7

After completing this chapter you should be able to

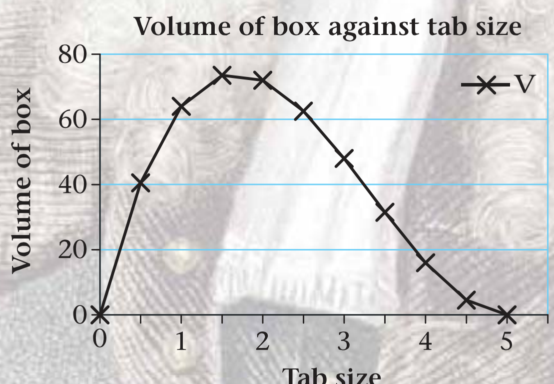
1. estimate the gradient of a curve
2. calculate the gradient function, $\frac{dy}{dx}$ for simple functions
3. calculate the gradient of a curve at any point
4. find the equation of the tangent and normal to a curve at a specified point
5. calculate the second differential $\frac{d^2y}{dx^2}$.

Differentiation

Sir Isaac Newton who, along with Gottfried Liebniz, devised the laws of calculus.

Did you know?

Differential calculus is an important part of A level Mathematics and is widely used in many branches of Science, Engineering and Business. Understanding it will help you to sketch a function by finding the maximum and minimum values.



Differentiation enables us to find the exact value where the volume of the box described in Chapter 4 is maximised.

Successful businesses maximise profits and minimise costs.
A simple example to explain this might be a drinks manufacturer using cans that hold 330 ml. If the surface area of the can is as small as possible, then profits are maximised as the amount of aluminium used is minimised.

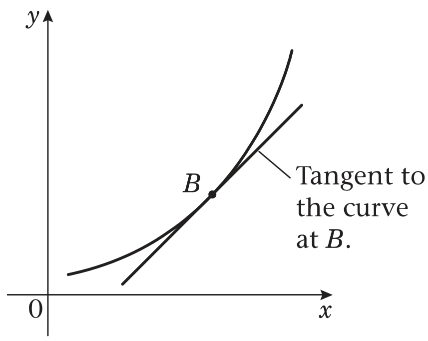
7.1 You can calculate an estimate of the gradient of a tangent.

In Section 5.1, you found the gradient of a **straight line** by calculation and by inspection of its equation.

The gradient of a curve changes as you move along it, and so:

- **The gradient of a curve at a specific point is defined as being the same as the gradient of the tangent to the curve at that point.**

The tangent is a straight line, which touches, but does not cut, the curve. You cannot calculate the gradient of the tangent directly, as you know only one point on the tangent and you require two points to calculate the gradient of a line.



To find the gradient of the tangent at a point *B* on a curve with a known equation, you can find the gradient of chords joining *B* to other points close to *B* on the curve. You can then investigate the values of these gradients as the other points become closer to *B*. You should find the values become very close to a limiting value, which is the value of the gradient of the tangent, and is also the gradient of the curve at the point *B*.

Example 1

The points shown on the curve with equation $y = x^2$, are $O(0, 0)$, $A(\frac{1}{2}, \frac{1}{4})$, $B(1, 1)$, $C(1.5, 2.25)$ and $D(2, 4)$.

a Calculate the gradients of:

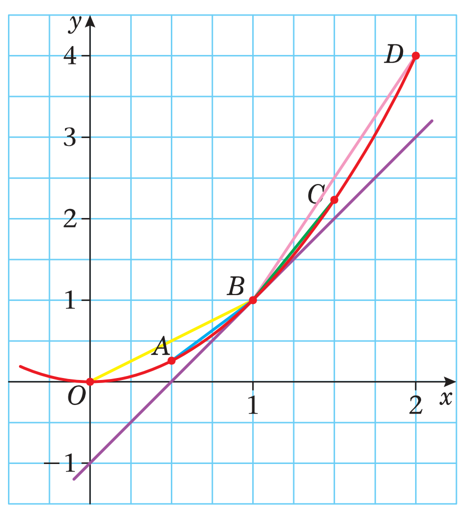
i *OB*

ii *AB*

iii *BC*

iv *BD*

b What do you deduce about the gradient of the tangent at the point B?



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a **i** *Gradient of the chord OB*

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{1 - 0}{1 - 0}$$

$$= 1$$

ii *Gradient of the chord AB*

$$= \frac{1 - \frac{1}{4}}{1 - \frac{1}{2}}$$

$$= \frac{\frac{3}{4}}{\frac{1}{2}}$$

$$= 1.5$$

iii *Gradient of the chord BC*

$$= \frac{2.25 - 1}{1.5 - 1}$$

$$= \frac{1.25}{0.5}$$

$$= 2.5$$

iv *Gradient of the chord BD*

$$= \frac{4 - 1}{2 - 1}$$

$$= 3$$

b *The gradient of the tangent at the point B is between 1.5 and 2.5.*

The formula for the gradient of a straight line is used.

(x_1, y_1) is (0, 0) and (x_2, y_2) is (1, 1).

The same formula is used with (x_1, y_1) as $(\frac{1}{2}, \frac{1}{4})$ and (x_2, y_2) as (1, 1).

$\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \times \frac{2}{1} = \frac{3}{2} = 1.5$.

