After completing this chapter you should be able to

1. plot the graph of a quadratic function
2. solve a quadratic function using factorisation
3. complete the square of a quadratic function
4. solve a quadratic equation by using the quadratic formula
5. calculate the discriminant of a quadratic expression
6. sketch the graph of a quadratic function.

The above techniques will enable you to solve many types of equation and inequality. The ability to spot and solve a quadratic equation is extremely important in A level Mathematics.

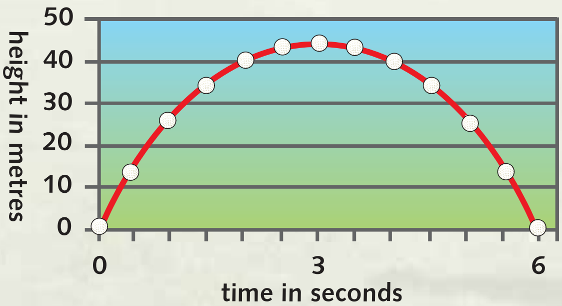
2

Quadratic functions

Did you know?

...that the path of a golf ball can be modelled by a quadratic function?

graph showing height of a golf ball against time in seconds



2.1 You need to be able to plot graphs of quadratic equations.

- **The general form of a quadratic equation is**

  $y = ax^2 + bx + c$

  where a, b and c are constants and $a \neq 0$.

This could also be written as $f(x) = ax^2 + bx + c$.

Example 1

a Draw the graph with equation $y = x^2 - 3x - 4$ for values of x from -2 to $+5$.

b Write down the minimum value of y and the value of x for this point.

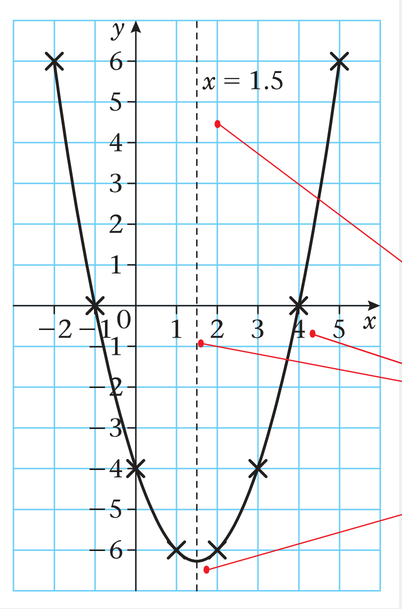
c Label the line of symmetry.

a

x	-2	-1	0	1	2	3	4	5
x^2	4	1	0	1	4	9	16	25
$-3x$	$+6$	$+3$	0	-3	-6	-9	-12	-15
-4	-4	-4	-4	-4	-4	-4	-4	-4
y	6	0	-4	-6	-6	-4	0	6

① First draw a table of values. Remember any number squared is positive.

② Look at the table to determine the extent of the y-axis. Use values of y from -6 to $+6$.



③ Plot the points and then join all the points together with a smooth curve.

The general shape of the curve is a ∪, it is called a parabola.

This is the line of symmetry. It is always half-way between the x-axis crossing points. It has equation $x = 1.5$.

This is the minimum.

b Minimum value is $y = -6.3$ when $x = 1.5$.

c See graph.

Exercise 2A

Draw graphs with the following equations, taking values of x from -4 to $+4$.

For each graph write down the equation of the line of symmetry.

1 $y = x^2 - 3$

2 $y = x^2 + 5$

3 $y = \frac{1}{2}x^2$

4 $y = -x^2$

5 $y = (x - 1)^2$

6 $y = x^2 + 3x + 2$

7 $y = 2x^2 + 3x - 5$

8 $y = x^2 + 2x - 6$

9 $y = (2x + 1)^2$

Hint: The general shape for question **4** is an upside down ∪-shape. i.e. ∩.

2.2 You can solve quadratic equations using factorisation.

Quadratic equations have two solutions or roots. (In some cases the two roots are equal.) To solve a quadratic equation, put it in the form $ax^2 + bx + c = 0$.

Example 2

Solve the equation $x^2 = 9x$

$x^2 = 9x$

$x^2 - 9x = 0$ — Rearrange in the form $ax^2 + bx + c = 0$.

$x(x - 9) = 0$ — Factorise by x (factorising is in Chapter 1). Then either part of the product could be zero.

