After completing this chapter you should be able to

1 use a graphical method to find the number of roots of the equation $f(x) = 0$

2 prove that a root lies within a given interval $[a, b]$

3 use iteration to find an approximation to the root of the equation $f(x) = 0$

4 express your answer to an appropriate degree of accuracy.

4

Numerical methods

The branch of Mathematics called Numerical Analysis predates the invention of modern computers by many centuries. Many equations don't have exact solutions and iterative methods form successive approximations that *converge* to the exact solution.

Weather forecasters use numerical methods to predict storms.

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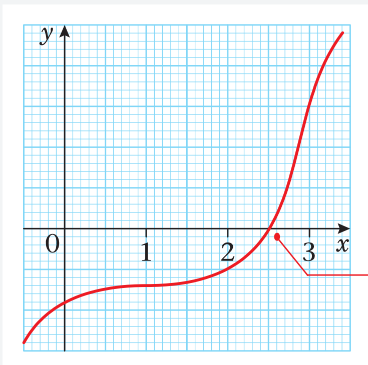
4.1 You can find approximations for the roots of the equation f(x) = 0 graphically.

Example 1

Show that the equation $x^3 - 3x^2 + 3x - 4 = 0$ has a root between $x = 2$ and $x = 3$.



Draw $y = x^3 - 3x^2 + 3x - 4$.

Find the point on the curve $y = x^3 - 3x^2 + 3x - 4$ at which $y = 0$.

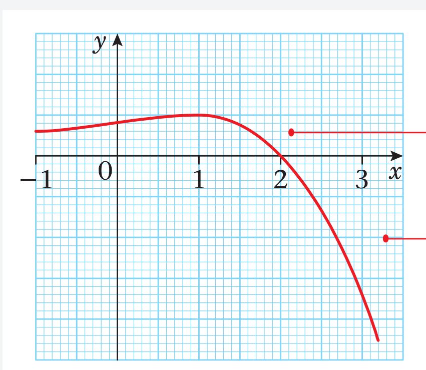
Remember the line $y = 0$ is the x-axis. So find where $y = x^3 - 3x^2 + 3x - 4$ crosses the x-axis.

The graph crosses the x-axis between $x = 2$ and $x = 3$, so there is a root of the equation between $x = 2$ and $x = 3$.

There is a root of the equation $y = x^3 - 3x^2 + 3x - 4$ between $x = 2$ and $x = 3$.

Example 2

Show that the values of y for points on the graph of $y = 4 + 2x - x^3$ change sign as the graph crosses the x-axis.



Draw $y = 4 + 2x - x^3$.

The graph meets the x-axis at $x = 2$.

To the left of $x = 2$ the values of y are positive because the curve lies above the x-axis at these points.

To the right of $x = 2$ the values of y are negative because the curve lies below the x-axis at these points.

There is a change of sign as the graph crosses the x-axis.

Example 3

Show that $e^x + 2x - 3 = 0$ has a root between $x = 0.5$ and $x = 0.6$.

Let $f(x) \equiv e^x + 2x - 3$

$f(0.5) = e^{0.5} + 2(0.5) - 3$

$= 1.648 \ldots + 1 - 3$

$= -0.351 \ldots$

$f(0.6) = e^{0.6} + 2(0.6) - 3$

$= 1.822 \ldots + 1.2 - 3$

$= 0.022 \ldots$

There is a root between $x = 0.5$ and $x = 0.6$.

Show that the graph of $y = f(x)$ crosses the x-axis between $x = 0.5$ and $x = 0.6$.

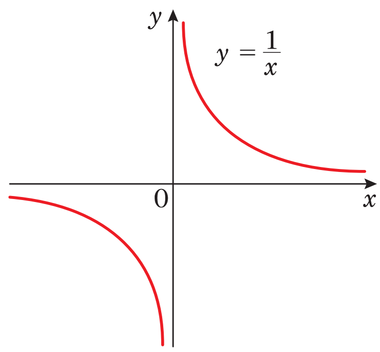
Substitute $x = 0.5$ and $x = 0.6$ into the function.

$f(0.5) < 0$ and $f(0.6) > 0$, so there is a change of sign.