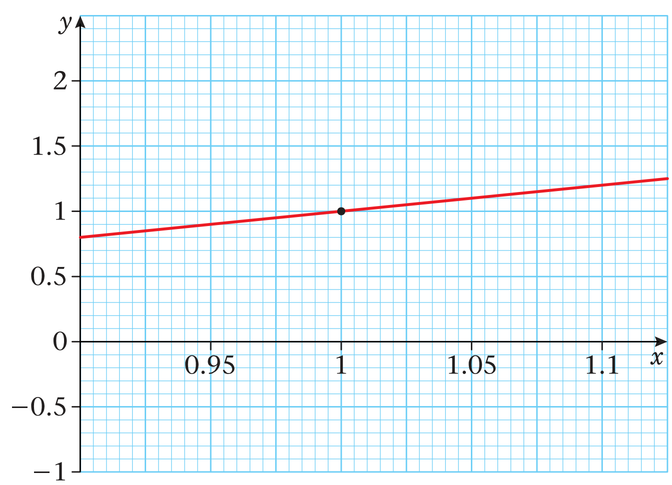
This time (x_1, y_1) is (1, 1) and (x_2, y_2) is (1.5, 2.25).

$\frac{1.25}{0.5} = \frac{12.5}{5} = 2.5$.

Note that the chords are steeper and the gradients are larger as you move along the curve.

The gradient of the tangent at *B* is less than the gradient of the chord *BC*, but is greater than the gradient of the chord *AB*.

You can now 'zoom in' on to the section of the curve near to the point *B*(1, 1). This section, shown below, is almost a straight line and is close in gradient to the tangent at the point *B*.



Example 2

a For the same curve as Example 1, find the gradient of the chord BP when P has coordinates:
i (1.1, 1.21) **ii** (1.01, 1.0201) **iii** (1.001, 1.002 001) **iv** $(1 + h, (1 + h)^2)$

b What do you deduce about the gradient of the tangent at the point B?

a i Gradient of the chord joining (1, 1) to (1.1, 1.21)

$$= \frac{1.21 - 1}{1.1 - 1}$$

$$= \frac{0.21}{0.1}$$

$$= 2.1$$

When $x = 1.1$, $y = 1.1^2 = 1.21$

The gradient is $\frac{0.21}{0.1} = \frac{0.21 \times 10}{0.1 \times 10} = \frac{2.1}{1}$

ii Gradient of the chord joining (1, 1) to (1.01, 1.0201)

$$= \frac{1.0201 - 1}{1.01 - 1}$$

$$= \frac{0.0201}{0.01}$$

$$= 2.01$$

This point is closer to (1, 1) than (1.1, 1.21).

This gradient is closer to 2.

iii Gradient of the chord joining (1, 1) to (1.001, 1.002 001)

$$= \frac{1.002\,001 - 1}{1.001 - 1}$$

$$= \frac{0.002\,001}{0.001}$$

$$= 2.001$$

The point $(1.001, 1.001^2)$ is very close to (1, 1).

The gradient is very close to 2.

iv Gradient of the chord joining (1, 1) to $(1 + h, (1 + h)^2)$

$$= \frac{(1 + h)^2 - 1}{(1 + h) - 1}$$

$$= \frac{1 + 2h + h^2 - 1}{1 + h - 1}$$

$$= \frac{2h + h^2}{h}$$

$$= 2 + h$$

h is a constant.

$(1 + h)^2 = (1 + h)(1 + h) = 1 + 2h + h^2$.

This becomes $\frac{h(2 + h)}{h}$

You can apply this formula to the chords in **i**, **ii** and **iii**,
e.g. $(1.1, 1.21) = (1 + 0.1), (1 + 0.1^2)$.
So $h = 0.1$ and the gradient of chord BP is $2 + 0.1 = 2.1$.

b When h is small the gradient of the chord is close to the gradient of the tangent, and $2 + h$ is close to the value 2. So we deduce that the gradient of the tangent at the point (1, 1) is 2.

If you let h become very close to zero, the gradient is very close to 2.

Exercise 7A

Questions like these will not appear in the examination papers.

1 F is the point with co-ordinates (3, 9) on the curve with equation $y = x^2$.

a Find the gradients of the chords joining the point F to the points with coordinates:

i (4, 16) **ii** (3.5, 12.25) **iii** (3.1, 9.61)

iv (3.01, 9.0601) **v** $(3 + h, (3 + h)^2)$

b What do you deduce about the gradient of the tangent at the point (3, 9)?

2 G is the point with coordinates (4, 16) on the curve with equation $y = x^2$.

a Find the gradients of the chords joining the point G to the points with coordinates:

i (5, 25) **ii** (4.5, 20.25) **iii** (4.1, 16.81)

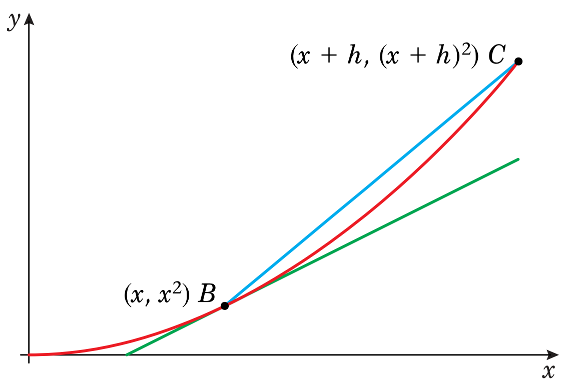
iv (4.01, 16.0801) **v** $(4 + h, (4 + h)^2)$

b What do you deduce about the gradient of the tangent at the point (4, 16)?

7.2 You can find the formula for the gradient of the function $f(x) = x^2$ and other functions of the form $f(x) = x^n$, $n \in \mathbb{R}$.

Examples 2 to 4 show you how to derive the formulae and will not be tested.

In the following sketch, the gradient of the tangent $y = f(x)$ at a point B is found by starting with the gradient of a chord BC.



■ **The gradient of the tangent at any particular point is the rate of change of y with respect to x.**

The point B is the point with coordinates (x, x^2) and the point C is the point near to B with coordinates $(x + h, (x + h^2))$.