Then either $x = 0$

or $x - 9 = 0 \Rightarrow x = 9$

So $x = 0$ or $x = 9$ are the two solutions of the equation $x^2 = 9x$.

A quadratic equation has two solutions (roots). In some cases the two roots are equal.

Example 3

Solve the equation $x^2 - 2x - 15 = 0$

$x^2 - 2x - 15 = 0$

$(x + 3)(x - 5) = 0$ — Factorise.

Then either $x + 3 = 0 \Rightarrow x = -3$

or $x - 5 = 0 \Rightarrow x = 5$

The solutions are $x = -3$ or $x = 5$.

Example 4

Solve the equation $6x^2 + 13x - 5 = 0$

$6x^2 + 13x - 5 = 0$

$(3x - 1)(2x + 5) = 0$

Then either $3x - 1 = 0 \Rightarrow x = \frac{1}{3}$

or $2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$

The solutions are $x = \frac{1}{3}$ or $x = -\frac{5}{2}$.

Factorise.

The solutions can be fractions or any other type of number.

Example 5

Solve the equation $x^2 - 5x + 18 = 2 + 3x$

$x^2 - 5x + 18 = 2 + 3x$

$x^2 - 8x + 16 = 0$

$(x - 4)(x - 4) = 0$

Then either $x - 4 = 0 \Rightarrow x = 4$

or $x - 4 = 0 \Rightarrow x = 4$

$\Rightarrow x = 4$

Rearrange in the form $ax^2 + bx + c = 0$.

Factorise.

Here $x = 4$ is the only solution, i.e. the two roots are equal.

Example 6

Solve the equation $(2x - 3)^2 = 25$

$(2x - 3)^2 = 25$

$2x - 3 = \pm 5$

$2x = 3 \pm 5$

Then either $2x = 3 + 5 \Rightarrow x = 4$

or $2x = 3 - 5 \Rightarrow x = -1$

The solutions are $x = 4$ or $x = -1$.

This is a special case.

Take the square root of both sides.

Remember $\sqrt{25} = +5$ or -5.

Add 3 to both sides.

Example 7

Solve the equation $(x - 3)^2 = 7$

$(x - 3)^2 = 7$

$x - 3 = \pm\sqrt{7}$

$x = +3 \pm \sqrt{7}$

Then either $x = 3 + \sqrt{7}$

or $x = 3 - \sqrt{7}$

The solutions are $x = 3 + \sqrt{7}$ or $x = 3 - \sqrt{7}$.

Square root. (If you do not have a calculator, leave this in surd form.)

Exercise 2B

Solve the following equations:

1 $x^2 = 4x$

2 $x^2 = 25x$

3 $3x^2 = 6x$

4 $5x^2 = 30x$

5 $x^2 + 3x + 2 = 0$

6 $x^2 + 5x + 4 = 0$

7 $x^2 + 7x + 10 = 0$

8 $x^2 - x - 6 = 0$

9 $x^2 - 8x + 15 = 0$

10 $x^2 - 9x + 20 = 0$

11 $x^2 - 5x - 6 = 0$

12 $x^2 - 4x - 12 = 0$

13 $2x^2 + 7x + 3 = 0$

14 $6x^2 - 7x - 3 = 0$

15 $6x^2 - 5x - 6 = 0$

16 $4x^2 - 16x + 15 = 0$

17 $3x^2 + 5x = 2$

18 $(2x - 3)^2 = 9$

19 $(x - 7)^2 = 36$

20 $2x^2 = 8$

21 $3x^2 = 5$

22 $(x - 3)^2 = 13$

23 $(3x - 1)^2 = 11$

24 $5x^2 - 10x^2 = -7 + x + x^2$

25 $6x^2 - 7 = 11x$

26 $4x^2 + 17x = 6x - 2x^2$

2.3 You can write quadratic expressions in another form by completing the square.

$$x^2 + 2bx + b^2 = (x + b)^2$$
$$x^2 - 2bx + b^2 = (x - b)^2$$

These are both perfect squares.

