6663/01 Unit C1
May/June 2015
Mark Scheme

Q1. (a) $(2\sqrt{5})^2 = (2\times\sqrt{5})^2 = 2\times\sqrt{5}\times 2\times\sqrt{5} = (2\times 2)\times\left(\sqrt{5}\times\sqrt{5}\right) = 4\times 5 = 20$

(b) $\frac{\sqrt{2}}{2\sqrt{5}-3\sqrt{2}} = \frac{\sqrt{2}}{2\sqrt{5}-3\sqrt{2}}\times\frac{2\sqrt{5}+3\sqrt{2}}{2\sqrt{5}+3\sqrt{2}}$ (Rationalise)

$= \frac{(\sqrt{2}\times 2\sqrt{5})+(\sqrt{2}\times 3\sqrt{2})}{(2\sqrt{5})^2-(3\sqrt{2})^2}$ (Multiply the numerator and denominator)

$= \frac{(2\sqrt{10})+(3\times 2)}{20-(3\times\sqrt{2}\times 3\times\sqrt{2})}$ $(2\sqrt{5})^2 = 20\ (from\ part\ a)$

$= \frac{2\sqrt{10}+6}{20-(9\times 2)}$ (apply the rule of indices: $a^m\times a^m = a^{2m}$

$\therefore 2^{\frac{1}{2}}\times 2^{\frac{1}{2}} = 2$

$= \frac{2\sqrt{10}+6}{20-18}$

$= \frac{2\sqrt{10}+6}{2}$

$= \frac{2(\sqrt{10}+3)}{2}$ (factorise the numerator)

$= \sqrt{10}+3$

Q2. Solve the simultaneous equations.

$y - 2x - 4 = 0$ (Equation 1)
$4x^2 + y^2 + 20x = 0$ (Equation 2)

$\Rightarrow$ $y = 2x + 4$ (Equation 3: Rearrange equation 1 to obtain equation 3)

$\Rightarrow$ $4x^2 + (2x+4)^2 + 20x = 0$ (Substitute y value of equ. 3 into equation 2)

$\Rightarrow$ $4x^2 + (4x^2 + 16x + 16) + 20x = 0$ Multiply: $(2x+4)(2x+4) = 4x^2 + 8x + 8x + 16$
$4x^2 + 16x + 16$

$\Rightarrow$ $8x^2 + 36x + 16 = 0$ (Bring like terms together to simplify the equation)

$\Rightarrow$ $4(2x^2 + 9x + 4) = 0$ (Factorise the quadratic equation)

$\Rightarrow$ $2x^2 + 9x + 4 = 0$ (Divide the equation by 4 to simplify it)

$\Rightarrow$ $2x^2 + 8x + x + 4 = 0$ (Factorise the quadratic equation)

$\Rightarrow$ $2x(x+4) + 1(x+4) = 0$

$\Rightarrow$ $(2x+1)(x+4) = 0$ (Solve the equation and find the value of x)

$\Rightarrow$ $2x + 1 = 0$ *or* $x + 4 = 0$

$\Rightarrow$ $2x = -1$ *or* $x = -4$

$\Rightarrow$ $x = -\frac{1}{2}$ *or* $x = -4$

$\Rightarrow$ $y = 2\left(-\frac{1}{2}\right) + 4$ $y = 2(-4) + 4$ (Substitute each x value into equation 3)

$\Rightarrow$ $y = -1 + 4$ $y = -8 + 4$

$\Rightarrow$ $y = 3$ *or* $y = -4$

$x = -\frac{1}{2}, \; y = 3$

$x = -4, \; y = -4$

Q3. (a)

$$y = 4x^3 - \frac{5}{x^2} \quad \text{(Linearize the equation)}$$

$$\Rightarrow y = 4x^3 - 5x^{-2}$$

$$\Rightarrow \frac{dy}{dx} = (4 \times 3) \times x^{3-1} - (5 \times -2) \times x^{-2-1} \quad \text{(Differentiate the equation)}$$

$$\Rightarrow \frac{dy}{dx} = 12x^2 - (-10)x^{-3} \quad \text{(Simplify the equation)}$$

