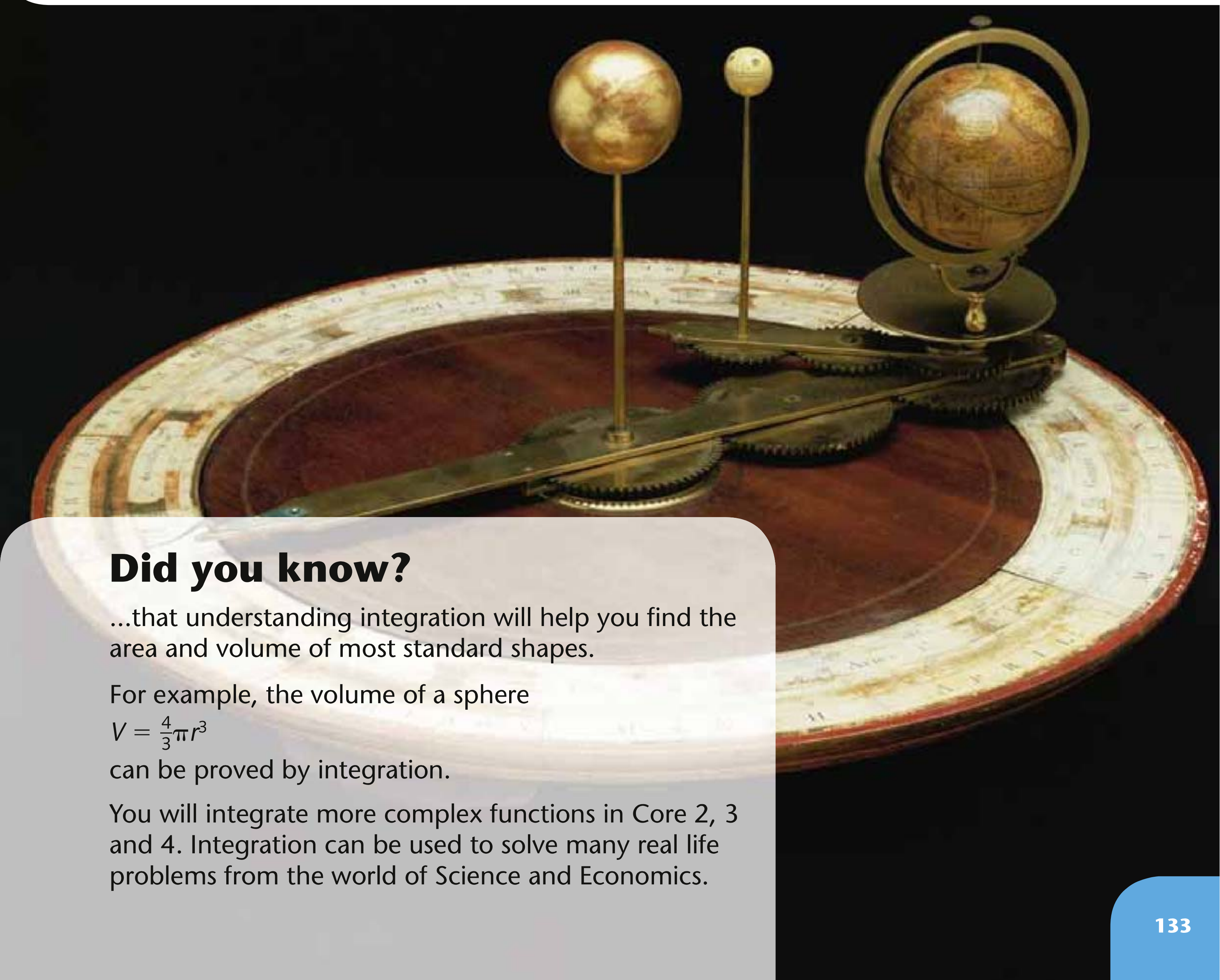
After completing this chapter you should be able to

1. integrate simple functions
2. understand the symbol $\int dx$
3. find the constant of integration by substituting in a given point (x, y).

8

Integration



Did you know?

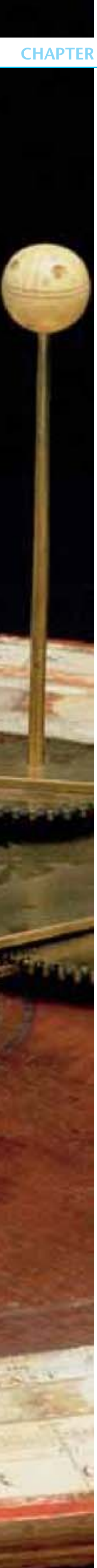
...that understanding integration will help you find the area and volume of most standard shapes.