To complete the square of the function $x^2 + 2bx$ you need a further term b^2. So the completed square form is

$$x^2 + 2bx = (x + b)^2 - b^2$$

Similarly

$$x^2 - 2bx = (x - b)^2 - b^2$$

Example 8

Complete the square for the expression $x^2 + 8x$

$x^2 + 8x$

$= (x + 4)^2 - 4^2$

$= (x + 4)^2 - 16$

$2b = 8$, so $b = 4$

In general

■ **Completing the square:** $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

Example 9

Complete the square for the expressions

a $x^2 + 12x$ **b** $2x^2 - 10x$

a $x^2 + 12x$

$= (x + 6)^2 - 6^2$

$= (x + 6)^2 - 36$

$2b = 12$, so $b = 6$

b $2x^2 - 10x$

$= 2(x^2 - 5x)$

$= 2[(x - \frac{5}{2})^2 - (\frac{5}{2})^2]$

$= 2(x - \frac{5}{2})^2 - \frac{25}{2}$

Here the coefficient of x^2 is 2.
So take out the coefficient of x^2.
Complete the square on $(x^2 - 5x)$.
Use $b = -5$.

Exercise 2C

Complete the square for the expressions:

1 $x^2 + 4x$	**2** $x^2 - 6x$	**3** $x^2 - 16x$	**4** $x^2 + x$
5 $x^2 - 14x$	**6** $2x^2 + 16x$	**7** $3x^2 - 24x$	**8** $2x^2 - 4x$
9 $5x^2 + 20x$	**10** $2x^2 - 5x$	**11** $3x^2 + 9x$	**12** $3x^2 - x$

2.4 You can solve quadratic equations by completing the square.

Example 10

Solve the equation $x^2 + 8x + 10 = 0$ by completing the square.

$x^2 + 8x + 10 = 0$ — Check coefficient of $x^2 = 1$.

$x^2 + 8x = -10$ — Subtract 10 to get LHS in the form $ax^2 + b$.

$(x + 4)^2 - 4^2 = -10$ — Complete the square for $(x^2 + 8x)$.

$(x + 4)^2 = -10 + 16$ — Add 4^2 to both sides.

$(x + 4)^2 = 6$

$(x + 4) = \pm\sqrt{6}$ — Square root both sides.

$x = -4 \pm \sqrt{6}$ — Subtract 4 from both sides.

Then the solutions (roots) of

$x^2 + 8x + 10 = 0$ are either

$x = -4 + \sqrt{6}$ or $x = -4 - \sqrt{6}$.

Leave your answer in surd form as this is a non-calculator question.

Example 11

Solve the equation $2x^2 - 8x + 7 = 0$.

$2x^2 - 8x + 7 = 0$	The coefficient of $x^2 = 2$.
$x^2 - 4x + \frac{7}{2} = 0$	So divide by 2.
$x^2 - 4x = -\frac{7}{2}$	Subtract $\frac{7}{2}$ from both sides.
$(x - 2)^2 - (2)^2 = -\frac{7}{2}$	Complete the square for $x^2 - 4x$.
$(x - 2)^2 = -\frac{7}{2} + 4$	Add $(2)^2$ to both sides.
$(x - 2)^2 = \frac{1}{2}$	Combine the RHS.
$x - 2 = \pm\sqrt{\frac{1}{2}}$	Square root both sides.
$x = 2 \pm \frac{1}{\sqrt{2}}$	Add 2 to both sides.

So the roots are either

$$x = 2 + \frac{1}{\sqrt{2}}$$

or $x = 2 - \frac{1}{\sqrt{2}}$

Note: Sometimes $b^2 - 4ac$ is negative, and there are then no real solutions.

Exercise 2D

Solve these quadratic equations by completing the square (remember to leave your answer in surd form):

1 $x^2 + 6x + 1 = 0$

2 $x^2 + 12x + 3 = 0$

3 $x^2 - 10x = 5$

4 $x^2 + 4x - 2 = 0$

5 $x^2 - 3x - 5 = 0$

6 $2x^2 - 7 = 4x$

7 $4x^2 - x = 8$

8 $10 = 3x - x^2$

9 $15 - 6x - 2x^2 = 0$

10 $5x^2 + 8x - 2 = 0$

2.5 You can solve quadratic equations $ax^2 + bx + c = 0$ by using the formula

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Example 12

Show that the solutions of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

To do this complete the square.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

The coefficient x^2 is a so divide by a.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Subtract $\frac{c}{a}$ from both sides.

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = -\frac{c}{a}$$

Complete the square.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

Add $\frac{b^2}{4a^2}$ to both sides.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Combine the RHS.

$$x + \frac{b}{2a} = \frac{\pm\sqrt{(b^2 - 4ac)}}{2a}$$

Square root.

Thus $$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Subtract $\frac{b}{2a}$ from both sides.

$b^2 - 4ac$ is called the discriminant.

