -3 + -7 + \ldots$ (14 terms)
e $5 + 7 + 9 + \ldots + 75$
f $4 + 7 + 10 + \ldots + 91$
g $34 + 29 + 24 + 19 + \ldots + -111$
h $(x + 1) + (2x + 1) + (3x + 1) + \ldots + (21x + 1)$

2 Find how many terms of the following series are needed to make the given sum:

a $5 + 8 + 11 + 14 + \ldots = 670$
b $3 + 8 + 13 + 18 + \ldots = 1575$
c $64 + 62 + 60 + \ldots = 0$
d $34 + 30 + 26 + 22 + \ldots = 112$

3 Find the sum of the first 50 even numbers.

4 Carol starts a new job on a salary of £20 000. She is given an annual wage rise of £500 at the end of every year until she reaches her maximum salary of £25 000. Find the total amount she earns (assuming no other rises), **a** in the first 10 years and **b** over 15 years.

5 Find the sum of the multiples of 3 less than 100. Hence or otherwise find the sum of the numbers less than 100 which are not multiples of 3.

6 James decides to save some money during the six-week holiday. He saves 1p on the first day, 2p on the second, 3p on the third and so on. How much will he have at the end of the holiday (42 days)? If he carried on, how long would it be before he has saved £100?

7 The first term of an arithmetic series is 4. The sum to 20 terms is −15. Find, in any order, the common difference and the 20th term.

8 The sum of the first three numbers of an arithmetic series is 12. If the 20th term is −32, find the first term and the common difference.

9 Show that the sum of the first $2n$ natural numbers is $n(2n + 1)$.

10 Prove that the sum of the first n odd numbers is n^2.

6.7 You can use $\sum$ to signify 'the sum of'.

For example:

$\sum_{n=1}^{10} 2n$ means sum of $2n$ from $n = 1$ to $n = 10$

$$= 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20$$

$$\sum_{n=1}^{10} U_n = U_1 + U_2 + U_3 + \ldots + U_{10}$$

$\sum_{r=0}^{10} (2 + 3r)$ means the sum of $2 + 3r$ from $r = 0$ to $r = 10$

$$= 2 + 5 + 8 + \ldots + 32$$

$\sum_{r=5}^{r=15} (10 - 2r)$ means the sum of $(10 - 2r)$ from $r = 5$ to $r = 15$

$$= 0 + -2 + -4 + \ldots + -20$$

Example 18

Calculate $\sum_{r=1}^{r=20} 4r + 1$

$$\sum_{r=1}^{20} (4r + 1)$$

$= 5 + 9 + 13 + \ldots + 81$

Substitute $r = 1, 2$, etc. to find terms in series.

$S = \frac{n}{2}[2a + (n - 1)d]$

Substitute $a = 5$, $d = 4$ and $n = 20$ into $S = \frac{n}{2}[2a + (n - 1)d]$.

$= \frac{20}{2}[2 \times 5 + (20 - 1)4]$

$= 10[10 + (19) \times 4]$

$= 10 \times 86$

$= 860$

Exercise 6G

1 Rewrite the following sums using Σ notation:

a $4 + 7 + 10 + \ldots + 31$

b $2 + 5 + 8 + 11 + \ldots + 89$

c $40 + 36 + 32 + \ldots + 0$

d The multiples of 6 less than 100

2 Calculate the following:

a $\sum_{r=1}^{5} 3r$

b $\sum_{r=1}^{10} (4r - 1)$

c $\sum_{r=1}^{20} (5r - 2)$

d $\sum_{r=0}^{5} r(r + 1)$

3 For what value of n does $\sum_{r=1}^{n} (5r + 3)$ first exceed 1000?

4 For what value of n would $\sum_{r=1}^{n} (100 - 4r) = 0$?

Mixed exercise 6H

1 The rth term in a sequence is $2 + 3r$. Find the first three terms of the sequence.

2 The rth term in a sequence is $(r + 3)(r - 4)$. Find the value of r for the term that has the value 78.

3 A sequence is formed from an inductive relationship:

$$U_{n+1} = 2U_n + 5$$

Given that $U_1 = 2$, find the first four terms of the sequence.