The graph of $y = f(x)$ crosses the x-axis between $x = 0.5$ and $x = 0.6$, so there is a root between $x = 0.5$ and $x = 0.6$.

■ **In general, if you find an interval in which $f(x)$ changes sign, then the interval must contain a root of the equation $f(x) = 0$.**

The only exception to this is when $f(x)$ has a discontinuity in the interval, e.g. $f(x) = \frac{1}{x}$ has a discontinuity at $x = 0$.



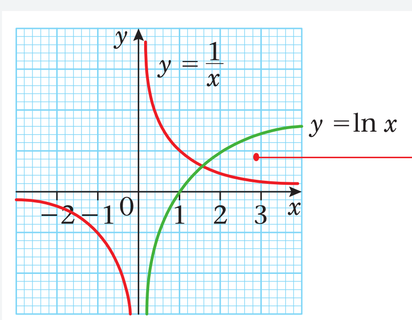
The graph shows that to the left of $x = 0$, $f(x) < 0$, and to the right of $x = 0$, $f(x) > 0$.
So the function changes sign in any interval that contains $x = 0$ but $x = 0$ is not a root of the equation $f(x) = 0$. There is a discontinuity at $x = 0$.

Example 4

a Using the same axes, sketch the graphs of $y = \ln x$ and $y = \frac{1}{x}$. Hence show that the equation $\ln x = \frac{1}{x}$ has only one root.

b Show that this root lies in the interval $1.7 < x < 1.8$.

a



The equation $\ln x = \frac{1}{x}$ has only one root.

Draw $y = \ln x$ and $y = \frac{1}{x}$ on the same axes.

The curves meet where $\ln x = \frac{1}{x}$.

The curves meet at only one point, so there is only one value of x that satisfies the equation $\ln x = \frac{1}{x}$.

b $\ln x = \frac{1}{x}$

$\ln x - \frac{1}{x} = 0$

Let $f(x) \equiv \ln x - \frac{1}{x}$

$f(1.7) = \ln 1.7 - \frac{1}{1.7}$

$= 0.5306\ldots - 0.5882\ldots$

$= -0.0576\ldots$

$f(1.8) = \ln 1.8 - \frac{1}{1.8}$

$= 0.5877\ldots - 0.5555\ldots$

$= 0.0322\ldots$

There is a change of sign between f(1.7) and f(1.8), so the root of the equation $\ln x = \frac{1}{x}$ lies in the interval $1.7 < x < 1.8$.

Rearrange the equation into the form $f(x) = 0$.
Subtract $\frac{1}{x}$ from each side.

Show that $f(x) = 0$ has a root between $x = 1.7$ and $x = 1.8$.

Substitute $x = 1.7$ and $x = 1.8$ into the function.

$f(1.7) < 0$ and $f(1.8) > 0$, so there is a change of sign.

The graph of $y = f(x)$ crosses the x-axis between $x = 1.7$ and $x = 1.8$, so there is a root between $x = 1.7$ and $x = 1.8$.

Exercise 4A

1 Show that each of these equations $f(x) = 0$ has a root in the given interval(s):

a $x^3 - x + 5 = 0 \quad -2 < x < -1$.

b $3 + x^2 - x^3 = 0 \quad 1 < x < 2$.

c $x^2 - \sqrt{x} - 10 = 0 \quad 3 < x < 4$.

d $x^3 - \frac{1}{x} - 2 = 0 \quad -0.5 < x < -0.2$ and $1 < x < 2$.

e $x^5 - 5x^3 - 10 = 0 \quad -2 < x < -1.8$, $-1.8 < x < -1$ and $2 < x < 3$.

f $\sin x - \ln x = 0 \quad 2.2 < x < 2.3$

g $e^x - \ln x - 5 = 0 \quad 1.65 < x < 1.75$.

h $\sqrt[3]{x} - \cos x = 0 \quad 0.5 < x < 0.6$.

For parts **f** and **h**, remember to use radians.

2 Given that $f(x) = x^3 - 5x^2 + 2$, show that the equation $f(x) = 0$ has a root near to $x = 5$.

3 Given that $f(x) \equiv 3 - 5x + x^3$, show that the equation $f(x) = 0$ has a root $x = a$, where a lies in the interval $1 < a < 2$.