$$\Rightarrow \frac{dy}{dx} = 12x^2 + 10x^{-3}$$

$$\Rightarrow \frac{dy}{dx} = 12x^2 + \frac{10}{x^3}$$

(b)

$$y = 4x^3 - \frac{5}{x^2}$$

$$\int y\, dx$$

$$\Rightarrow \int 4x^3 - \frac{5}{x^2} dx \quad \text{(Linearize the equation)}$$

$$\Rightarrow \int 4x^3 - 5x^{-2}\, dx$$

$$\Rightarrow [\int 4x^3 dx\,] - [\int 5x^{-2}\, dx] \quad \text{(Integrate the equation)}$$

$$\Rightarrow \left[4 \times \frac{x^{3+1}}{4}\right] - \left[5 \times \frac{x^{-2+1}}{-1}\right] + C \quad \text{(Simplify the equation)}$$

$$\Rightarrow x^4 + 5x^{-1} + C$$

$$\Rightarrow x^4 + \frac{5}{x} + C$$

Q4. (i) A sequence $U_{1,}\ U_2, U_{3,\ldots\ldots}$ is defined by

$$U_{n+2} = 2U_{n+1} - U_n$$

$$U_1 = 4\ and\ U_2 = 4$$

(a) $$U_3 = U_{1+2} \quad \therefore \quad n = 1$$

$$\Rightarrow U_{1+2} = 2U_{1+1} - U_1$$

$\Rightarrow U_{1+2} = 2U_2 - U_1$ (Substitute the value of $U_2 = 4$ and $U_1 = 4$)

$$\Rightarrow U_{1+2} = 2\times 4 - 4$$

$$\Rightarrow U_{1+2} = 8 - 4$$

$$\Rightarrow U_{1+2} = 4$$

$$\Rightarrow U_3 = 4$$

(b)

$$\sum_{n=1}^{20} U_n \qquad \therefore\ This\ means\ sum\ of\ all\ 20\ U\ terms\ from\ U_1\ to\ U_{20}$$

$$We\ know\ that\ U_1 = 4, U_2 = 4, U_3 = 4$$

$$\therefore This\ equation\ U_{n+2} = 2U_{n+1} - U_n\ would\ give\ answer\ 4\ for\ any\ value\ of\ n$$

$$For\ example: Let's\ try\ U_4 = U_{2+2} = 2U_2 - U_2 = (2\times 4) - 4 = 8 - 4 = 4$$

$$\therefore U_4 = 4$$

$$Hence, \sum_{n=1}^{20} U_n \quad = 20 \times 4 = 80$$

(ii) Another sequence $V_{1,}\ V_2, V_{3,\ldots\ldots}$is defined by

$$V_{n+2} = 2V_{n+1} - V_n$$
$V_1 = k \quad and \quad V_2 = 2k$ where k is a constant

(a) $$V_3 = V_{1+2} \qquad \therefore \quad n = 1$$

$$\Rightarrow V_{1+2} = 2V_{1+1} - V_1$$

$$\Rightarrow V_{1+2} = 2V_2 - V_1$$

$$\Rightarrow V_{1+2} = 2(2k) - k$$

$$\Rightarrow V_{1+2} = 4k - k$$

$$\Rightarrow V_{1+2} = 3k$$

$$\Rightarrow V_3 = 3k$$

$$V_4 = V_{2+2} \qquad \therefore \quad n = 2$$

$$\Rightarrow V_{2+2} = 2V_{2+1} - V_2$$

$$\Rightarrow V_{2+2} = 2V_3 - V_2$$

$$\Rightarrow V_{2+2} = 2(3k) - 2k$$

$$\Rightarrow V_{2+2} = 6k - 2k$$

$$\Rightarrow V_{2+2} = 4k$$

$$\Rightarrow V_4 = 4k$$

(b) Given that

$$\sum_{n=1}^{5} V_n = 165 \quad (this\ means\ sum\ of\ all\ 5\ V\ terms\ from\ V_1\ to\ V_5\ is\ 165)$$

