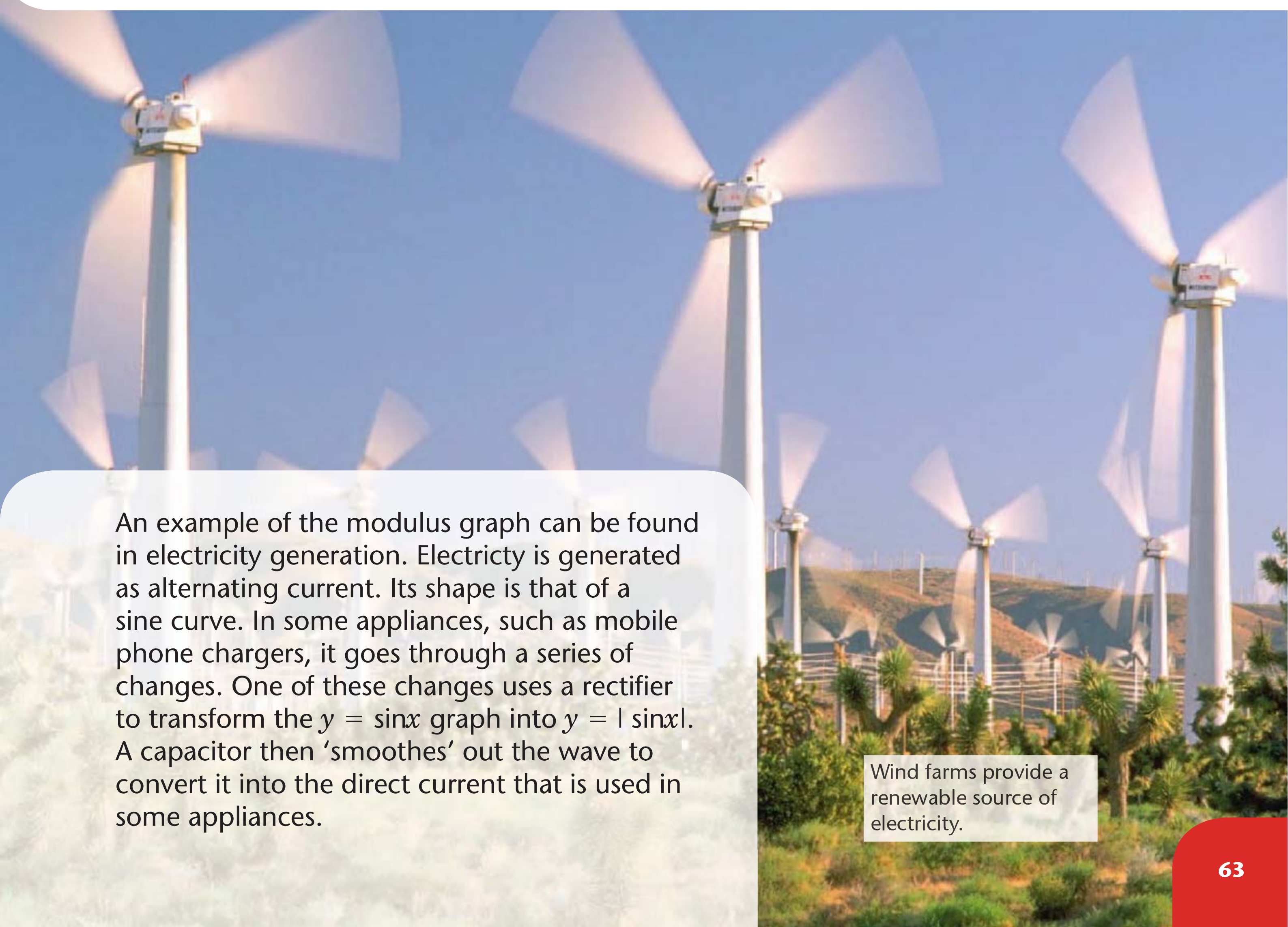
After completing this chapter you should be able to

1 sketch the graph of the modulus function $y = |f(x)|$
2 sketch the graph of the function $y = f(|x|)$
3 solve equations involving the modulus function
4 apply a combination of two (or more) transformations to the same curve
5 sketch transformations of the graph $y = f(x)$.

5

Transforming graphs of functions



An example of the modulus graph can be found in electricity generation. Electricty is generated as alternating current. Its shape is that of a sine curve. In some appliances, such as mobile phone chargers, it goes through a series of changes. One of these changes uses a rectifier to transform the $y = \sin x$ graph into $y = |\sin x|$. A capacitor then 'smoothes' out the wave to convert it into the direct current that is used in some appliances.

Wind farms provide a renewable source of electricity.

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5.1 You need to be able to sketch the graph of the modulus function $y = |f(x)|$.