4 Given that $f(x) \equiv e^x \sin x - 1$, show that the equation $f(x) = 0$ has a root $x = r$, where r lies in the interval $0.5 < r < 0.6$.

5 It is given that $f(x) \equiv x^3 - 7x + 5$.

a Copy and complete the table below.

x	-3	-2	-1	0	1	2	3
f(x)							

b Given that the negative root of the equation $x^3 - 7x + 5 = 0$ lies between α and $\alpha + 1$, where α is an integer, write down the value of α.

6 Given that $f(x) \equiv x - (\sin x + \cos x)^{\frac{1}{2}}$, $0 \leqslant x \leqslant \frac{3}{4}\pi$, show that the equation $f(x) = 0$ has a root lying between $\frac{\pi}{3}$ and $\frac{\pi}{2}$.

7 **a** Using the same axes, sketch the graphs of $y = e^{-x}$ and $y = x^2$.

b Explain why the equation $e^{-x} = x^2$ has only one root.

c Show that the equation $e^{-x} = x^2$ has a root between $x = 0.70$ and $x = 0.71$.

8 **a** On the same axes, sketch the graphs of $y = \ln x$ and $y = e^x - 4$.

b Write down the number of roots of the equation $\ln x = e^x - 4$.

c Show that the equation $\ln x = e^x - 4$ has a root in the interval (1.4, 1.5).

9 **a** On the same axes, sketch the graphs of $y = \sqrt{x}$ and $y = \frac{2}{x}$.

b Using your sketch, write down the number of roots of the equation $\sqrt{x} = \frac{2}{x}$.

c Given that $f(x) \equiv \sqrt{x} - \frac{2}{x}$, show that $f(x) = 0$ has a root r, where r lies between $x = 1$ and $x = 2$.

d Show that the equation $\sqrt{x} = \frac{2}{x}$ may be written in the form $x^p = q$, where p and q are integers to be found.

e Hence write down the exact value of the root of the equation $\sqrt{x} - \frac{2}{x} = 0$.

10 **a** On the same axes, sketch the graphs of $y = \frac{1}{x}$ and $y = x + 3$.

b Write down the number of roots of the equation $\frac{1}{x} = x + 3$.

c Show that the positive root of the equation $\frac{1}{x} = x + 3$ lies in the interval (0.30, 0.31).

d Show that the equation $\frac{1}{x} = x + 3$ may be written in the form $x^2 + 3x - 1 = 0$.

e Use the quadratic formula to find the positive root of the equation $x^2 + 3x - 1 = 0$ to 3 decimal places.

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4.2 You can use iteration to find an approximation for the root of the equation f(x) = 0.

Example 5

a Show that $x^2 - 4x + 1 = 0$ can be written in the form $x = 4 - \frac{1}{x}$.

b Use the iteration formula $x_{n+1} = 4 - \frac{1}{x_n}$ to find, to 2 decimal places, a root of the equation $x^2 - 4x + 1 = 0$. Start with $x_0 = 3$.

c Show graphically the first two iterations of this formula.

a $x^2 - 4x + 1 = 0$ — Rearrange the equation.

$x^2 + 1 = 4x$ — Add $4x$ to each side.

$x^2 = 4x - 1$ — Subtract 1 from each side.

$x = 4 - \frac{1}{x}$ — Divide each term by x, so that
$x^2 \div x = x$
$4x \div x = 4$
$-1 \div x = -\frac{1}{x}$

b $x_1 = 4 - \frac{1}{x_0}$ — Use $x_{n+1} = 4 - \frac{1}{x_n}$. Here $n = 0$.

$= 4 - \frac{1}{3}$ — Substitute $x_0 = 3$.

$= 3.666\,666\,667$

$x_2 = 4 - \frac{1}{x_1}$ — Use $x_{n+1} = 4 - \frac{1}{x_n}$. Here $n = 1$.

$= 4 - \frac{1}{3.666\,666\,667}$ — Substitute $x_1 = 3.666\,666\,667$.