For example, the volume of a sphere

$V = \frac{4}{3}\pi r^3$

can be proved by integration.

You will integrate more complex functions in Core 2, 3 and 4. Integration can be used to solve many real life problems from the world of Science and Economics.



8.1 You can integrate functions of the form $f(x) = ax^n$ where $n \in \mathbb{R}$ and a is a constant.

In Chapter 7 you saw that if $y = x^2$

then $\frac{dy}{dx} = 2x$.

Also if $y = x^2 + 1$

then $\frac{dy}{dx} = 2x$.

So if $y = x^2 + c$ where c is some constant

then $\frac{dy}{dx} = 2x$.

Integration is the process of finding y when you know $\frac{dy}{dx}$.

If $\frac{dy}{dx} = 2x$

then $y = x^2 + c$ where c is some constant.

■ **If $\frac{dy}{dx} = x^n$, then $y = \frac{1}{n+1}x^{n+1} + c$, $n \neq -1$.**

Hint: This is called indefinite integration because you cannot find the constant.

Example 1

Find y for the following:

a $\frac{dy}{dx} = x^4$ **b** $\frac{dy}{dx} = x^{-5}$

a $\frac{dy}{dx} = x^4$

$y = \frac{x^5}{5} + c$

$\frac{dy}{dx} = x^n$ where $n = 4$.

So use $y = \frac{1}{n+1}x^{n+1} + c$ for $n = 4$.

Raise the power by 1.

Divide by the new power and don't forget to add c.

b $\frac{dy}{dx} = x^{-5}$

$y = \frac{x^{-4}}{-4} + c$

$= -\frac{1}{4}x^{-4} + c$

Remember raising the power by 1 gives $-5 + 1 = -4$.

Divide by the new power (-4) and add c.

Example 2

Find y for the following:

a $\frac{dy}{dx} = 2x^3$ **b** $\frac{dy}{dx} = 3x^{\frac{1}{2}}$

a

$$\frac{dy}{dx} = 2x^3$$
$$= 2 \times x^3$$

So $y = 2 \times \frac{x^4}{4} + c$

$$= \frac{x^4}{2} + c$$

Use the formula first with $n = 3$.

Then simplify the $\frac{2}{4}$ to $\frac{1}{2}$.

Check $\frac{dy}{dx} = \frac{4x^3}{2} = 2x^3$.

b

$$\frac{dy}{dx} = 3x^{\frac{1}{2}}$$

So $y = 3 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$$= 2x^{\frac{3}{2}} + c$$

It is always worth writing down this line as sometimes marks are given for unsimplified expressions.

Remember $3 \div \frac{3}{2} = 3 \times \frac{2}{3} = 2$.

It is always worth simplifying your answers as you may need to use this expression in a later part of the question.

Notice that you treat $\frac{dy}{dx} = x^n$ and $\frac{dy}{dx} = kx^n$ in the same way. You only consider the x^n term when integrating.

So in general

■ **If $\frac{dy}{dx} = kx^n$, then $y = \frac{kx^{n+1}}{n+1} + c$, $n \neq -1$.**

Exercise 8A

Find an expression for y when $\frac{dy}{dx}$ is the following:

1 x^5	**2** $10x^4$	**3** $3x^2$
4 $-x^{-2}$	**5** $-4x^{-3}$	**6** $x^{\frac{2}{3}}$
7 $4x^{\frac{1}{2}}$	**8** $-2x^6$	**9** $3x^5$
10 $3x^{-4}$	**11** $x^{-\frac{1}{2}}$	**12** $5x^{-\frac{3}{2}}$
13 $-2x^{-\frac{3}{2}}$	**14** $6x^{\frac{1}{3}}$	**15** $36x^{11}$
16 $-14x^{-8}$	**17** $-3x^{-\frac{2}{3}}$	**18** -5
19 $6x$	**20** $2x^{-0.4}$	

8.2 You can apply the principle of integration separately to each term of $\frac{dy}{dx}$.

Example 3

Given $\frac{dy}{dx} = 6x + 2x^{-3} - 3x^{\frac{1}{2}}$, find y.

$$y = \frac{6x^2}{2} + \frac{2}{-2}x^{-2} - \frac{3}{\frac{3}{2}}x^{\frac{3}{2}} + c$$

$$= 3x^2 - x^{-2} - 2x^{\frac{3}{2}} + c$$

Apply the rule from Section 8.1 to each term of the expression.

Then simplify each term and don't forget to add c.

In Chapter 7 you saw that if $y = \text{f}(x)$, then $\frac{dy}{dx} = \text{f}'(x)$.