- **The modulus of a number a, written as $|a|$, is its positive numerical value.**

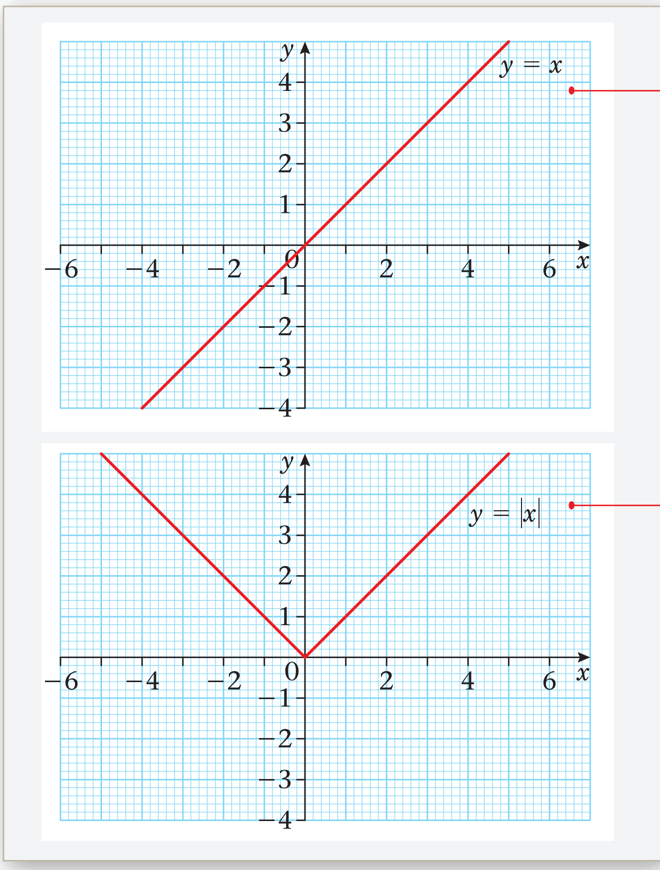
So, for example, $|5| = 5$ and also $|-5| = 5$.

It is sometimes known as the **absolute value**, and is shown on the display of some calculators as, for example, 'Abs −5' or 'Abs(−5)'. If your calculator has a modulus or absolute value button, make sure you understand how to use it.

- **A modulus function is, in general, a function of the type $y = |f(x)|$.**
  When $f(x) \geqslant 0$, $|f(x)| = f(x)$.
  When $f(x) < 0$, $|f(x)| = -f(x)$.

Example 1

Sketch the graph of $y = |x|$.



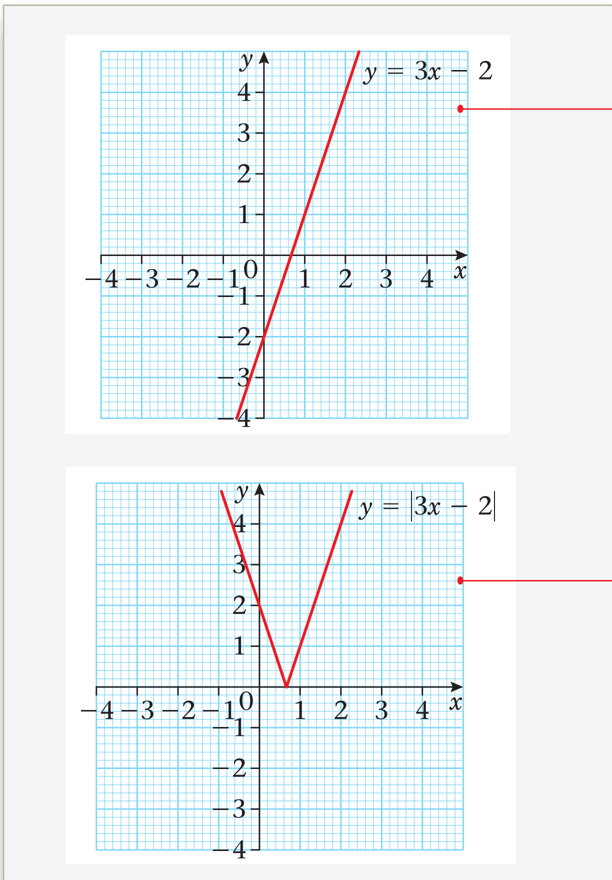
Step 1
Sketch the graph of $y = x$.
(Ignore the modulus.)

Step 2
For the part of the line below the x-axis (the negative values of y), reflect in the x-axis.
For example this will change the y-value -3 into the y-value 3.

Important
If you do steps 1 and 2 above on the same diagram, make sure that you clearly show that you have deleted the part of the graph below the x-axis.

Example 2

Sketch the graph of $y = |3x - 2|$.

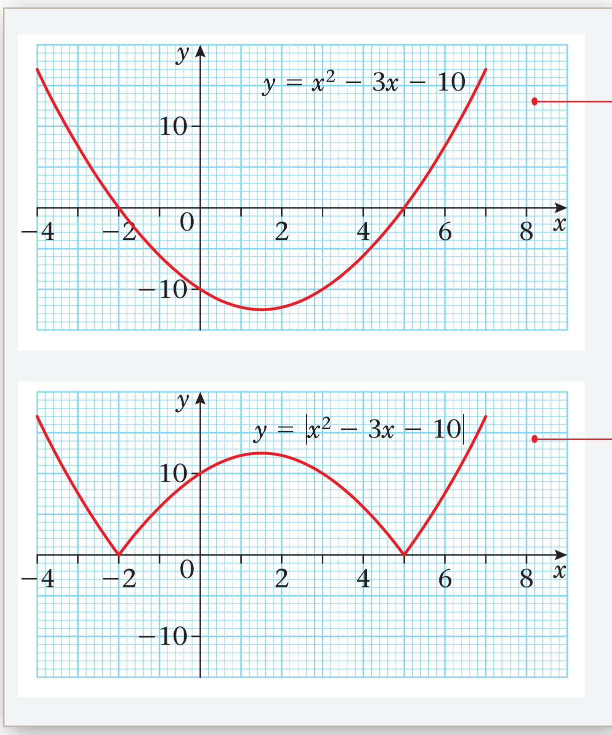


Step 1
Sketch the graph of $y = 3x - 2$.
(Ignore the modulus.)