$= 3.727\,272\,727$

$x_3 = 4 - \frac{1}{x_2}$ — Use $x_{n+1} = 4 - \frac{1}{x_n}$. Here $n = 2$.

$= 4 - \frac{1}{3.727\,272\,727}$ — Substitute $x_2 = 3.727\,272\,727$.

$= 3.731\,707\,317$

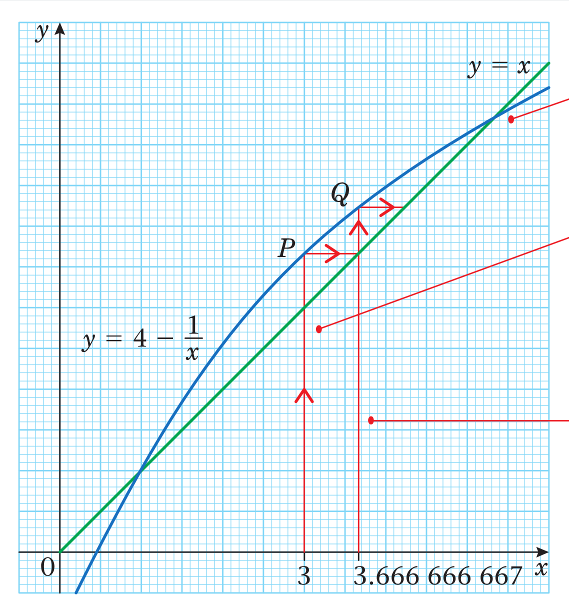
Similarly,

$x_4 = 3.732\,026\,144$ — Each iteration gets closer to the root.

$x_5 = 3.732\,049\,037$

So a root is 3.73 to 2 decimal places.

c



Draw the graphs of $y = x$ and $y = 4 - \frac{1}{x}$.

The graphs intersect when $x = 4 - \frac{1}{x}$.

Substitute $x = 3$ into $y = 4 - \frac{1}{x}$ so that $y = 3.666\,666\,667$. This is the same as moving vertically from $x = 3$ to P.

Let $x = 3.666\,666\,667$. This is the same as moving horizontally from P to the line $y = x$.

Substitute $x = 3.666\,666\,667$ into $y = 4 - \frac{1}{x}$ so that $y = 3.727\,272\,727$. This is the same as moving vertically from $x = 3.666\,666\,667$ to Q.

Let $x = 3.727\,272\,727$. This is the same as moving horizontally from Q to the line $y = x$.

Further iterations take you closer to the root.

The example above is a particular instance of this general result:

■ **To solve an equation of the form f(x) = 0 by an iterative method, rearrange f(x) = 0 into a form x = g(x) and use the iterative formula $x_{n+1} = \mathrm{g}(x_n)$.**

Example 6

a Show that $x^2 - 5x - 3 = 0$ can be written in the form

i $x = \sqrt{5x + 3}$ **ii** $x = \dfrac{x^2 - 3}{5}$

b Show that the iteration formulae

i $x_{n+1} = \sqrt{5x_n + 3}$ **ii** $x_{n+1} = \dfrac{x_n^2 - 3}{5}$

give different roots of the equation $x^2 - 5x - 3 = 0$. Start each iteration with $x_0 = 5$.

a i $x^2 - 5x - 3 = 0$

$x^2 - 3 = 5x$

$x^2 = 5x + 3$

So $x = \sqrt{5x + 3}$

Rearrange the equation.
Add $5x$ to each side.
Add 3 to each side.
Square root each side.

ii $x^2 - 5x - 3 = 0$

$x^2 - 3 = 5x$

$\dfrac{x^2 - 3}{5} = x$

So $x = \dfrac{x^2 - 3}{5}$

Rearrange the equation.
Add $5x$ to each side.
Divide each side by 5.

b **i** $x_1 = \sqrt{5x_0 + 3}$

$= \sqrt{5(5) + 3}$

$= 5.291\,502\,622$

$x_2 = \sqrt{5x_1 + 3}$

$= \sqrt{5(5.291\,502\,622) + 3}$

$= 5.427\,477\,601$

Similarly,

$x_3 = 5.489\,753$

$x_4 = 5.518\,039\,96$

$x_5 = 5.530\,840\,786$

$x_6 = 5.536\,623\,875$

So a root is 5.5 to 1 decimal place.