The gradient of the chord BC is $\dfrac{(x + h)^2 - x^2}{(x + h) - x}$

This can be written as $\dfrac{(x^2 + 2hx + h^2) - x^2}{x + h - x}$

which simplifies to give $\dfrac{2hx + h^2}{h}$

$= \dfrac{h(2x + h)}{h}$

$= 2x + h.$

Hint:
Use the gradient formula for a straight line.

Expand $(x + h)(x + h)$.

Factorise the numerator.

Cancel the factor h.

As h becomes smaller the gradient of the chord becomes closer to the gradient of the tangent to the curve at the point B.

The gradient of the tangent at the point B to the curve with equation $y = x^2$ is therefore given by the formula: gradient = $2x$.

In general you will find that the gradients of the tangents to a given curve can be expressed by a formula related to the equation of the curve.

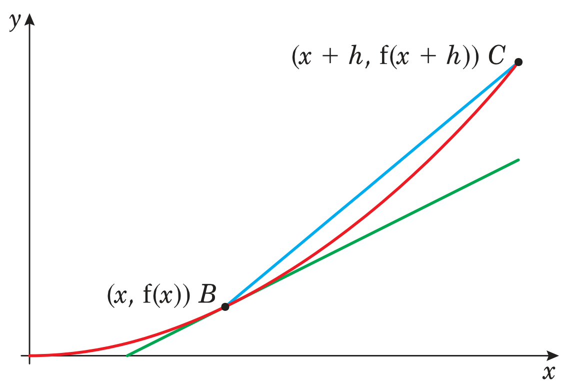
- **The gradient formula for $y = \mathrm{f}(x)$ is given by the equation: gradient = $\mathrm{f}'(x)$, where $\mathrm{f}'(x)$ is called the derived function.**

$\mathrm{f}'(x)$ is defined as the gradient of the curve $y = \mathrm{f}(x)$ at the general point $(x, \mathrm{f}(x))$. It is also the gradient of the tangent to the curve at that point.

So far you have seen that when $\mathrm{f}(x) = x^2$, $\mathrm{f}'(x) = 2x$.

You can use this result to determine the gradient of the curve $y = x^2$ at any specified point on the curve.

You can also use a similar approach to establish a gradient formula for the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x)$ is a power of x, i.e. $\mathrm{f}(x) = x^n$, where n is any real number.



Again you need to consider the gradient of a chord joining two points which are close together on the curve and determine what happens when the points become very close together.

This time the point B has coordinates $(x, \mathrm{f}(x))$ and the point C is the point near to B with coordinates $(x + h, \mathrm{f}(x + h))$.

The gradient of BC is

$$\frac{\mathrm{f}(x + h) - \mathrm{f}(x)}{(x + h) - x}$$

So as h becomes small and the gradient of the chord becomes close to the gradient of the tangent, the definition of $\mathrm{f}'(x)$ is given as

$$\lim_{h \to 0} \left[\frac{\mathrm{f}(x + h) - \mathrm{f}(x)}{h} \right]$$

Using this definition you can differentiate a function of the form $\mathrm{f}(x) = x^n$.

Example 3

Find, from the definition of the derived function, an expression for $f'(x)$ when $f(x) = x^3$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - (x)^3}{(x+h) - x}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{(x+h) - x}$$

$(x+h)^3 = (x+h)(x+h)^2$
$= (x+h)(x^2 + 2hx + h^2)$
which expands to give
$x^3 + 3x^2h + 3xh^2 + h^3$.

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

Factorise the numerator.

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

The $3xh$ term and the h^2 term become zero.

As $h \to 0$ the limiting value is $3x^2$.

So when $f(x) = x^3$, $f'(x) = 3x^2$.

Example 4

Find, from the definition of the derived function, an expression for $f'(x)$ when $f(x) = \frac{1}{x}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)} - \frac{1}{x}}{(x+h) - x}$$

$$= \lim_{h \to 0} \frac{\frac{x - (x+h)}{x(x+h)}}{(x+h) - x}$$

Use a common denominator.

$$= \lim_{h \to 0} \frac{-h}{x(x+h)} \div h$$

A fraction over a denominator h is the same as the fraction divided by h, and the h then cancels.

$$= \lim_{h \to 0} -\frac{1}{x^2 + xh}$$

The xh term becomes zero.

As $h \to 0$ the limiting value is $-\frac{1}{x^2} = -x^{-2}$.

So when $f(x) = x^{-1}$, $f'(x) = (-1)x^{-2}$.

You have found that:

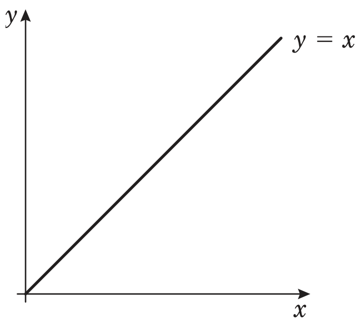
$f(x) = x^2$ gives $f'(x) = 2x^{2-1}$

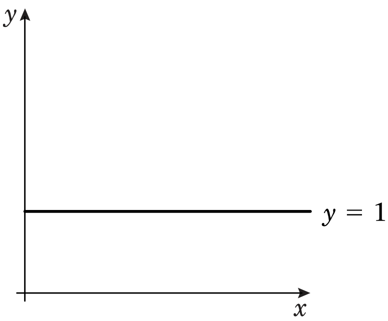
$f(x) = x^3$ gives $f'(x) = 3x^{3-1}$

$f(x) = x^{-1}$ gives $f'(x) = -1x^{-1-1}$

Hint: Notice the pattern in these results is the same each time.

Also, you know that the gradient of the straight line $y = x$ is 1, and that the gradient of the straight line $y = 1$ is 0.