Step 2
For the part of the line below the x-axis (the negative values of y), reflect in the x-axis. For example, this will change the y-value -2 into the y-value 2.

Example 3

Sketch the graph of $y = |x^2 - 3x - 10|$.



Step 1
Sketch the graph of $y = x^2 - 3x - 10$.
(Ignore the modulus.)

Step 2
For the part of the curve below the x-axis (the negative values of y), reflect in the x-axis. For example, this will change the y-value -3 into the y-value 3.

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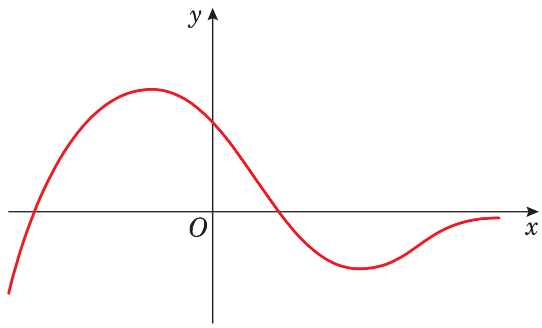
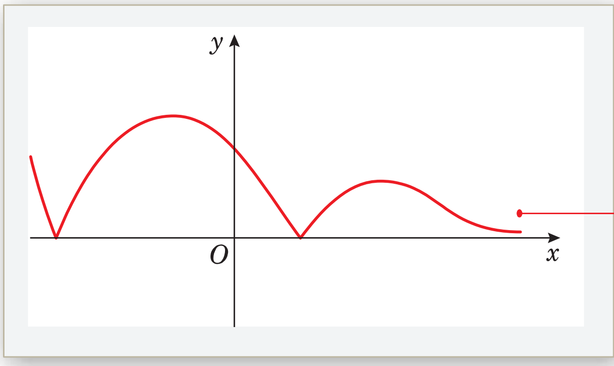
Example 4

The diagram on the right shows the graph of $y = f(x)$.
Sketch the graph of $y = |f(x)|$.

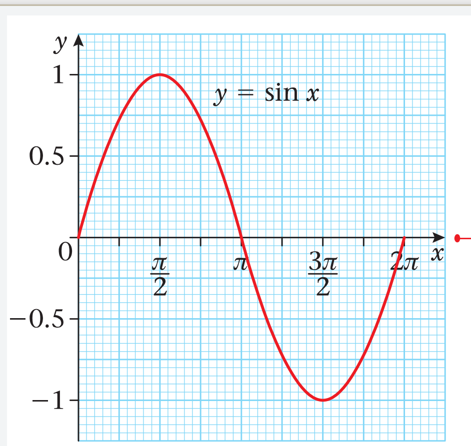




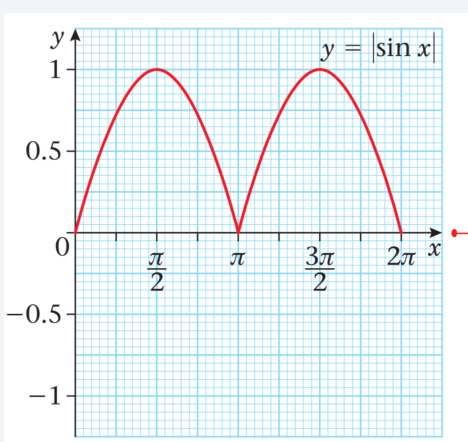
As in the previous examples, the part of the curve below the x-axis must be reflected in the x-axis. The graph of $y = |f(x)|$ looks like this.

Example 5

Sketch the graph of $y = |\sin x|$, $0 \leqslant x \leqslant 2\pi$.



First draw the graph of $y = \sin x$.



As before, reflect the part of the curve below the x-axis in the x-axis.

Exercise 5A

1 Sketch the graph of each of the following. In each case, write down the coordinates of any points at which the graph meets the coordinate axes.

a $y = |x - 1|$ **b** $y = |2x + 3|$

c $y = |\frac{1}{2}x - 5|$ **d** $y = |7 - x|$

e $y = |x^2 - 7x - 8|$ **f** $y = |x^2 - 9|$

g $y = |x^3 + 1|$ **h** $y = \left|\frac{12}{x}\right|$

i $y = -|x|$ **j** $y = -|3x - 1|$

2 Sketch the graph of each of the following. In each case, write down the coordinates of any points at which the graph meets the coordinate axes.

a $y = |\cos x|,\ 0 \leqslant x \leqslant 2\pi$ **b** $y = |\ln x|,\ x > 0$ **c** $y = |2^x - 2|$

d $y = |100 - 10^x|$ **e** $y = |\tan 2x|,\ 0 < x < 2\pi$

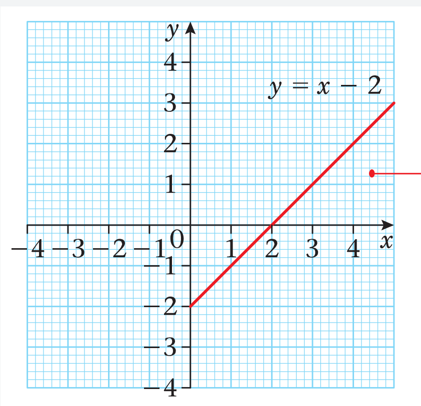
5.2 You need to be able to sketch the graph of the function $y = \mathrm{f}(|x|)$.