Use $x_{n+1} = \sqrt{5x_n + 3}$. Here $n = 0$.

Substitute $x_0 = 5$.

Use $x_{n+1} = \sqrt{5x_n + 3}$. Here $n = 1$.

Substitute $x_1 = 5.291\,502\,622$.

Each iteration gets closer to a root, so the sequence $x_0, x_1, x_2, x_3, x_4 \ldots$ is **convergent**.

ii $x_1 = \dfrac{x_0^2 - 3}{5}$

$= \dfrac{(5)^2 - 3}{5}$

$= 4.4$

$x_2 = \dfrac{x_1^2 - 3}{5}$

$= \dfrac{(4.4)^2 - 3}{5}$

$= 3.272$

Similarly,

$x_3 = 1.541\,196\,8$

$x_4 = -0.124\,942\,484\,7$

$x_5 = -0.596\,877\,875\,1$

$x_6 = -0.528\,747\,360\,4$

$x_7 = -0.544\,085\,245\,8$

$x_8 = -0.540\,794\,249\,1$

So a root is -0.5 to 1 decimal place.

Each iteration formula gives a sequence that converges to a different root of the equation.

Use $x_{n+1} = \dfrac{x_n^2 - 3}{5}$. Here $n = 0$.

Substitute $x_0 = 5$.

Use $x_{n+1} = \dfrac{x_n^2 - 3}{5}$. Here $n = 1$.

Substitute $x_1 = 4.4$.

The sequence $x_0, x_1, x_2, x_3, x_4 \ldots$ converges to a root.

■ **Example 6 shows that different rearrangements of the equation f(x) = 0 give iteration formulae that may lead to different roots of the equation.**

Example 7

a Show that $x^3 - 3x^2 - 2x + 5 = 0$ has a root in the interval $3 < x < 4$.

b Use the iteration formula $x_{n+1} = \sqrt{\dfrac{x_n^3 - 2x_n + 5}{3}}$ to find an approximation for the root of the equation $x^3 - 3x^2 - 2x + 5 = 0$. Start with **i** $x_0 = 3$ **ii** $x_0 = 4$.

a Let $f(x) \equiv x^3 - 3x^2 - 2x + 5$

$f(3) = (3)^3 - 3(3)^2 - 2(3) + 5$
$= 27 - 27 - 6 + 5$
$= -1$

$f(4) = (4)^3 - 3(4)^2 - 2(4) + 5$
$= 64 - 48 - 8 + 5$
$= 13$

$f(3) < 0$ and $f(4) > 0$, so there is a change of sign

There is a root in the interval $3 < x < 4$.

> Show that $x^3 - 3x^2 - 2x + 5 = 0$ has a root between $x = 3$ and $x = 4$.
>
> Substitute $x = 3$ and $x = 4$ into the function.
>
> The graph crosses the x-axis between $x = 3$ and $x = 4$.

b i $x_1 = \sqrt{\dfrac{x_0^3 - 2x_0 - + 5}{3}}$

$= \sqrt{\dfrac{(3)^3 - 2(3) + 5}{3}}$

$= 2.943\,920\,289$

$x_2 = \sqrt{\dfrac{x_1^3 - 2x_1 + 5}{3}}$

$= \sqrt{\dfrac{(2.943\ldots)^3 - (2.94\ldots) + 5}{3}}$

$= 2.865\,084\,947$

Similarly,

$x_3 = 2.756\,113\,603$

$x_4 = 2.609\,192\,643$ etc.

$x_{14} = 1.203\,042\,309$

$x_{16} = 1.202\,094\,215$

So a root is 1.2 to 1 decimal place.