So $f(x) = x^1$ gives $f'(x) = 1x^{1-1}$

and $f(x) = x^0$ gives $f'(x) = 0x^{0-1}$

Hint: Notice the same pattern for these linear functions.

■ **In general it can be shown that if**

$$f(x) = x^n,\ n \in \mathbb{R} \text{ then } f'(x) = nx^{n-1}$$

So the original power multiplies the expression and the power of x is reduced by 1.

Example 5

Find the derived function when f(x) equals:

a x^6 **b** $x^{\frac{1}{2}}$ **c** x^{-2} **d** $\frac{x}{x^5}$ **e** $x^2 \times x^3$

a $6x^5$

The power 6 is reduced to power 5 and the 6 multiplies the answer.

b $f(x) = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

The power $\frac{1}{2}$ is reduced to $\frac{1}{2} - 1 = -\frac{1}{2}$, and the $\frac{1}{2}$ multiplies the answer. This is then rewritten in an alternative form.

c $f(x) = x^{-2}$

$f'(x) = -2x^{-3}$

$= -\frac{2}{x^3}$

The power -2 is reduced to -3 and the -2 multiplies the answer. This is also rewritten in an alternative form using knowledge of negative powers.

d Let $f(x) = x \div x^5$
$= x^{-4}$

So $f'(x) = -4x^{-5}$
$= -\frac{4}{x^5}$

Simplify using rules of powers to give one simple power, i.e. subtract $1 - 5 = -4$.

Reduce the power -4 to give -5, then multiply your answer by -4.

e Let $f(x) = x^2 \times x^3$
$= x^5$

So $f'(x) = 5x^4$

Add the powers this time to give $2 + 3 = 5$.

Reduce the power 5 to 4 and multiply your answer by 5.

Exercise 7B

Find the derived function, given that f(x) equals:

1 x^7 **2** x^8 **3** x^4

4 $x^{\frac{1}{3}}$ **5** $x^{\frac{1}{4}}$ **6** $\sqrt[3]{x}$

7 x^{-3} **8** x^{-4} **9** $\frac{1}{x^2}$

10 $\frac{1}{x^5}$ **11** $\frac{1}{\sqrt[3]{x}}$ **12** $\frac{1}{\sqrt{x}}$

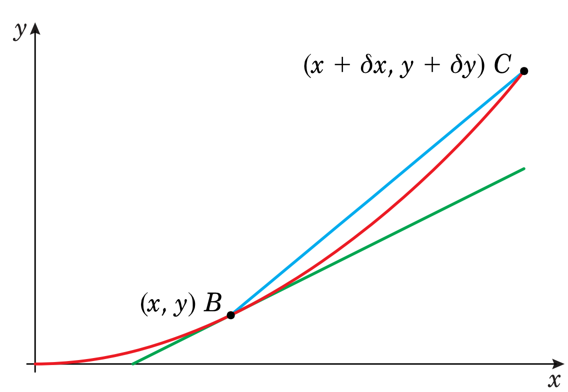
13 $\frac{x^2}{x^4}$ **14** $\frac{x^3}{x^2}$ **15** $\frac{x^6}{x^3}$

16 $x^3 \times x^6$ **17** $x^2 \times x^3$ **18** $x \times x^2$

7.3 You can find the gradient formula for a function such as $f(x) = 4x^2 - 8x + 3$ and other functions of the form $f(x) = ax^2 + bx + c$, where a, b and c are constants.

You can use an alternative notation when finding the gradient function.

Again, you find the gradient of the tangent at a point B by starting with the gradient of a chord BC. This time the point B is the point with coordinates (x, y) and the point C is the point near to B with coordinates $(x + \delta x, y + \delta y)$. δx is called delta x and is a single symbol which stands for a small change in the value of x. This was denoted by h in Section 7.2. Also δy is called 'delta y' and is a single symbol which stands for a small change in the value of y.



The gradient of the chord BC is then

$$\frac{y + \delta y - y}{x + \delta x - x} = \frac{\delta y}{\delta x}$$

But both B and C lie on the curve with equation $y = \text{f}(x)$ and so B is the point $(x, \text{f}(x))$ and C is the point $(x + \delta x, \text{f}(x + \delta x))$.

So the gradient of BC can also be written as

$$\frac{\text{f}(x + \delta x) - \text{f}(x)}{(x + \delta x) - x} = \frac{\text{f}(x + \delta x) - \text{f}(x)}{\delta x}$$

You can make the value of δx very small and you will find that the smaller the value of δx, the smaller the value of δy will be.

The limiting value of the gradient of the chord is the gradient of the tangent at B, which is also the gradient of the curve at B.

This is called the rate of change of y with respect to x at the point B and is denoted by $\frac{\text{d}y}{\text{d}x}$.

$$\frac{\text{d}y}{\text{d}x} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$$

$$= \lim_{\delta x \to 0} \frac{\text{f}(x + \delta x) - \text{f}(x)}{\delta x}$$

$\frac{\text{d}y}{\text{d}x}$ is called the derivative of y with respect to x.

Also $\frac{\text{d}y}{\text{d}x} = \text{f}'(x)$.

The process of finding $\frac{\text{d}y}{\text{d}x}$ when y is given is called differentiation.

■ **When $y = x^n$, $\frac{\text{d}y}{\text{d}x} = nx^{n-1}$ for all real values of n.**

You can also differentiate the general quadratic equation $y = ax^2 + bx + c$.