$$We\ know\ that\ V_1 = k, V_2 = 2k, V_3 = 3k, V_4 = 4k$$

$$For\ example: Let's\ try\ V_5 = V_{3+2} = 2V_4 - V_3 = (2\times 4k) - 3k = 8k - 3k = 5k$$

$$\therefore\ V_5 = 5k$$

$$\Rightarrow \sum_{n=1}^{5} V_n = 165$$

$$\Rightarrow V_1 + V_2 + V_3 + V_4 + V_5 = 165$$

$$\Rightarrow k + 2k + 3k + 4k + 5k = 165$$

$$\Rightarrow 15k = 165$$

$$\Rightarrow k = \frac{165}{15} = \frac{33}{3} = 11$$

Q5. The equation

$$(p-1)x^2 + 4x + (p-5) = 0, where\ p\ is\ a\ constant\ has\ no\ real\ roots$$

(a) $For\ quadratic\ equation\ with\ no\ real\ roots:\quad b^2 - 4ac < 0$

$$a = (p-1) \qquad b = 4 \qquad c = (p-5)$$

$\Rightarrow 4^2 - \left(4 \times (p-1) \times (p-5)\right) < 0$ (Substitute a, b & c value into discriminant)

$\Rightarrow 16 - (4 \times (p^2 - 5p - p + 5)) < 0$ (Multiply: $(p-1)(p-5) = (p^2 - 5p - p + 5)$)

$$\Rightarrow 16 - (4 \times (p^2 - 6p + 5)) < 0$$

$\Rightarrow 16 - (4p^2 - 24p + 20) < 0$ (Expand the brackets)

$\Rightarrow -4p^2 + 24p - 4 < 0$ (Divide by -1. Don't forget to change inequality sign)

$$\Rightarrow 4p^2 - 24p + 4 > 0$$

$\Rightarrow 4(p^2 - 6p + 1) > 0$ (Divide by 4)

$$\Rightarrow p^2 - 6p + 1 > 0$$

(b)

$$\Rightarrow p^2 - 6p + 1 > 0$$

$$\Rightarrow (p-3)^2 - (-3)^2 + 1 > 0$$

$$\Rightarrow (p-3)^2 - 9 + 1 > 0$$

$$\Rightarrow (p-3)^2 - 8 > 0$$

$$\Rightarrow (p-3)^2 > 8$$

$$\Rightarrow p - 3 > \pm\sqrt{8}$$

$$\Rightarrow p > \pm\sqrt{8} + 3$$

$$\Rightarrow p > +\sqrt{8} + 3 \quad or \quad p > -\sqrt{8} + 3$$

$$We\ know\ that\ \sqrt{9}\ is\ 3, So\ \sqrt{8}\ will\ have\ an\ answer\ less\ than\ 3$$

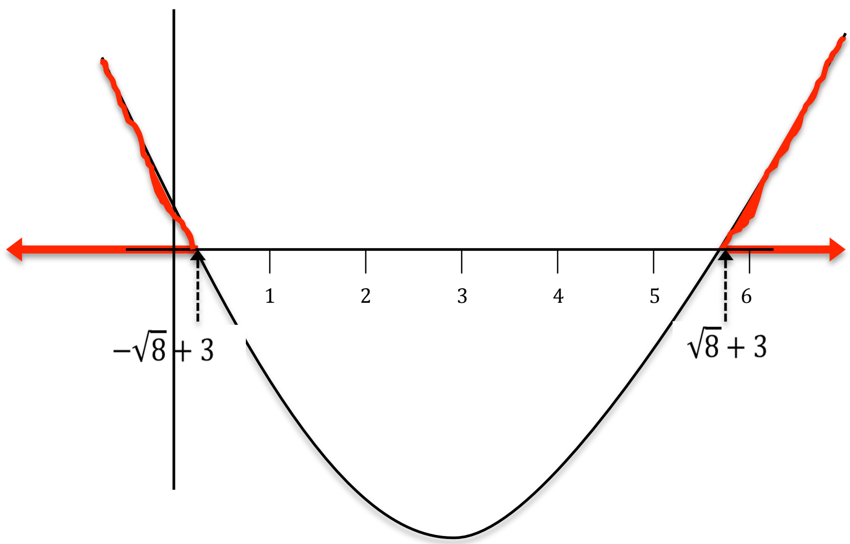
$$\therefore p > \ +\sqrt{8} + 3\ will\ be\ have\ a\ positive\ value\ close\ to\ 6$$

$$\therefore\ p > \ -\sqrt{8} + 3\ will\ have\ a\ positive\ value\ close\ to\ 0$$