> Use $x_{n+1} = \sqrt{\dfrac{x_n^3 - 2x_n + 5}{3}}$. Here $n = 0$.
>
> Substitute $x_0 = 3$.
>
> Use $x_{n+1} = \sqrt{\dfrac{x_n^3 - 2x_n + 5}{3}}$. Here $n = 1$.
>
> Substitute $x_1 = 2.943\,920\,289$.
>
> The sequence $x_0, x_1, x_2, x_3, x_4 \ldots$ converges slowly to a root.
>
> This root is not in the interval $3 < x < 4$.

ii $x_1 = \sqrt{\dfrac{x_0^3 - 2x_0 + 5}{3}}$

$= \sqrt{\dfrac{(4)^3 - 2(4) + 5}{3}}$

$= 4.509\,249\,753$

$x_2 = \sqrt{\dfrac{x_1^3 - 2x_1 + 5}{3}}$

> Use $x_{n+1} = \sqrt{\dfrac{x_n^3 - 2x_n + 5}{3}}$. Here $n = 0$.
>
> Substitute $x_0 = 4$.
>
> Use $x_{n+1} = \sqrt{\dfrac{x_n^3 - 2x_n + 5}{3}}$. Here $n = 1$.

$$= \sqrt{\frac{(4.509\ldots)^3 - 2(4.509\ldots) + 5}{3}}$$

Substitute $x_1 = 4.509\,249\,753$.

$= 5.405\,848\,031$

Similarly,

$x_3 = 7.121\,901\,523$
$x_4 = 10.831\,891\,06$
$x_5 = 20.447\,008\,93$
$x_6 = 53.268\,486\,74$

No root is found.

Each iteration gets further from a root, so the sequence $x_0, x_1, x_2, x_3, x_4 \ldots$ is **divergent**.

■ **Example 7 shows that even if you choose a value $x_0 = a$ that is close to a root, the sequence $x_0, x_1, x_2, x_3, x_4 \ldots$ does not necessarily converge to that root. In fact it might not converge to a root at all.**

Exercise 4B

1 Show that $x^2 - 6x + 2 = 0$ can be written in the form:

a $x = \dfrac{x^2 + 2}{6}$ **b** $x = \sqrt{6x - 2}$ **c** $x = 6 - \dfrac{2}{x}$

2 Show that $x^3 + 5x^2 - 2 = 0$ can be written in the form:

a $x = \sqrt[3]{2 - 5x^2}$ **b** $x = \dfrac{2}{x^2} - 5$ **c** $x = \sqrt{\dfrac{2 - x^3}{5}}$

3 Rearrange $x^3 - 3x + 4 = 0$ into the form $x = \dfrac{x^3}{3} + a$, where the value of a is to be found.

4 Rearrange $x^4 - 3x^3 - 6 = 0$ into the form $x = \sqrt[3]{px^4 - 2}$, where the value of p is to be found.

5 **a** Show that the equation $x^3 - x^2 + 7 = 0$ can be written in the form $x = \sqrt[3]{x^2 - 7}$.

b Use the iteration formula $x_{n+1} = \sqrt[3]{x_n^2 - 7}$, starting with $x_0 = 1$, to find x_2 to 1 decimal place.

6 **a** Show that the equation $x^3 + 3x^2 - 5 = 0$ can be written in the form $x = \sqrt{\dfrac{5}{x + 3}}$.

b Use the iteration formula $x_{n+1} = \sqrt{\dfrac{5}{x_n + 3}}$, starting with $x_0 = 1$, to find x_4 to 3 decimal places.

7 **a** Show that the equation $x^6 - 5x + 3 = 0$ has a root between $x = 1$ and $x = 1.5$.

b Use the iteration formula $x_{n+1} = \sqrt[5]{5 - \dfrac{3}{x_n}}$ to find an approximation for the root of the equation $x^6 - 5x + 3 = 0$, giving your answer to 2 decimal places.

8 **a** Rearrange the equation $x^2 - 6x + 1 = 0$ into the form $x = p - \frac{1}{x}$, where p is a constant to be found.

b Starting with $x_0 = 3$, use the iteration formula $x_{n+1} = p - \frac{1}{x_n}$ with your value of p, to find x_3 to 2 decimal places.

9 **a** Show that the equation $x^3 - x^2 + 8 = 0$ has a root in the interval $(-2, -1)$.

b Use a suitable iteration formula to find an approximation to 2 decimal places for the negative root of the equation $x^3 - x^2 + 8 = 0$.