Example 13

Solve $4x^2 - 3x - 2 = 0$ by using the formula.

$$x = \frac{-(-3) \pm \sqrt{[(-3)^2 - 4(4)(-2)]}}{2 \times 4}$$

Use $x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$ where $a = 4$, $b = -3$, $c = -2$.

$$x = \frac{+3 \pm \sqrt{(9 + 32)}}{8}$$

$-4 \times 4 \times -2 = +32$

$$x = \frac{+3 \pm \sqrt{41}}{8}$$

Then $x = \frac{+3 + \sqrt{41}}{8}$

or $x = \frac{+3 - \sqrt{41}}{8}$

Leave your answer in surd form.

Exercise 2E

Solve the following quadratic equations by using the formula, giving the solutions in surd form. Simplify your answers.

1 $x^2 + 3x + 1 = 0$

2 $x^2 - 3x - 2 = 0$

3 $x^2 + 6x + 6 = 0$

4 $x^2 - 5x - 2 = 0$

5 $3x^2 + 10x - 2 = 0$

6 $4x^2 - 4x - 1 = 0$

7 $7x^2 + 9x + 1 = 0$

8 $5x^2 + 4x - 3 = 0$

9 $4x^2 - 7x = 2$

10 $11x^2 + 2x - 7 = 0$

2.6 You need to be able to sketch graphs of quadratic equations and solve problems using the discriminant.

The steps to help you sketch the graphs are:

1 Decide on the shape.

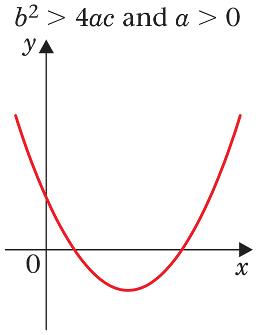
When a is >0 the curve will be a  shape.

When a is <0 the curve will be a  shape.

2 Work out the points where the curve crosses the x- and y-axes.
Put $y = 0$ to find the x-axis crossing points coordinates.
Put $x = 0$ to find the y-axis crossing points coordinates.

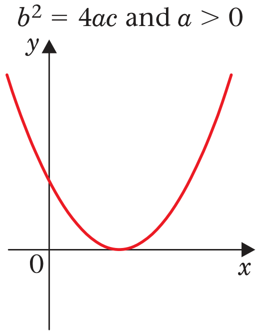
3 Check the general shape of curve by considering the discriminant, $b^2 - 4ac$.
When specific conditions apply, the general shape of the curve takes these forms:

$b^2 > 4ac$ and $a > 0$



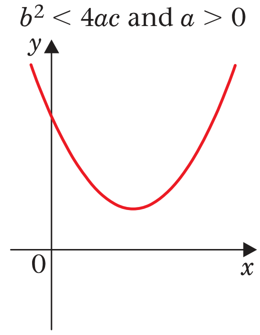
Here there are two different roots.

$b^2 = 4ac$ and $a > 0$



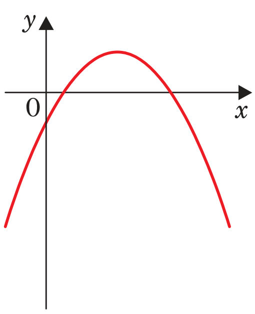
Here there are two equal roots.

$b^2 < 4ac$ and $a > 0$



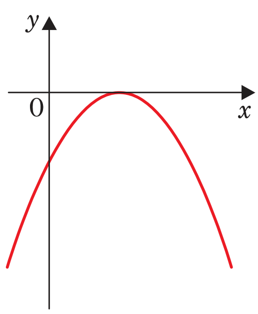
Here there are no real roots.

$b^2 > 4ac$ and $a < 0$



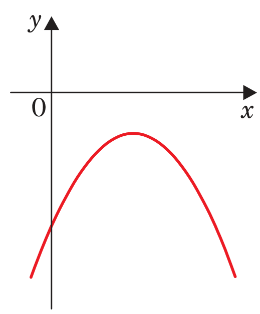
Here there are two different roots.

$b^2 = 4ac$ and $a < 0$



Here there are two equal roots.

$b^2 < 4ac$ and $a < 0$



Here there are no real roots.

You can use the discriminant to establish when a quadratic equation has

- equal roots: $b^2 = 4ac$
- real roots: $b^2 > 4ac$
- no real roots: $b^2 < 4ac$

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Example 14

Sketch the graph of $y = x^2 - 5x + 4$

$a > 0$ so it is a $\cup$ shape.