Plot the graph of $p^2 - 6p + 1 > 0$:

The coefficient of $p^2$is positive; hence shape of the curve is cup-up. ∪
The minimum point on the curve is (3, -8).
The two x values where the curve crosses the x-axis are $-\sqrt{8} + 3$ & $+\sqrt{8} + 3$.

Graph shows that the set of values of x for which $p^2 - 6p + 1 > 0$ are $p > +\sqrt{8} + 3$ and $p < -\sqrt{8} + 3$.



Q6 (a)

$$y = \frac{(x^2 + 4)(x - 3)}{(2x)}$$

$$\Rightarrow y = \frac{(x^3 - 3x^2 + 4x - 12)}{2x}$$

$$\Rightarrow y = \frac{x^3}{2x} - \frac{3x^2}{2x} + \frac{4x}{2x} - \frac{12}{2x}$$

$$\Rightarrow y = \frac{x^2}{2} - \frac{3x}{2} + 2 - 6x^{-1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \times 2 \times x^{2-1} - \frac{3}{2} - 6 \times -1 \times x^{-1-1}$$

$$\Rightarrow \frac{dy}{dx} = x - \frac{3}{2} + 6x^{-2}$$

$$\Rightarrow \frac{dy}{dx} = x - \frac{3}{2} + \frac{6}{x^2}$$

(b)

$$\Rightarrow \frac{dy}{dx} = x - \frac{3}{2} + \frac{6}{x^2}$$

$$Step\ 1: \ gradient\ at\ x = \ -1$$

$$\Rightarrow \frac{dy}{dx} = -1 - \frac{3}{2} + \frac{6}{(-1)^2}$$

$$\Rightarrow \frac{dy}{dx} = -1 - \frac{3}{2} + 6$$

$$\Rightarrow \frac{dy}{dx} = -1 - 1.5 + 6$$

$$\Rightarrow \frac{dy}{dx} = -2.5 + 6$$

$$\Rightarrow \frac{dy}{dx} = 3.5\ or\ \frac{7}{2}$$

$$Step\ 2: \ Find\ y\ value\ at\ x = \ -1$$

$$y = \frac{(x^2+4)(x-3)}{(2x)}$$

$$\Rightarrow y = \frac{(-1^2+4)(-1-3)}{(2\times -1)}$$

$$\Rightarrow y = \frac{(5)(-4)}{(-2)}$$

$$\Rightarrow y = \frac{-20}{-2}$$

$$\Rightarrow y = \ 10$$

$$Step\ 3: \ Find\ the\ equation\ of\ the\ line$$

$$\Rightarrow y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 10 = \frac{7}{2}(x - (-1))$$

$$\Rightarrow y - 10 = \frac{7}{2}(x + 1)$$

$$\Rightarrow 2(y - 10) = 7(x + 1)$$

$$\Rightarrow 2y - 20 = 7x + 7$$

$$\Rightarrow 2y - 7x - 27 = 0$$

Q7. (a)

$$Given\ that\ y = 2^x$$

$$\Rightarrow 4^x = (2^2)^x = (2^{2x}) = (2^x)^2 = y^2$$

(b) *Solve*

$$8(4^x) - 9(2^x) + 1 = 0$$

Since $4^x = y^2\ and\ 2^x = y$:

$$\Rightarrow 8(y^2) - 9(y) + 1 = 0$$

$$\Rightarrow 8y^2 - 9y + 1 = 0$$

$$\Rightarrow 8y^2 - 8y - y + 1 = 0$$

$$\Rightarrow 8y(y-1) - 1(y-1) = 0$$

$$\Rightarrow (8y-1)(y-1) = 0$$

$$\Rightarrow 8y - 1 = 0 \quad or \quad y - 1 = 0$$

$$\Rightarrow y = \frac{1}{8} \quad or \quad y = 1$$

Since y = 2^x;