For the function $y = \mathrm{f}(|x|)$, the value of y at, for example, $x = -5$ is the same as the value of y at $x = 5$. This is because $\mathrm{f}(|-5|) = \mathrm{f}(5)$.

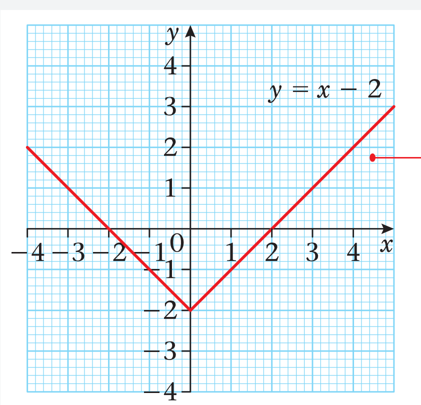
Example 6

Sketch the graph of $y = |x| - 2$.

Method 1

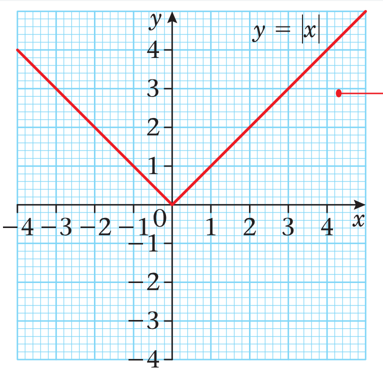


Step 1
Sketch the graph of $y = x - 2$ (ignore the modulus) for $x \geqslant 0$.

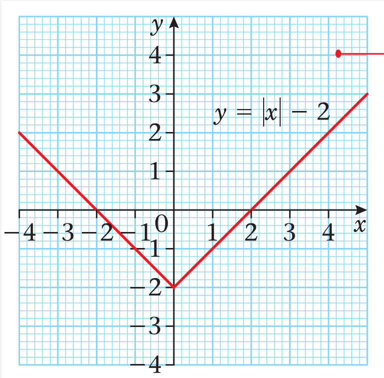


Step 2
Reflect in the y-axis.

Method 2



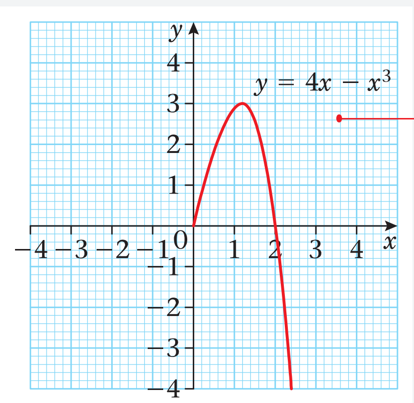
Step 1
Sketch the graph of $y = |x|$.



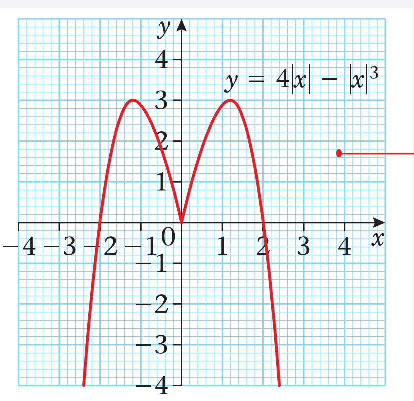
Step 2
Vertical translation of -2 units.
(See transformations of curves in Book C1, Chapter 4.)

Example 7

Sketch the graph of $y = 4|x| - |x|^3$.



Step 1
Sketch the graph of $y = 4x - x^3$ (ignore the modulus) for $x \geqslant 0$.



Step 2
Reflect in the y-axis.

Exercise 5B

Sketch the graph of each of the following. In each case, write down the coordinates of any points at which the graph meets the coordinate axes.