Hint: Both types of notation are used in the next exercise. Sometimes we say that the integral of $\frac{dy}{dx}$ is y or the integral of $\text{f}'(x)$ is $\text{f}(x)$.

Exercise 8B

1 Find y when $\frac{dy}{dx}$ is given by the following expressions. In each case simplify your answer:

a $4x - x^{-2} + 6x^{\frac{1}{2}}$

b $15x^2 + 6x^{-3} - 3x^{-\frac{5}{2}}$

c $x^3 - \frac{3}{2}x^{-\frac{1}{2}} - 6x^{-2}$

d $4x^3 + x^{-\frac{2}{3}} - x^{-2}$

e $4 - 12x^{-4} + 2x^{-\frac{1}{2}}$

f $5x^{\frac{2}{3}} - 10x^4 + x^{-3}$

g $-\frac{4}{3}x^{-\frac{4}{3}} - 3 + 8x$

h $5x^4 - x^{-\frac{3}{2}} - 12x^{-5}$

2 Find $\text{f}(x)$ when $\text{f}'(x)$ is given by the following expressions. In each case simplify your answer:

a $12x + \frac{3}{2}x^{-\frac{3}{2}} + 5$

b $6x^5 + 6x^{-7} - \frac{1}{6}x^{-\frac{7}{6}}$

c $\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

d $10x + 8x^{-3}$

e $2x^{-\frac{1}{3}} + 4x^{-\frac{5}{3}}$

f $9x^2 + 4x^{-3} + \frac{1}{4}x^{-\frac{1}{2}}$

g $x^2 + x^{-2} + x^{\frac{1}{2}}$

h $-2x^{-3} - 2x + 2x^{\frac{1}{2}}$

8.3 You need to be able to use the integral sign.

The integral of x^n is denoted by $\int x^n\,dx$ and the formula you met in Section 8.1 is:

■ $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

The elongated S means integrate.

The expression to be integrated.

This dx tells you which letter is the variable to integrate with respect to. See example 4.

Example 4

Find:

a $\int (x^{\frac{1}{2}} + 2x^3)\,dx$ **b** $\int (x^{-\frac{3}{2}} + 2)\,dx$

c $\int (3x^2 + p^2x^{-2} + q)\,dx$ **d** $\int (4t^2 + 6)\,dt$

a $\int (x^{\frac{1}{2}} + 2x^3)\,dx$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2x^4}{4} + c$$

First apply the rule term by term. Then simplify each term.

$$= \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^4 + c$$

b $\int (x^{-\frac{3}{2}} + 2)\,dx$

$$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + 2x + c$$

Remember $\frac{-3}{2} + 1 = -\frac{1}{2}$ and the integral of a constant like 2 is $2x$.

$$= -2x^{-\frac{1}{2}} + 2x + c$$

c $\int (3x^2 + p^2x^{-2} + q)\,dx$

The dx tells you to integrate with respect to the variable x, so any other letters must be treated as constants.

$$= \frac{3x^3}{3} + \frac{p^2}{-1}x^{-1} + qx + c$$

$$= x^3 - p^2x^{-1} + qx + c$$

d $\int (4t^2 + 6)\,dt$

The dt tells you that this time you must integrate with respect to t.

$$= \frac{4t^3}{3} + 6t + c$$

Exercise 8C

Find the following integrals.

1 $\int(x^3 + 2x)\mathrm{d}x$

2 $\int(2x^{-2} + 3)\mathrm{d}x$

3 $\int(5x^{\frac{3}{2}} - 3x^2)\mathrm{d}x$

4 $\int(2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + 4)\mathrm{d}x$

5 $\int(4x^3 - 3x^{-4} + r)\mathrm{d}x$

6 $\int(3t^2 - t^{-2})\mathrm{d}t$

7 $\int(2t^2 - 3t^{-\frac{3}{2}} + 1)\mathrm{d}t$

8 $\int(x + x^{-\frac{1}{2}} + x^{-\frac{3}{2}})\mathrm{d}x$

9 $\int(px^4 + 2t + 3x^{-2})\mathrm{d}x$

10 $\int(pt^3 + q^2 + px^3)\mathrm{d}t$

8.4 **You need to simplify an expression into separate terms of the form x^n, $n \in \mathbb{R}$, before you integrate.**

Example 5

Find the following integrals:

a $\int\left(\frac{2}{x^3} - 3\sqrt{x}\right)\mathrm{d}x$ **b** $\int x\left(x^2 + \frac{2}{x}\right)\mathrm{d}x$ **c** $\int\left[(2x)^2 + \frac{\sqrt{x} + 5}{x^2}\right]\mathrm{d}x$

a $\int\left(\frac{2}{x^3} - 3\sqrt{x}\right)dx$

$= \int(2x^{-3} - 3x^{\frac{1}{2}})dx$ — First write each term in the form x^n.