$$\Rightarrow 2^x = \frac{1}{8}$$

$$\Rightarrow 2^x = 2^{-3}$$

$$\therefore x = -3$$

$$\Rightarrow 2^x = 1$$

$$\Rightarrow 2^x = 2^0$$

$$\therefore x = 0$$

Q8. (a)

$$Factorise\ 9x - 4x^3$$

$$\Rightarrow x(9 - 4x^2)$$

$$\Rightarrow x(3^2 - (2x)^2)$$

$$\Rightarrow (x)(3 - 2x)(3 + 2x)$$

(b) Sketch the curve of $9x - 4x^3$.

$$Find\ the\ x - corrdinates\ where\ the\ curve\ crosses\ the\ x - axis$$

$$\Rightarrow y = (x)(3 - 2x)(3 + 2x) = 9x - 4x^3$$

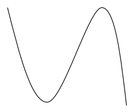
y = 0 where the curve crosses the x – axis, hence

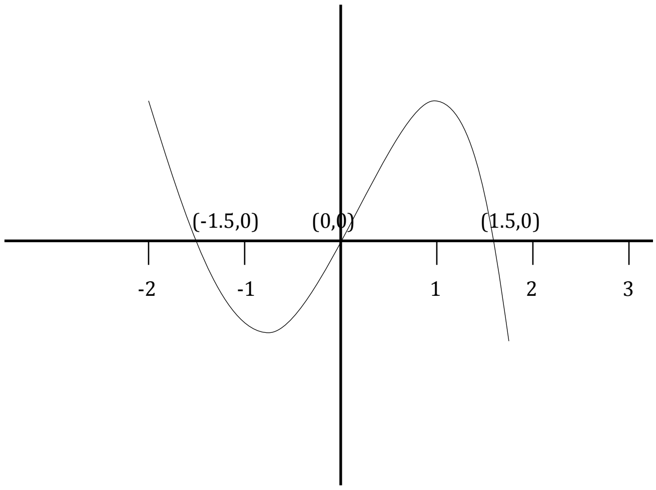
$$\Rightarrow 0 = (x)(3 - 2x)(3 + 2x)$$

$$\Rightarrow 0 = x\ or\ 3 - 2x = 0\ or\ 3 + 2x = 0$$

$$\Rightarrow x = 0\ or\ x = \frac{3}{2}\ or\ x = -\frac{3}{2}$$

Since the coefficient of x^3 is - 4 (a negative number), the curve has the following shape:





(c) Point A has x-coordinate -2 and point B has x-coordinate 1.

$$Find\ the\ y - corrdinates\ when\ x = -2\ and\ x = 1$$

$$y = 9x - 4x^3$$

$$\Rightarrow y = 9(-2) - 4(-2)^3 \quad or \quad y = 9(1) - 4(1)^3$$

$$\Rightarrow y = -18 + 32 \quad or \quad y = 9 - 4$$

$$\Rightarrow y = 14 \quad or \quad y = 5$$

$$\boldsymbol{Point\ A\ (-2, 14) and\ Point\ B\ (1,5)}$$

$$Distance\ AB = \sqrt{(5-14)^2 + (1-(-2))^2} = \sqrt{(-9)^2 + 9} = \sqrt{81+9} = \sqrt{90} = \sqrt{10} \times \sqrt{9} = 3\sqrt{10}$$

$$\mathrm{AB} = 3\sqrt{10} \qquad \text{and} \qquad \mathrm{k} = 3$$

Q. 9. (a)

$$First\ term\ = U_1 = £17{,}000$$

$$Second\ term = U_2 = £18{,}500$$

$$Third\ term\ = U_3 = £20{,}000$$

$$nth\ term\ = U_n = £32{,}000$$

$$difference\ between\ sucessive\ terms\ (d) = £18500 - £17000 = £1500$$

$$nth\ term = U_n = a + (n-1)d = £32{,}000$$

$$U_n \Rightarrow 17000 + (n-1)1500 = £32{,}000$$

$$\Rightarrow 17000 + 1500n - 1500 = £32{,}000$$

$$\Rightarrow 1500n + 15500 = £32{,}000$$

$$\Rightarrow 1500n = 16500$$

$$n = \frac{16500}{1500} = 11$$

(b) **Step1:** Find the total sum of the salary for the first 11 years that form the arithmetic series.