Using the definition that $\frac{\text{d}y}{\text{d}x} = \lim_{\delta x \to 0} \frac{\text{f}(x + \delta x) - \text{f}(x)}{\delta x}$

Then $\frac{\text{d}y}{\text{d}x} = \lim_{\delta x \to 0} \frac{a(x + \delta x)^2 + b(x + \delta x) + c - (ax^2 + bx + c)}{x + \delta x - x}$

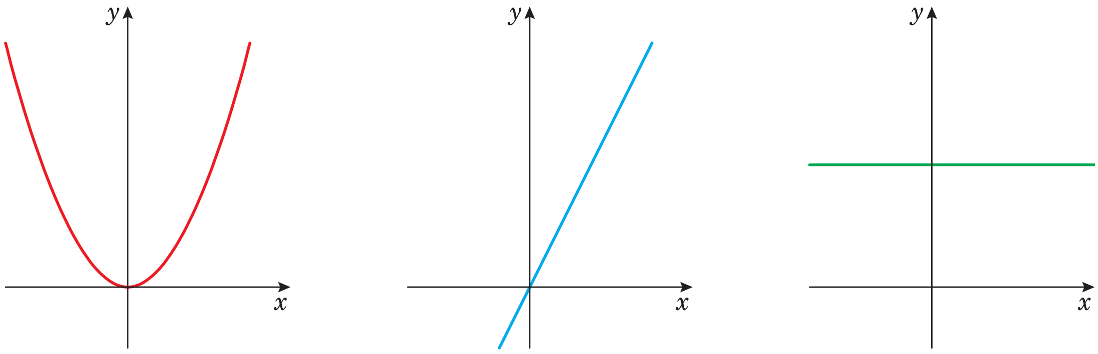
$$= \lim_{\delta x \to 0} \frac{2ax\delta x + a(\delta x)^2 + b\delta x}{\delta x}$$

$$= 2ax + b$$

Therefore when $y = ax^2 + bx + c$, $\frac{\text{d}y}{\text{d}x} = 2ax + b$.

Hint:
Factorise the numerator to give.
$\delta x(2ax + a\delta x + b)$
then simplify the fraction as δx is a common factor.
$a\delta x$ term becomes zero.

Consider the three sketches below:



$y = ax^2$
gradient $= a(2x)$

$y = bx$
gradient $= b$

$y = c$
gradient $= 0$

Combining these functions gives $y = ax^2 + bx + c$, with gradient given by $\frac{dy}{dx} = 2ax + b$.

Example 6

Find $\frac{dy}{dx}$ when y equals:

a x^2 **b** 4 **c** $12x + 3$ **d** $x^2 - 6x - 4$ **e** $3 - 5x^2$

a $2x$

b 0 — The line $y = 4$ has zero gradient.

c 12 — Using $y = mx + c$, the gradient is the value of m.

d $2x - 6$ — Use the result given above with $a = 1$, $b = -6$, $c = -4$.

e $-10x$ — $a = -5$, $b = 0$ and $c = 3$.

Example 7

Let $f(x) = 4x^2 - 8x + 3$.

a Find the gradient of $y = f(x)$ at the point $(\frac{1}{2}, 0)$.

b Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 8.

c Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 4x - 5$.

a As $y = 4x^2 - 8x + 3$

$\frac{dy}{dx} = f'(x) = 8x - 8 + 0$

So $f'(\frac{1}{2}) = -4$

First find $f'(x)$, the derived function, then substitute the x-coordinate value to obtain the gradient.

b $\frac{dy}{dx} = f'(x) = 8x - 8 + 0 = 8$

So $x = 2$

So $y = f(2) = 3$

The point where the gradient is 8 is $(2, 3)$.

Put the gradient function equal to 8. Then solve the equation you have obtained to give the value of x.

Substitute this value for x into $f(x)$ to give the value of y and interpret your answer in words.

c $4x^2 - 8x + 3 = 4x - 5$

$4x^2 - 12x + 8 = 0$

$x^2 - 3x + 2 = 0$

$(x - 2)(x - 1) = 0$

So $x = 1$ or $x = 2$

At $x = 1$ the gradient is 0.

At $x = 2$ the gradient is 8, as in part **b**.

Put $f(x) = 4x - 5$, then rearrange and collect terms to give a quadratic equation.

Divide by the common factor 4.

Solve the quadratic equation by factorising, or by using the quadratic formula

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Substitute the values of x into $f'(x) = 8x - 8$ to give the gradients at the specified points.

Exercise 7C

1 Find $\frac{dy}{dx}$ when y equals:

a $2x^2 - 6x + 3$

b $\frac{1}{2}x^2 + 12x$

c $4x^2 - 6$

d $8x^2 + 7x + 12$

e $5 + 4x - 5x^2$

2 Find the gradient of the curve whose equation is

a $y = 3x^2$ at the point $(2, 12)$

b $y = x^2 + 4x$ at the point $(1, 5)$

c $y = 2x^2 - x - 1$ at the point $(2, 5)$

d $y = \frac{1}{2}x^2 + \frac{3}{2}x$ at the point $(1, 2)$

e $y = 3 - x^2$ at the point $(1, 2)$

f $y = 4 - 2x^2$ at the point $(-1, 2)$

3 Find the y-coordinate and the value of the gradient at the point P with x-coordinate 1 on the curve with equation $y = 3 + 2x - x^2$.

4 Find the coordinates of the point on the curve with equation $y = x^2 + 5x - 4$ where the gradient is 3.

5 Find the gradients of the curve $y = x^2 - 5x + 10$ at the points A and B where the curve meets the line $y = 4$.

6 Find the gradients of the curve $y = 2x^2$ at the points C and D where the curve meets the line $y = x + 3$.

7.4 You can find the gradient formula for a function such as $f(x) = x^3 + x^2 - x^{\frac{1}{2}}$ where the powers of x are real numbers $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, where $a_n, a_{n-1}, \ldots, a_0$ are constants, $a_n \neq 0$ and $n \in \mathbb{R}$.

You know that if $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$.

This is true for all real values of n.