4 Find a rule that describes the following sequences:

a 5, 11, 17, 23, …

b 3, 6, 9, 12, …

c 1, 3, 9, 27, …

d 10, 5, 0, −5, …

e 1, 4, 9, 16, …

f 1, 1.2, 1.44, 1.728, …

Which of the above are arithmetic sequences?
For the ones that are, state the values of a and d.

5 For the arithmetic series $5 + 9 + 13 + 17 + \ldots$
Find **a** the 20th term, and **b** the sum of the first 20 terms.

6 **a** Prove that the sum of the first n terms in an arithmetic series is

$$S = \frac{n}{2}[2a + (n - 1)d]$$

where a = first term and d = common difference.

b Use this to find the sum of the first 100 natural numbers.

7 Find the least value of n for which $\sum_{r=1}^{n} (4r - 3) > 2000$.

8 A salesman is paid commission of £10 per week for each life insurance policy that he has sold. Each week he sells one new policy so that he is paid £10 commission in the first week, £20 commission in the second week, £30 commission in the third week and so on.

a Find his total commission in the first year of 52 weeks.

b In the second year the commission increases to £11 per week on new policies sold, although it remains at £10 per week for policies sold in the first year. He continues to sell one policy per week. Show that he is paid £542 in the second week of his second year.

c Find the total commission paid to him in the second year. *E*

9 The sum of the first two terms of an arithmetic series is 47.
The thirtieth term of this series is −62. Find:

a the first term of the series and the common difference

b the sum of the first 60 terms of the series. *E*

10 **a** Find the sum of the integers which are divisible by 3 and lie between 1 and 400.

b Hence, or otherwise, find the sum of the integers, from 1 to 400 inclusive, which are **not** divisible by 3. *E*

11 A polygon has 10 sides. The lengths of the sides, starting with the smallest, form an arithmetic series. The perimeter of the polygon is 675 cm and the length of the longest side is twice that of the shortest side. Find, for this series:

a the common difference

b the first term. *E*

12 A sequence of terms $\{U_n\}$ is defined for $n \geqslant 1$, by the recurrence relation $U_{n+2} = 2kU_{n+1} + 15U_n$, where k is a constant. Given that $U_1 = 1$ and $U_2 = -2$:

a find an expression, in terms of k, for U_3

b hence find an expression, in terms of k, for U_4

c given also that $U_4 = -38$, find the possible values of k. *E*

13 Prospectors are drilling for oil. The cost of drilling to a depth of 50 m is £500. To drill a further 50 m costs £640 and, hence, the total cost of drilling to a depth of 100 m is £1140. Each subsequent extra depth of 50 m costs £140 more to drill than the previous 50 m.

a Show that the cost of drilling to a depth of 500 m is £11 300.

b The total sum of money available for drilling is £76 000. Find, to the nearest 50 m, the greatest depth that can be drilled. *E*

14 Prove that the sum of the first $2n$ multiples of 4 is $4n(2n + 1)$. *E*

15 A sequence of numbers $\{U_n\}$ is defined, for $n \geqslant 1$, by the recurrence relation $U_{n+1} = kU_n - 4$, where k is a constant. Given that $U_1 = 2$:

a find expressions, in terms of k, for U_2 and U_3

b given also that $U_3 = 26$, use algebra to find the possible values of k. *E*

16 Each year, for 40 years, Anne will pay money into a savings scheme. In the first year she pays in £500. Her payments then increase by £50 each year, so that she pays in £550 in the second year, £600 in the third year, and so on.

a Find the amount that Anne will pay in the 40th year.

b Find the total amount that Anne will pay in over the 40 years.

c Over the same 40 years, Brian will also pay money into the savings scheme. In the first year he pays in £890 and his payments then increase by £d each year. Given that Brian and Anne will pay in exactly the same amount over the 40 years, find the value of d.

17 The fifth term of an arithmetic series is 14 and the sum of the first three terms of the series is -3.

a Use algebra to show that the first term of the series is -6 and calculate the common difference of the series.

b Given that the nth term of the series is greater than 282, find the least possible value of n.

18 The fourth term of an arithmetic series is $3k$, where k is a constant, and the sum of the first six terms of the series is $7k + 9$.

a Show that the first term of the series is $9 - 8k$.

b Find an expression for the common difference of the series in terms of k.

Given that the seventh term of the series is 12, calculate:

c the value of k

d the sum of the first 20 terms of the series.