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$S_n = \frac{11}{2}(2(17000) + (11-1)1500)$$

$$S_n = \frac{11}{2}(34000 + 15000)$$

$$S_n = \frac{11}{2}(49000)$$

$$S_n = \frac{11\times 49\times 1000}{2} = \left(((10\times 49) + 49\right)\times\frac{1000}{2} = (490 + 49)\times\frac{1000}{2} = \frac{539}{2}\times 1000 = 269.5\times 1000 = \mathbf{269500}$$

Step 2: Find the sum of the salary for the remaining 9 years. From 11th year onwards the salary stays constant at £32000 per annum.

$$9 \times 32000 = 32 \times 9 \times 1000 = 288000$$

$$Total\ sum\ earned\ in\ 20\ years = 269500 + 288000 = £557500$$

Q10. (a)

$$f'(x) = \frac{3\sqrt{x}}{2} - \frac{9}{4\sqrt{x}} + 2$$

$$\int \frac{3\sqrt{x}}{2} - \frac{9}{4\sqrt{x}} + 2\,dx$$

$$\int \frac{3\sqrt{x}}{2} - \frac{9}{4\sqrt{x}} + 2\,dx$$

$$\int \frac{3}{2}x^{\frac{1}{2}} - \frac{9}{4}x^{-\frac{1}{2}} + 2\,dx$$

$$\Rightarrow (\frac{3}{2} \div \frac{3}{2})\times x^{\frac{1}{2}+1} - (\frac{9}{4} \div \frac{1}{2})\times x^{-\frac{1}{2}+1} + 2x + C$$

$$\Rightarrow \mathrm{f(x)} = \mathrm{x}^{\frac{3}{2}} - \frac{9}{2}\mathrm{x}^{\frac{1}{2}} + 2\mathrm{x} + \mathrm{C}$$

Given the point (4,9) lies on the curve,

$$\Rightarrow 9 = 4^{\frac{3}{2}} - \frac{9}{2}(4)^{\frac{1}{2}} + 2(4) + C$$

$$\Rightarrow 9 = 8 - 9 + 8 + C$$

$$\Rightarrow 9 = 7 + C$$

$$\Rightarrow 9 - 7 = C$$

$$\Rightarrow 2 = C$$

$$\Rightarrow \mathrm{f(x)} = \mathrm{x}^{\frac{3}{2}} - \frac{9}{2}\mathrm{x}^{\frac{1}{2}} + 2\mathrm{x} + 2$$

(b) **Step 1: Find the gradient of the tangent**

$$Normal\ to\ the\ curve\ at\ P \Rightarrow 2y + x = 0$$

$$\Rightarrow y = -\frac{x}{2} \qquad \therefore \quad gradient\ of\ normal\ (m') = -\frac{1}{2}$$

$$gradient\ of\ tangent:\ m = 2$$

Step 2: Find the x-value of the tangent on the curve giving the gradient 2

$$f'(x) = \frac{3\sqrt{x}}{2} - \frac{9}{4\sqrt{x}} + 2$$

$$\Rightarrow 2 = \frac{3\sqrt{x}}{2} - \frac{9}{4\sqrt{x}} + 2$$

$$\Rightarrow 0 = \frac{3\sqrt{x}}{2} - \frac{9}{4\sqrt{x}}$$

(Multiply both sides by $4\sqrt{x}$)

$$\Rightarrow 0 \times 4\sqrt{x} = (\frac{3\sqrt{x}}{2})4\sqrt{x} - \frac{9}{4\sqrt{x}}4\sqrt{x}$$

$$\Rightarrow 0 = \frac{12x}{2} - 9$$

$$\Rightarrow 9 = 6x$$

$$\Rightarrow \frac{3}{2} = x$$

End of mark scheme