When $y = 0$,

$0 = x^2 - 5x + 4$

$0 = (x - 4)(x - 1)$

$x = 4$ or $x = 1$

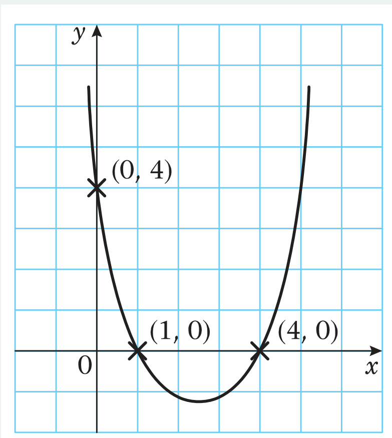
So x-axis crossing points are $(4, 0)$ and $(1, 0)$.

When $x = 0$, $y = 4$, so y-axis crossing point $= (0, 4)$

$b^2 = 25$, $4ac = 16$

So $b^2 > 4ac$ and $a > 0$.

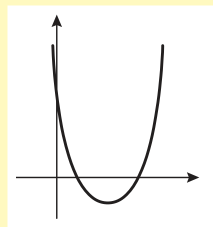
So sketch of the graph is:



Factorise to solve the equation. (You may need to use the formula or complete the square.)

$a = 1$, $b = -5$, $c = 4$

Remember general shape:



Label the crossing points.

Example 15

Find the values of k for which $x^2 + kx + 9 = 0$ has equal roots.

$x^2 + kx + 9 = 0$

Here $a = 1$, $b = k$ and $c = 9$

$k^2 = 4 \times 1 \times 9$

So $k = \pm 6$

For equal roots use $b^2 = 4ac$

Exercise 2F

1 Sketch the graphs of the following equations:

a $y = x^2 + 3x + 2$ **b** $y = x^2 - 3x + 10$ **c** $y = x^2 + 2x - 15$ **d** $y = 2x^2 + 7x + 3$

e $y = 2x^2 + x - 3$ **f** $y = 6x^2 - 19x + 10$ **g** $y = 3x^2 - 2x - 5$ **h** $y = 3x^2 - 13x$

i $y = -x^2 + 6x + 7$ **j** $y = 4 - 7x - 2x^2$

2 Find the values of k for which $x^2 + kx + 4 = 0$ has equal roots.

3 Find the values of k for which $kx^2 + 8x + k = 0$ has equal roots.

Mixed exercise 2G

1 Draw the graphs with the following equations, choosing appropriate values for x. For each graph write down the equation of the line of symmetry.

a $y = x^2 + 6x + 5$ **b** $y = 2x^2 - 3x - 4$

2 Solve the following equations:

a $y^2 + 3y + 2 = 0$ **b** $3x^2 + 13x - 10 = 0$

c $5x^2 - 10x = 4x + 3$ **d** $(2x - 5)^2 = 7$

3 Solve the following equations by:
i completing the square
ii using the formula.

a $x^2 + 5x + 2 = 0$ **b** $x^2 - 4x - 3 = 0$

c $5x^2 + 3x - 1 = 0$ **d** $3x^2 - 5x = 4$

4 Sketch graphs of the following equations:

a $y = x^2 + 5x + 4$ **b** $y = 2x^2 + x - 3$

c $y = 6 - 10x - 4x^2$ **d** $y = 15x - 2x^2$

5 Given that for all values of x:

$$3x^2 + 12x + 5 = p(x + q)^2 + r$$

a find the values of p, q and r

b solve the equation $3x^2 + 12x + 5 = 0$. *E*

6 Find, as surds, the roots of the equation:

$$2(x + 1)(x - 4) - (x - 2)^2 = 0$$

Hint: Remember roots mean solutions.

7 Use algebra to solve $(x - 1)(x + 2) = 18$. *E*

Summary of key points

1 The general form of a quadratic equation is
$y = ax^2 + bx + c$ where a, b, c are constants and $a \neq 0$.

2 Quadratic equations can be solved by:

- factorisation
- completing the square:

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

- using the formula

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

3 A quadratic equation has two solutions, which may be equal.

4 To sketch a quadratic graph:

- decide on the shape:

  $a > 0$ ∪

  $a < 0$ ∩

- work out the x-axis and y-axis crossing points
- check the general shape by considering the discriminant $b^2 - 4ac$.