$= \frac{2}{-2}x^{-2} - \frac{3}{\frac{3}{2}}x^{\frac{3}{2}} + c$ — Apply the rule term by term.

$= -x^{-2} - 2x^{\frac{3}{2}} + c$ — Then simplify each term.

or $= -\frac{1}{x^2} - 2\sqrt{x^3} + c$ — Sometimes it is helpful to write the answer in the same form as the question.

b $\int x\left(x^2 + \frac{2}{x}\right)dx$

$= \int(x^3 + 2)dx$ — First multiply out the bracket.

$= \frac{x^4}{4} + 2x + c$ — Then apply the rule to each term.

c $\int\left[(2x)^2 + \dfrac{\sqrt{x}+5}{x^2}\right]dx$

$= \int\left[4x^2 + \dfrac{x^{\frac{1}{2}}}{x^2} + \dfrac{5}{x^2}\right]dx$ — Simplify $(2x)^2$ and write $\sqrt{x}$ as $x^{\frac{1}{2}}$.

$= \int(4x^2 + x^{-\frac{3}{2}} + 5x^{-2})dx$ — Write each term in the x^n form.

$= \dfrac{4}{3}x^3 + \dfrac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + \dfrac{5x^{-1}}{-1} + c$ — Apply the rule term by term.

$= \dfrac{4}{3}x^3 - 2x^{-\frac{1}{2}} - 5x^{-1} + c$

or $= \dfrac{4}{3}x^3 - \dfrac{2}{\sqrt{x}} - \dfrac{5}{x} + c$ — Finally simplify the answer.

Exercise 8D

1 Find the following integrals:

a $\int(2x+3)x^2\,dx$ **b** $\int\dfrac{(2x^2+3)}{x^2}\,dx$ **c** $\int(2x+3)^2\,dx$

d $\int(2x+3)(x-1)\,dx$ **e** $\int(2x+3)\sqrt{x}\,dx$

2 Find $\int f(x)dx$ when $f(x)$ is given by the following:

a $(x+2)^2$ **b** $\left(x+\dfrac{1}{x}\right)^2$ **c** $(\sqrt{x}+2)^2$

d $\sqrt{x}(x+2)$ **e** $\left(\dfrac{x+2}{\sqrt{x}}\right)$ **f** $\left(\dfrac{1}{\sqrt{x}}+2\sqrt{x}\right)$

3 Find the following integrals:

a $\int\left(3\sqrt{x}+\dfrac{1}{x^2}\right)dx$ **b** $\int\left(\dfrac{2}{\sqrt{x}}+3x^2\right)dx$

c $\int\left(x^{\frac{2}{3}}+\dfrac{4}{x^3}\right)dx$ **d** $\int\left(\dfrac{2+x}{x^3}+3\right)dx$

e $\int(x^2+3)(x-1)dx$ **f** $\int\left(\dfrac{2}{\sqrt{x}}+3x\sqrt{x}\right)dx$

g $\int(x-3)^2\,dx$ **h** $\int\dfrac{(2x+1)^2}{\sqrt{x}}\,dx$

i $\int\left(3+\dfrac{\sqrt{x}+6x^3}{x}\right)dx$ **j** $\int\sqrt{x}(\sqrt{x}+3)^2\,dx$

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8.5 You can find the constant of integration, c, when you are given any point (x, y) that the curve of the function passes through.

Example 6

The curve C with equation $y = \text{f}(x)$ passes through the point (4, 5). Given that $\text{f}'(x) = \frac{x^2 - 2}{\sqrt{x}}$, find the equation of C.

$$\text{f}'(x) = \frac{x^2 - 2}{\sqrt{x}}$$

$$= x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}$$

First write $\text{f}'(x)$ in a form suitable for integration.

So $$\text{f}(x) = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

Integrate as normal and don't forget the $+c$.

$$= \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + c$$

But $\text{f}(4) = 5$

Use the fact that the curve passes through (4, 5).

So $$5 = \frac{2}{5} \times 2^5 - 4 \times 2 + c$$

Remember $4^{\frac{5}{2}} = 2^5$.

$$5 = \frac{64}{5} - 8 + c$$

$$5 = \frac{24}{5} + c$$

So $$c = \frac{1}{5}$$

Solve for c.