It can also be shown that

- **if $y = ax^n$. where a is a constant then $\frac{dy}{dx} = anx^{n-1}$.**

Also

- **if $y = f(x) \pm g(x)$ then $\frac{dy}{dx} = f'(x) \pm g'(x)$.**

These standard results can be assumed without proof at A Level.

Hint: Note that you again reduce the power by 1 and the original power multiplies the expression.

Example 8

Use standard results to differentiate:

a $x^3 + x^2 - x^{\frac{1}{2}}$ **b** $2x^{-3}$ **c** $\frac{1}{3}x^{\frac{1}{2}} + 4x^2$

a $y = x^3 + x^2 - x^{\frac{1}{2}}$

So $\frac{dy}{dx} = 3x^2 + 2x - \frac{1}{2}x^{-\frac{1}{2}}$

Differentiate each term as you come to it. First x^3, then x^2, then $-x^{\frac{1}{2}}$.

b $y = 2x^{-3}$

So $\frac{dy}{dx} = -6x^{-4}$

$= -\frac{6}{x^4}$

Differentiate x^{-3}, then multiply the answer by 2.

c $x = \frac{1}{3}x^{\frac{1}{2}} + 4x^2$

So $\frac{dy}{dx} = \frac{1}{3} \times \frac{1}{2}x^{-\frac{1}{2}} + 8x$

$= \frac{1}{6} \times x^{-\frac{1}{2}} + 8x$

Take each term as you come to it, and treat each term as a multiple.

Exercise 7D

1 Use standard results to differentiate:

a $x^4 + x^{-1}$ **b** $\frac{1}{2}x^{-2}$ **c** $2x^{-\frac{1}{2}}$

2 Find the gradient of the curve with equation $y = f(x)$ at the point A where:

a $f(x) = x^3 - 3x + 2$ and A is at $(-1, 4)$ **b** $f(x) = 3x^2 + 2x^{-1}$ and A is at $(2, 13)$

3 Find the point or points on the curve with equation $y = f(x)$, where the gradient is zero:

a $f(x) = x^2 - 5x$ **b** $f(x) = x^3 - 9x^2 + 24x - 20$

c $f(x) = x^{\frac{3}{2}} - 6x + 1$ **d** $f(x) = x^{-1} + 4x$

7.5 You can expand or simplify polynomial functions so that they are easier to differentiate.

Example 9

Use standard results to differentiate:

a $\dfrac{1}{4\sqrt{x}}$ **b** $x^3(3x + 1)$ **c** $\dfrac{x - 2}{x^2}$

a Let $y = \dfrac{1}{4\sqrt{x}}$

$= \frac{1}{4}x^{-\frac{1}{2}}$

Therefore $\dfrac{dy}{dx} = -\frac{1}{8}x^{-\frac{3}{2}}$

Express the 4 in the denominator as a multiplier of $\frac{1}{4}$ and express the x term as power $-\frac{1}{2}$.

Then differentiate by reducing the power of x and multiplying $\frac{1}{4}$ by $-\frac{1}{2}$.

b Let $y = x^3(3x + 1)$

$= 3x^4 + x^3$

Therefore $\dfrac{dy}{dx} = 12x^3 + 3x^2$

$= 3x^2(4x + 1)$

Multiply out the brackets to give a polynomial function.

Differentiate each term.

c Let $y = \dfrac{x - 2}{x^2}$

$= \dfrac{1}{x} - \dfrac{2}{x^2}$

$= x^{-1} - 2x^{-2}$

Therefore $\dfrac{dy}{dx} = -x^{-2} + 4x^{-3}$

$= -\dfrac{1}{x^2} + \dfrac{4}{x^3}$

$= \dfrac{-(x - 4)}{x^3}$

Express the single fraction as two separate fractions, and simplify $\dfrac{x}{x^2}$ as $\dfrac{1}{x}$.

Then express the rational expressions as negative powers of x, and differentiate.

Simplify by using a common denominator.

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Exercise 7E

1 Use standard results to differentiate:

a $2\sqrt{x}$ **b** $\frac{3}{x^2}$ **c** $\frac{1}{3x^3}$

d $\frac{1}{3}x^3(x - 2)$ **e** $\frac{2}{x^3} + \sqrt{x}$ **f** $\sqrt[3]{x} + \frac{1}{2x}$

g $\frac{2x + 3}{x}$ **h** $\frac{3x^2 - 6}{x}$ **i** $\frac{2x^3 + 3x}{\sqrt{x}}$

j $x(x^2 - x + 2)$ **k** $3x^2(x^2 + 2x)$ **l** $(3x - 2)\left(4x + \frac{1}{x}\right)$

2 Find the gradient of the curve with equation $y = f(x)$ at the point A where:

a $f(x) = x(x + 1)$ and A is at $(0, 0)$ **b** $f(x) = \frac{2x - 6}{x^2}$ and A is at $(3, 0)$

c $f(x) = \frac{1}{\sqrt{x}}$ and A is at $(\frac{1}{4}, 2)$ **d** $f(x) = 3x - \frac{4}{x^2}$ and A is at $(2, 5)$

7.6 You can repeat the process of differentiation to give a second order derivative.

■ **A second order derivative is written as $\frac{d^2y}{dx^2}$, or $f''(x)$ using function notation.**

Example 10

Given that $y = 3x^5 + \frac{4}{x^2}$ find:

a $\frac{dy}{dx}$ **b** $\frac{d^2y}{dx^2}$

a $y = 3x^5 + \frac{4}{x^2}$

$= 3x^5 + 4x^{-2}$ — Express the fraction as a negative power.

So $\frac{dy}{dx} = 15x^4 - 8x^{-3}$ — Differentiate a first time.