1 $y = 2|x| + 1$

2 $y = |x|^2 - 3|x| - 4$

3 $y = \sin|x|,\ -2\pi \leqslant x \leqslant 2\pi$

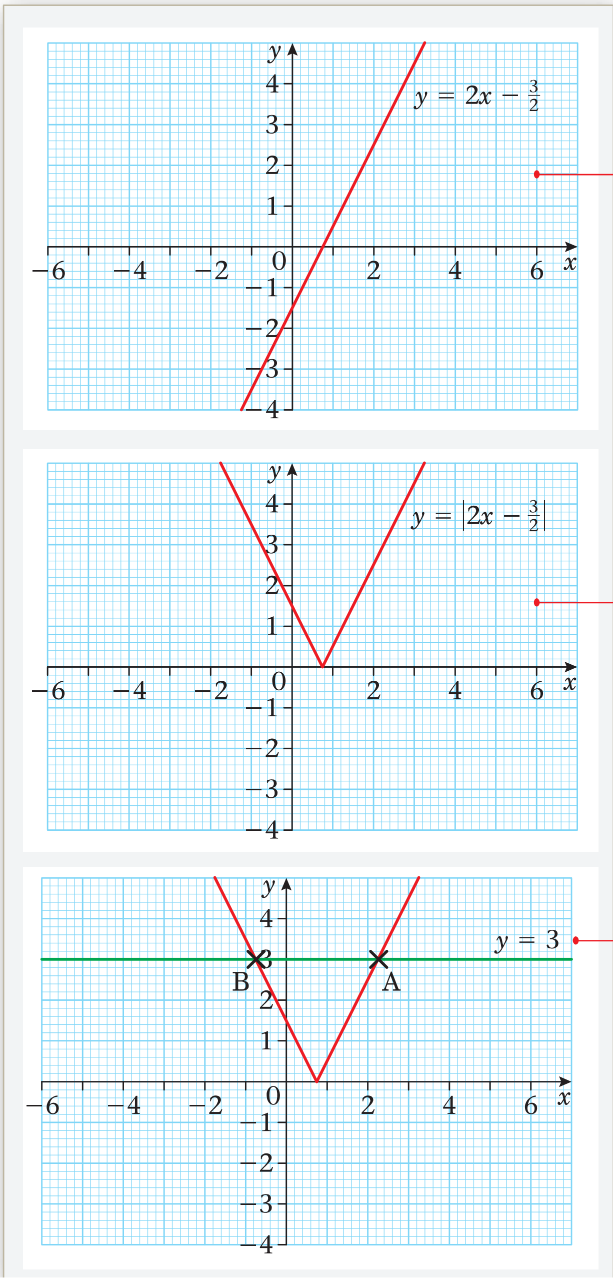
4 $y = 2^{|x|}$

5.3 You need to be able to solve equations involving a modulus.

Solutions can come from either the 'original' or the 'reflected' part of the graph.

Example 8

Solve the equation $|2x - \frac{3}{2}| = 3$.



Sketch the graph of $y = 2x - \frac{3}{2}$. (Ignore the modulus.)

For the part of the line below the x-axis, reflect in the x-axis.

Draw the line $y = 3$ on the same sketch.

The solutions are the values of x where the graphs cross (A and B).

A is on the original graph of $y = 2x - \frac{3}{2}$. B is on the reflected part.

At A, $2x - \frac{3}{2} = 3$

$2x = \frac{9}{2}$

$x = \frac{9}{4} = 2\frac{1}{4}$

At B, $-(2x - \frac{3}{2}) = 3$

$-2x + \frac{3}{2} = 3$

$-2x = \frac{3}{2}$

$x = -\frac{3}{4}$

The solutions to the equation are $x = -\frac{3}{4}$ and $x = \frac{9}{4}$.

Original: Use $2x - \frac{3}{2}$.

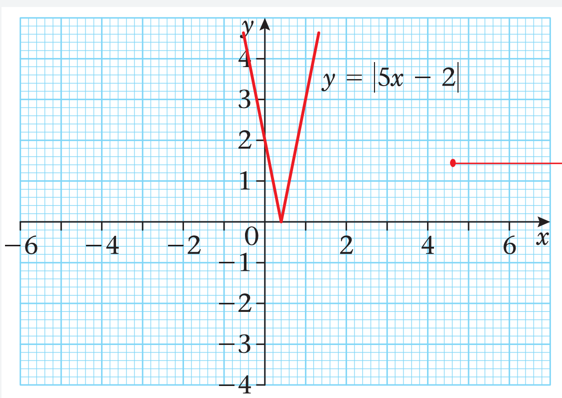
When $f(x) < 0$, $|f(x)| = -f(x)$, so, as it is reflected, use $-(2x - \frac{3}{2})$.

Example 9

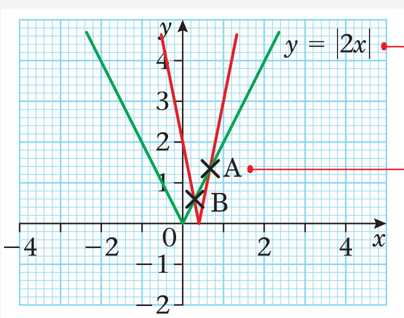
a On the same diagram, sketch the graphs of $y = |5x - 2|$ and $y = |2x|$.

b Solve the equation $|5x - 2| = |2x|$.

a



Sketch the graph of $y = |5x - 2|$. (As usual, for the part of $y = 5x - 2$ that is below the x-axis, reflect in the x-axis.)



On the same diagram, sketch the graph of $y = |2x|$.

The solutions for part **b** are the values of x where the 2 graphs intersect.

Intersection point A is on the original graph of $y = 5x + 2$, and on the original graph of $y = 2x$.

Intersection point B is on the reflected part of $y = 5x - 2$, and on the original graph of $y = 2x$.

b At A, $5x - 2 = 2x$

$3x = 2$

$x = \frac{2}{3}$

At B, $-(5x - 2) = 2x$

$-5x + 2 = 2x$

$-7x = -2$

$x = \frac{2}{7}$

Original: Use $5x - 2$ and $2x$.

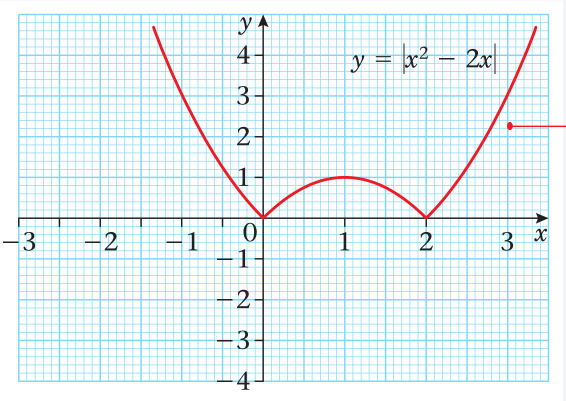
Reflected: Use $-(5x - 2)$
Original: Use $2x$.