10 **a** Show that $x^7 - 5x^2 - 20 = 0$ has a root in the interval (1.6, 1.7).

b Use a suitable iteration formula to find an approximation to 3 decimal places for the root of $x^7 - 5x^2 - 20 = 0$ in the interval (1.6, 1.7).

Mixed exercise 4C

1 **a** Rearrange the cubic equation $x^3 - 6x - 2 = 0$ into the form $x = \pm\sqrt{a + \frac{b}{x}}$. State the values of the constants a and b.

b Use the iterative formula $x_{n+1} = \sqrt{a + \frac{b}{x_n}}$ with $x_0 = 2$ and your values of a and b to find the approximate positive solution x_4 of the equation, to an appropriate degree of accuracy. Show all your intermediate answers. *E*

2 **a** By sketching the curves with equations $y = 4 - x^2$ and $y = e^x$, show that the equation $x^2 + e^x - 4 = 0$ has one negative root and one positive root.

b Use the iteration formula $x_{n+1} = -(4 - e^{x_n})^{\frac{1}{2}}$ with $x_0 = -2$ to find in turn x_1, x_2, x_3 and x_4 and hence write down an approximation to the negative root of the equation, giving your answer to 4 decimal places.

An attempt to evaluate the positive root of the equation is made using the iteration formula $x_{n+1} = (4 - e^{x_n})^{\frac{1}{2}}$ with $x_0 = 1.3$.

c Describe the result of such an attempt. *E*

3 **a** Show that the equation $x^5 - 5x - 6 = 0$ has a root in the interval (1, 2).

b Stating the values of the constants p, q and r, use an iteration of the form $x_{n+1} = (px_n + q)^{\frac{1}{r}}$ an appropriate number of times to calculate this root of the equation $x^5 - 5x - 6 = 0$ correct to 3 decimal places. Show sufficient working to justify your final answer. *E*

4 $f(x) \equiv 5x - 4\sin x - 2$, where x is in radians.

a Evaluate, to 2 significant figures, f(1.1) and f(1.15).

b State why the equation $f(x) = 0$ has a root in the interval (1.1, 1.15).

An iteration formula of the form $x_{n+1} = p\sin x_n + q$ is applied to find an approximation to the root of the equation $f(x) = 0$ in the interval (1.1, 1.15).

c Stating the values of p and q, use this iteration formula with $x_0 = 1.1$ to find x_4 to 3 decimal places. Show the intermediate results in your working. *E*

5 $f(x) \equiv 2\sec x + 2x - 3$, where x is in radians.

a Evaluate f(0.4) and f(0.5) and deduce the equation $f(x) = 0$ has a solution in the interval $0.4 < x < 0.5$.

b Show that the equation $f(x) = 0$ can be arranged in the form $x = p + \frac{q}{\cos x}$, where p and q are constants, and state the value of p and the value of q.

c Using the iteration formula $x_{n+1} = p + \frac{q}{\cos x_n}$, $x_0 = 0.4$, with the values of p and q found in part **b**, calculate x_1, x_2, x_3 and x_4, giving your final answer to 4 decimal places. *E*

6 $f(x) \equiv e^{0.8x} - \frac{1}{3 - 2x}$, $x \neq \frac{3}{2}$

a Show that the equation $f(x) = 0$ can be written as $x = 1.5 - 0.5e^{-0.8x}$.

b Use the iteration formula $x_{n+1} = 1.5 - 0.5e^{-0.8x_n}$ with $x_0 = 1.3$ to obtain x_1, x_2 and x_3. Give the value of x_3, an approximation to a root of $f(x) = 0$, to 3 decimal places.

c Show that the equation $f(x) = 0$ can be written in the form $x = p\ln(3 - 2x)$, stating the value of p.

d Use the iteration formula $x_{n+1} = p\ln(3 - 2x_n)$ with $x_0 = -2.6$ and the value of p found in part **c** to obtain x_1, x_2 and x_3. Give the value of x_3, an approximation to the second root of $f(x) = 0$, to 3 decimal places. *E*

7 **a** Use the iteration $x_{n+1} = (3x_n + 3)^{\frac{1}{3}}$ with $x_0 = 2$ to find, to 3 significant figures, x_4.