$= 15x^4 - \frac{8}{x^3}$

b $\frac{d^2y}{dx^2} = 60x^3 + 24x^{-4}$ — Differentiate a second time.

$= 60x^3 + \frac{24}{x^4}$

Example 11

Given that $f(x) = 3\sqrt{x} + \frac{1}{2x}$, find:

a $f'(x)$ **b** $f''(x)$

a $f(x) = 3\sqrt{x} + \frac{1}{2\sqrt{x}}$

$= 3x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$

Express the roots as fractional powers.

$f'(x) = \frac{3}{2}x^{-\frac{1}{2}} - \frac{1}{4}x^{-\frac{3}{2}}$

Multiply 3 by a half and reduce power of x.

Multiply a half by negative a half and reduce power of x.

b $f''(x) = -\frac{3}{4}x^{-\frac{3}{2}} + \frac{3}{8}x^{-\frac{5}{2}}$

Note that $\frac{1}{4} \times \frac{3}{2} = \frac{3}{8}$ and the product of two negatives is positive.

Exercise 7F

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when y equals:

1 $12x^2 + 3x + 8$

2 $15x + 6 + \frac{3}{x}$

3 $9\sqrt{x} - \frac{3}{x^2}$

4 $(5x + 4)(3x - 2)$

5 $\frac{3x + 8}{x^2}$

7.7 You can find the rate of change of a function f at a particular point by using $f'(x)$ and substituting in the value of x.

The variables in the relationship $y = f(x)$ are such that x is the independent variable and y is the dependent variable.
These variables often stand for quantities, where it is more meaningful to use letters, other than x and y, to suggest what these quantities are.

For example, it is usual to substitute t for time, V for volume, P for population, A for area, r for radius, s for displacement, h for height, v for velocity, θ for temperature, etc.

So $\frac{dV}{dt}$ might represent the gradient in a graph of volume against time. It therefore would represent the rate of change of volume with respect to time.

Also $\frac{dA}{dr}$ might represent the gradient in a graph of area against radius. It therefore would represent the rate of change of area with respect to radius.

You should know that the rate of change of velocity with respect to time is acceleration, and that the rate of change of displacement with respect to time is velocity.

Example 12

Given that the volume (V cm^3) of an expanding sphere is related to its radius (r cm) by the formula $V = \frac{4}{3}\pi r^3$, find the rate of change of volume with respect to radius at the instant when the radius is 5 cm.

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$ — Differentiate V with respect to r.

When $r = 5$, $\frac{dV}{dr} = 4\pi \times 5^2$ — Substitute $r = 5$.

$= 314$

So the rate of change is 314 cm^3 per cm. — Interpret the answer with units.

Exercise 7G

1 Find $\frac{d\theta}{dt}$ where $\theta = t^2 - 3t$

2 Find $\frac{dA}{dr}$ where $A = 2\pi r$

3 Find $\frac{dr}{dt}$ where $r = \frac{12}{t}$

4 Find $\frac{dv}{dt}$ where $v = 9.8t + 6$

5 Find $\frac{dR}{dr}$ where $R = r + \frac{5}{r}$

6 Find $\frac{dx}{dt}$ where $x = 3 - 12t + 4t^2$

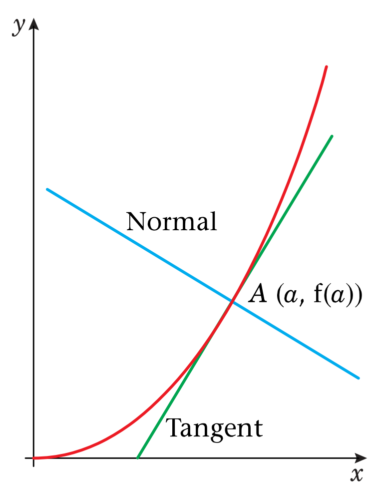
7 Find $\frac{dA}{dx}$ where $A = x(10 - x)$

7.8 You can use differentiation to find the gradient of a tangent to a curve and you can then find the equation of the tangent and normal to that curve at a specified point.

The tangent at the point A $(a, f(a))$ has gradient $f'(a)$. You can use the formula for the equation of a straight line, $y - y_1 = m(x - x_1)$, to obtain the equation of the tangent at $(a, f(a))$.

■ **The equation of the tangent to a curve at a point $(a, f(a))$ is $y - f(a) = f'(a)(x - a)$.**

The normal to the curve at the point A is defined as being the straight line through A which is perpendicular to the tangent at A (see sketch alongside).



The gradient of the normal is $-\frac{1}{f'(a)}$, because the product of the gradients of lines which are at right angles is -1.

■ **The equation of the normal at point A is $y - f(a) = -\frac{1}{f'(a)}(x - a)$.**

Example 13

Find the equation of the tangent to the curve $y = x^3 - 3x^2 + 2x - 1$ at the point (3, 5).

$y = x^3 - 3x^2 + 2x - 1$

$\frac{dy}{dx} = 3x^2 - 6x + 2$

When $x = 3$, the gradient is 11.

So the equation of the tangent at $(3, 5)$ is

$y - 5 = 11(x - 3)$

$y = 11x - 28$

First differentiate to determine the gradient of the curve and therefore the gradient of the tangent.

Then substitute for x to calculate the value of the gradient of the curve and of the tangent when $x = 3$.

You can now use the line equation and simplify.

Example 14

Find the equation of the normal to the curve with equation $y = 8 - 3\sqrt{x}$ at the point where $x = 4$.

$y = 8 - 3\sqrt{x}$

$= 8 - 3x^{\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{3}{2}x^{-\frac{1}{2}}$

Where $x = 4$, $y = 2$ and gradient of curve and of tangent $= -\frac{3}{4}$.

So gradient of normal is $\frac{4}{3}$.