Example 10

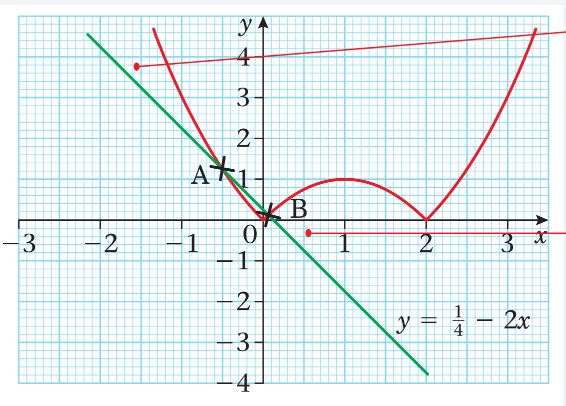
a On the same diagram, sketch the graphs of $y = |x^2 - 2x|$ and $y = \frac{1}{4} - 2x$.

b Solve the equation $|x^2 - 2x| = \frac{1}{4} - 2x$.

a



Sketch the graph of $y = |x^2 - 2x|$. (As usual, for the part of $y = x^2 - 2x$ that is below the x-axis, reflect in the x-axis.)



On the same diagram, sketch the graph of $y = \frac{1}{4} - 2x$.

The solutions for part **b** are the values of x where the 2 graphs intersect.

Intersection point A is on the original part of both graphs.

Intersection point B is on the original graph of $y = \frac{1}{4} - 2x$ and on the reflected part of $y = x^2 - 2x$.

b At A, $x^2 - 2x = \frac{1}{4} - 2x$

$x^2 = \frac{1}{4}$

$x = -\frac{1}{2}$ (A)

or $x = \frac{1}{2}$ (not valid)

Original: Use $x^2 - 2x$ and $\frac{1}{4} - 2x$.

This is not valid, since $x < 0$.

At B, $\frac{1}{4} - 2x = -(x^2 - 2x)$

$x^2 - 4x + \frac{1}{4} = 0$

$$x = \frac{4 \pm \sqrt{16 - 1}}{2}$$

$x = 3.94$ (2 d.p.) (not valid)

or $x = 0.06$ (2 d.p.) (B)

Reflected: Use $-(x^2 - 2x)$.
Original: Use $\frac{1}{4} - 2x$.

You need to reject any invalid 'solutions'.

The complete set of solutions is $x = -\frac{1}{2}$ and $x = 2 - \frac{1}{2}\sqrt{15}$ (≈ 0.06).

Exercise 5C

1 On the same diagram, sketch the graphs of $y = -2x$ and $y = |\frac{1}{2}x - 2|$. Solve the equation $-2x = |\frac{1}{2}x - 2|$.

2 On the same diagram, sketch the graphs of $y = |x|$ and $y = |-4x - 5|$. Solve the equation $|x| = |-4x - 5|$.

3 On the same diagram, sketch the graphs of $y = 3x$ and $y = |x^2 - 4|$. Solve the equation $3x = |x^2 - 4|$.

4 On the same diagram, sketch the graphs of $y = |x| - 1$ and $y = -|3x|$. Solve the equation $|x| - 1 = -|3x|$.

5 On the same diagram, sketch the graphs of $y = 24 + 2x - x^2$ and $y = |5x - 4|$. Solve the equation $24 + 2x - x^2 = |5x - 4|$. (Answers to 2 d.p. where appropriate).

5.4 You need to be able to apply a combination of two (or more) transformations to the same curve.

In Book C1, Chapter 4, you saw how to apply various transformations to curves. To summarise these:

- ① $f(x + a)$ is a horizontal translation of $-a$
  ② $f(x) + a$ is a vertical translation of $+a$
  ③ $f(ax)$ is a horizontal stretch of scale factor $\frac{1}{a}$
  ④ $af(x)$ is a vertical stretch of scale factor a

Example 11

Sketch the graph of $y = (x - 2)^2 + 3$.

Start with $f(x) = x^2$

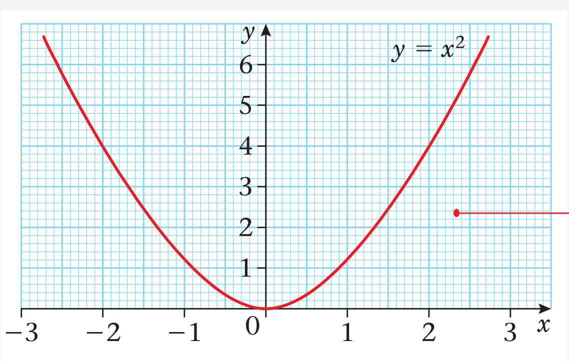
$f(x - 2) = (x - 2)^2$

Calling this $g(x)$, $g(x) = (x - 2)^2$

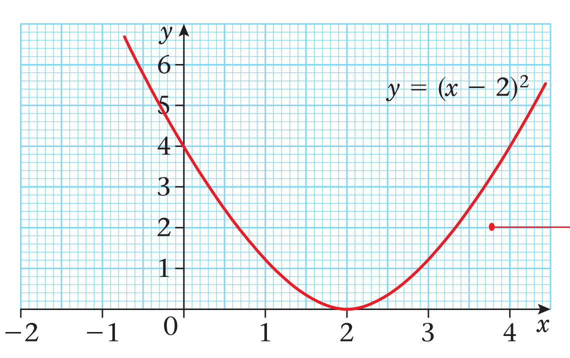
$g(x) + 3 = (x - 2)^2 + 3$

Step 1 using ①:
Horizontal translation of $+2$.