So the equation of the curve is

$$y = \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + \frac{1}{5}.$$

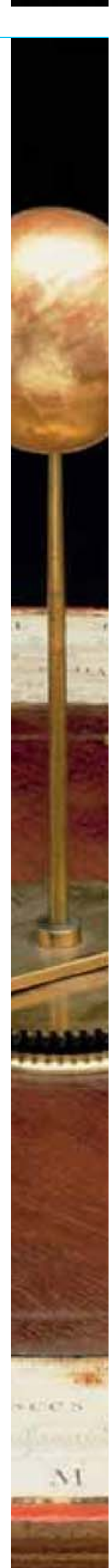
Finally write down the equation of the curve.

Exercise 8E

1 Find the equation of the curve with the given derivative of y with respect to x that passes through the given point:

a $\frac{\text{d}y}{\text{d}x} = 3x^2 + 2x$; point (2, 10)

b $\frac{\text{d}y}{\text{d}x} = 4x^3 + \frac{2}{x^3} + 3$; point (1, 4)



c $\dfrac{dy}{dx} = \sqrt{x} + \dfrac{1}{4}x^2$; point (4, 11)

d $\dfrac{dy}{dx} = \dfrac{3}{\sqrt{x}} - x$; point (4, 0)

e $\dfrac{dy}{dx} = (x + 2)^2$; point (1, 7)

f $\dfrac{dy}{dx} = \dfrac{x^2 + 3}{\sqrt{x}}$; point (0, 1)

2 The curve C, with equation $y = f(x)$, passes through the point (1, 2) and $f'(x) = 2x^3 - \dfrac{1}{x^2}$.
Find the equation of C in the form $y = f(x)$.

3 The gradient of a particular curve is given by $\dfrac{dy}{dx} = \dfrac{\sqrt{x} + 3}{x^2}$. Given that the curve passes through the point (9, 0), find an equation of the curve.

4 A set of curves, that each pass through the origin, have equations $y = f_1(x)$, $y = f_2(x)$, $y = f_3(x)$ … where $f_n'(x) = f_{n-1}(x)$ and $f_1(x) = x^2$.

a Find $f_2(x)$, $f_3(x)$.

b Suggest an expression for $f_n(x)$.

5 A set of curves, with equations $y = f_1(x)$, $y = f_2(x)$, $y = f_3(x)$ … all pass through the point (0, 1) and they are related by the property $f_n'(x) = f_{n-1}(x)$ and $f_1(x) = 1$.
Find $f_2(x)$, $f_3(x)$, $f_4(x)$.

Mixed exercise 8F

1 Find:

a $\int (x + 1)(2x - 5)\,dx$ **b** $\int (x^{\frac{1}{3}} + x^{-\frac{1}{3}})\,dx$.

2 The gradient of a curve is given by $f'(x) = x^2 - 3x - \dfrac{2}{x^2}$. Given that the curve passes through the point (1, 1), find the equation of the curve in the form $y = f(x)$.

3 Find:

a $\int (8x^3 - 6x^2 + 5)\,dx$ **b** $\int (5x + 2)x^{\frac{1}{2}}\,dx$.

4 Given $y = \dfrac{(x + 1)(2x - 3)}{\sqrt{x}}$, find $\int y\,dx$.

5 Given that $\dfrac{dx}{dt} = 3t^2 - 2t + 1$ and that $x = 2$ when $t = 1$, find the value of x when $t = 2$.

6 Given $y = 3x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}$, $x > 0$, find $\int y\,dx$.

7 Given that $\dfrac{dx}{dt} = (t + 1)^2$ and that $x = 0$ when $t = 2$, find the value of x when $t = 3$.

8 Given that $y^{\frac{1}{2}} = x^{\frac{1}{3}} + 3$:

a show that $y = x^{\frac{2}{3}} + Ax^{\frac{1}{3}} + B$, where A and B are constants to be found

b hence find $\int y\,\mathrm{d}x$. *E*

9 Given that $y = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$ $(x > 0)$:

a find $\frac{\mathrm{d}y}{\mathrm{d}x}$

b find $\int y\,\mathrm{d}x$. *E*

10 Find $\int (x^{\frac{1}{2}} - 4)(x^{-\frac{1}{2}} - 1)\mathrm{d}x$. *E*

Summary of key points

1 If $\frac{\mathrm{d}y}{\mathrm{d}x} = x^n$, then $y = \frac{1}{n+1}x^{n+1} + c$ $(n \neq -1)$.

2 If $\frac{\mathrm{d}y}{\mathrm{d}x} = kx^n$, then $y = \frac{kx^{n+1}}{n+1} + c$ $(n \neq -1)$.

3 $\int x^n\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$ $(n \neq -1)$.