Equation of normal is

$y - 2 = \frac{4}{3}(x - 4)$

$3y - 6 = 4x - 16$

$3y - 4x + 10 = 0$

Express the function simply as powers of x, and differentiate to obtain the gradient function.

You find the y-coordinate when $x = 4$ by substituting into the equation of the curve and calculating $8 - 3\sqrt{4}$.

Then find the gradient of the curve, by calculating

$\frac{dy}{dx} = -\frac{3}{2}(4)^{-\frac{1}{2}} = -\frac{3}{2 \times 2}$.

Use normal gradient

$= -\frac{1}{\text{gradient of curve}} = -\frac{1}{-\frac{3}{4}} = +\frac{4}{3}$.

Then simplify by multiplying both sides by 3 and collecting terms.

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Exercise 7H

1 Find the equation of the tangent to the curve:

a $y = x^2 - 7x + 10$ at the point $(2, 0)$

b $y = x + \frac{1}{x}$ at the point $(2, 2\frac{1}{2})$

c $y = 4\sqrt{x}$ at the point $(9, 12)$

d $y = \frac{2x - 1}{x}$ at the point $(1, 1)$

e $y = 2x^3 + 6x + 10$ at the point $(-1, 2)$

f $y = x^2 + \frac{-7}{x^2}$ at the point $(1, -6)$

2 Find the equation of the normal to the curves:

a $y = x^2 - 5x$ at the point $(6, 6)$

b $y = x^2 - \frac{8}{\sqrt{x}}$ at the point $(4, 12)$

3 Find the coordinates of the point where the tangent to the curve $y = x^2 + 1$ at the point $(2, 5)$ meets the normal to the same curve at the point $(1, 2)$.

4 Find the equations of the normals to the curve $y = x + x^3$ at the points $(0, 0)$ and $(1, 2)$, and find the coordinates of the point where these normals meet.

5 For $f(x) = 12 - 4x + 2x^2$, find an equation of the tangent and normal at the point where $x = -1$ on the curve with equation $y = f(x)$. *E*

Mixed exercise 7I

1 A curve is given by the equation $y = 3x^2 + 3 + \frac{1}{x^2}$, where $x > 0$. At the points A, B and C on the curve, $x = 1$, 2 and 3 respectively. Find the gradients at A, B and C. *E*

2 Taking $f(x) = \frac{1}{4}x^4 - 4x^2 + 25$, find the values of x for which $f'(x) = 0$. *E*

3 A curve is drawn with equation $y = 3 + 5x + x^2 - x^3$. Find the coordinates of the two points on the curve where the gradient of the curve is zero. *E*

4 Calculate the x-coordinates of the points on the curve with equation $y = 7x^2 - x^3$ at which the gradient is equal to 16. *E*

5 Find the x-coordinates of the two points on the curve with equation $y = x^3 - 11x + 1$ where the gradient is 1. Find the corresponding y-coordinates. *E*

6 The function f is defined by $f(x) = x + \frac{9}{x}$, $x \in \mathbb{R}$, $x \neq 0$.

a Find $f'(x)$. **b** Solve $f'(x) = 0$. *E*

7 Given that

$$y = x^{\frac{3}{2}} + \frac{48}{x}, \quad x > 0,$$

find the value of x and the value of y when $\frac{dy}{dx} = 0$. *E*

8 Given that

$$y = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}, \quad x > 0,$$

find $\frac{dy}{dx}$. *E*

9 A curve has equation $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$.

a Show that $\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$

b Find the coordinates of the point on the curve where the gradient is zero. *E*

10 **a** Expand $(x^{\frac{3}{2}} - 1)(x^{-\frac{1}{2}} + 1)$.

b A curve has equation $y = (x^{\frac{3}{2}} - 1)(x^{-\frac{1}{2}} + 1)$, $x > 0$. Find $\frac{dy}{dx}$.

c Use your answer to **b** to calculate the gradient of the curve at the point where $x = 4$. *E*

11 Differentiate with respect to x:

$$2x^3 + \sqrt{x} + \frac{x^2 + 2x}{x^2}$$ *E*

12 The volume, V cm^3, of a tin of radius r cm is given by the formula $V = \pi(40r - r^2 - r^3)$. Find the positive value of r for which $\frac{dV}{dr} = 0$, and find the value of V which corresponds to this value of r. *E*

13 The total surface area of a cylinder A cm^2 with a fixed volume of 1000 cubic cm is given by the formula $A = 2\pi x^2 + \frac{2000}{x}$, where x cm is the radius. Show that when the rate of change of the area with respect to the radius is zero, $x^3 = \frac{500}{\pi}$. *E*

14 The curve with equation $y = ax^2 + bx + c$ passes through the point $(1, 2)$. The gradient of the curve is zero at the point $(2, 1)$. Find the values of a, b and c. *E*

15 A curve C has equation $y = x^3 - 5x^2 + 5x + 2$.

a Find $\frac{dy}{dx}$ in terms of x.

b The points P and Q lie on C. The gradient of C at both P and Q is 2. The x-coordinate of P is 3.

i Find the x-coordinate of Q.

ii Find an equation for the tangent to C at P, giving your answer in the form $y = mx + c$, where m and c are constants.

iii If this tangent intersects the coordinate axes at the points R and S, find the length of RS, giving your answer as a surd. *E*

16 Find an equation of the tangent and the normal at the point where $x = 2$ on the curve with equation $y = \frac{8}{x} - x + 3x^2$, $x > 0$. *E*

17 The normals to the curve $2y = 3x^3 - 7x^2 + 4x$, at the points $O(0, 0)$ and $A(1, 0)$, meet at the point N.

a Find the coordinates of N.

b Calculate the area of triangle OAN. *E*