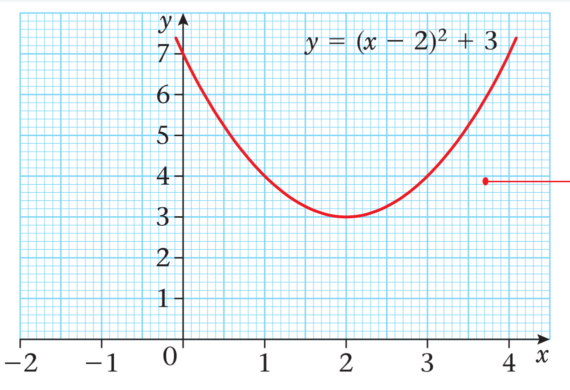
Step 2 using ②:
Vertical translation of $+3$.



Sketch the graph of $f(x) = x^2$.



Step 1
Horizontal translation of $+2$.



Step 2
Vertical translation of $+3$.

Example 12

Sketch the graph of $y = \dfrac{2}{x+5}$.

Start with $f(x) = \dfrac{1}{x}$

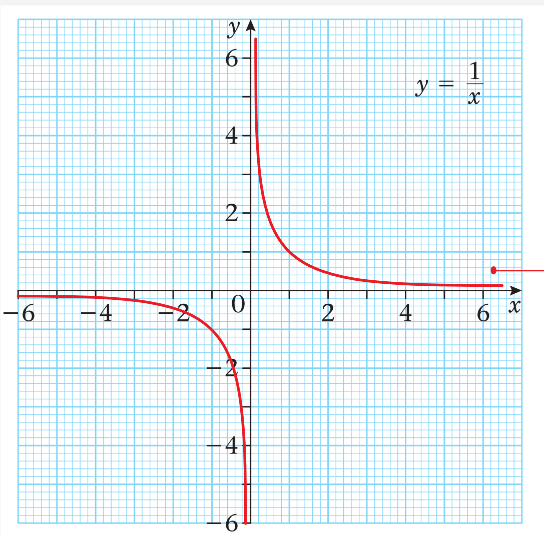
$$f(x+5) = \frac{1}{x+5}$$

Step 1 using ①:
Horizontal translation of -5

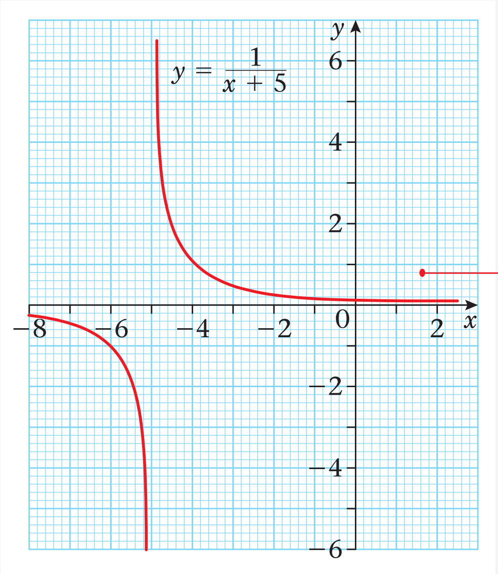
Calling this $g(x)$, $g(x) = \dfrac{1}{x+5}$

$$2g(x) = \frac{2}{x+5}$$

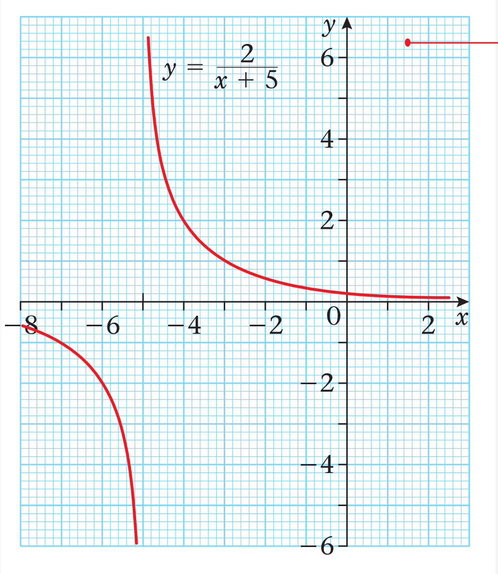
Step 2 using ④:
Vertical stretch, scale factor 2.



Sketch the graph of $f(x) = \dfrac{1}{x}$.



Step 1
Horizontal translation of −5.



Step 2
Vertical stretch, scale factor 2.

Notice what happens to a point such as (−4, 1) … . It goes to (−4, 2).

Example 13

Sketch the graph of $y = \cos 2x - 1$.