The only real root of the equation $x^3 - 3x - 3 = 0$ is α. It is given that, to 3 significant figures, $\alpha = x_4$.

b Use the substitution $y = 3^x$ to express $27^x - 3^{x+1} - 3 = 0$ as a cubic equation.

c Hence, or otherwise, find an approximate solution to the equation $27^x - 3^{x+1} - 3 = 0$, giving your answer to 2 significant figures. *E*

8 The equation $x^x = 2$ has a solution near $x = 1.5$.

a Use the iteration formula $x_{n+1} = 2^{\frac{1}{x_n}}$ with $x_0 = 1.5$ to find the approximate solution x_5 of the equation. Show the intermediate iterations and give your final answer to 4 decimal places.

b Use the iteration formula $x_{n+1} = 2x_n^{(1 - x_n)}$ with $x_0 = 1.5$ to find x_1, x_2, x_3, x_4. Comment briefly on this sequence. *E*

9 **a** Show that the equation $2^{1-x} = 4x + 1$ can be arranged in the form $x = \frac{1}{2}(2^{-x}) + q$, stating the value of the constant q.

b Using the iteration formula $x_{n+1} = \frac{1}{2}(2^{-x_n}) + q$ with $x_0 = 0.2$ and the value of q found in part **a**, find x_1, x_2, x_3 and x_4. Give the value of x_4, to 4 decimal places. *E*

10 The curve with equation $y = \ln(3x)$ crosses the x-axis at the point P $(p, 0)$.

a Sketch the graph of $y = \ln(3x)$, showing the exact value of p.

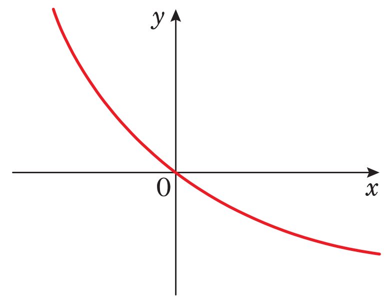
The normal to the curve at the point Q, with x-coordinate q, passes through the origin.

b Show that $x = q$ is a solution of the equation $x^2 + \ln 3x = 0$.

c Show that the equation in part **b** can be rearranged in the form $x = \frac{1}{3}e^{-x^2}$.

d Use the iteration formula $x_{n+1} = \frac{1}{3}e^{-x_n^2}$, with $x_0 = \frac{1}{3}$, to find x_1, x_2, x_3 and x_4. Hence write down, to 3 decimal places, an approximation for q. *E*

11 **a** Copy this sketch of the curve with equation $y = e^{-x} - 1$.
On the same axes sketch the graph of $y = \frac{1}{2}(x - 1)$, for $x \geqslant 1$, and $y = -\frac{1}{2}(x - 1)$, for $x < 1$. Show the coordinates of the points where the graph meets the axes.



The x-coordinate of the point of intersection of the graphs is α.

b Show that $x = \alpha$ is a root of the equation $x + 2e^{-x} - 3 = 0$.

c Show that $-1 < \alpha < 0$.

The iterative formula $x_{n+1} = -\ln[\frac{1}{2}(3 - x_n)]$ is used to solve the equation $x + 2e^{-x} - 3 = 0$.

d Starting with $x_0 = -1$, find the values of x_1 and x_2.

e Show that, to 2 decimal places, $\alpha = -0.58$. *E*

Summary of key points

1 If you find an interval in which f(x) changes sign, and f(x) is continuous in that interval, then the interval must contain a root of the equation f(x) = 0.

2 To solve an equation of the form f(x) = 0 by an iterative method, rearrange f(x) = 0 into a form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$.

3 Different rearrangements of the equation f(x) = 0 give iteration formulae that **may** lead to different roots of the equation.

4 If you choose a value $x_0 = a$ for the starting value in an iteration formula, and $x_0 = a$ is close to a root of the equation f(x) = 0, then the sequence x_0, x_1, x_2, x_3, x_4 … does not necessarily converge to that root. In fact it might not converge to a root at all.