Start with $f(x) = \cos x$

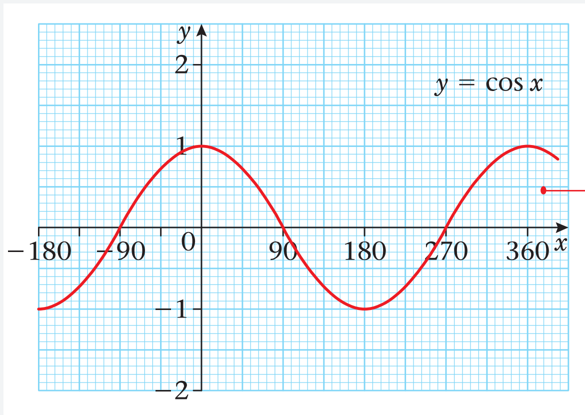
$f(2x) = \cos 2x$

Calling this $g(x)$, $g(x) = \cos 2x$

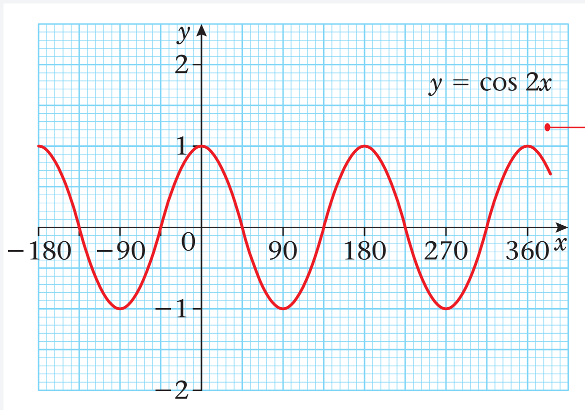
$g(x) - 1 = \cos 2x - 1$

Step 1 using (3):
Horizontal stretch, scale factor $\frac{1}{2}$.

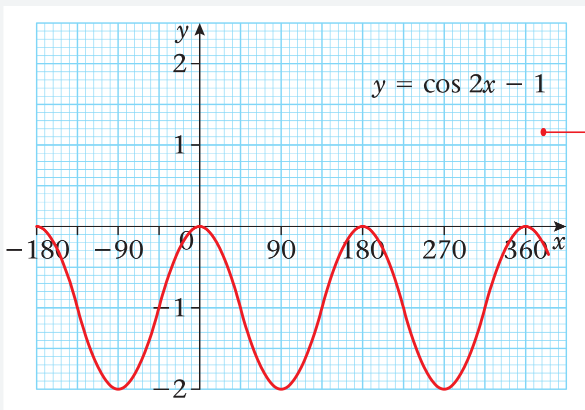
Step 2 using (2):
Vertical translation of −1.



Sketch the graph of $f(x) = \cos x$.



Step 1
Horizontal stretch, scale factor $\frac{1}{2}$.



Step 2
Vertical translation of −1.

Example 14

Sketch the graph of $y = 3|x - 1| - 2$.

Start with $f(x) = |x|$

$f(x - 1) = |x - 1|$

Calling this $g(x)$, $g(x) = |x - 1|$

$3g(x) = 3|x - 1|$

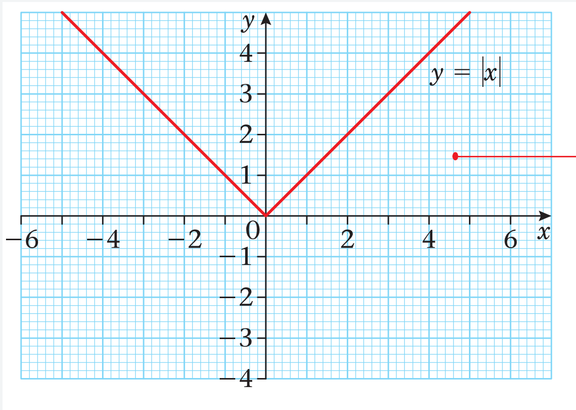
Calling this $h(x)$, $h(x) = 3|x - 1|$

$h(x) - 2 = 3|x - 1| - 2$

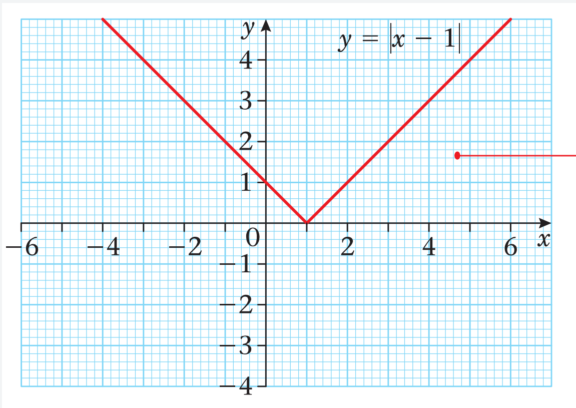
Step 1 using (1):
Horizontal translation of 1.

Step 2 using (4):
Vertical stretch, scale factor 3.

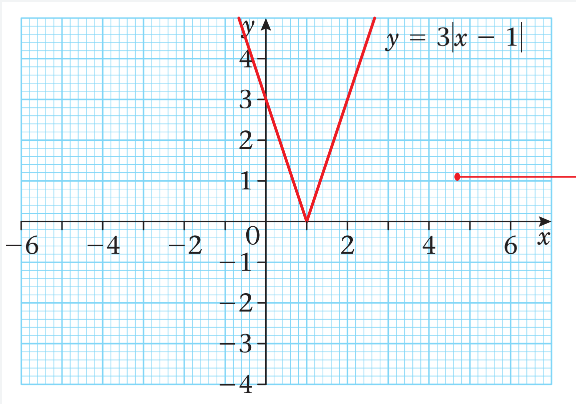
Step 3 using (2):
Vertical translation of −2.



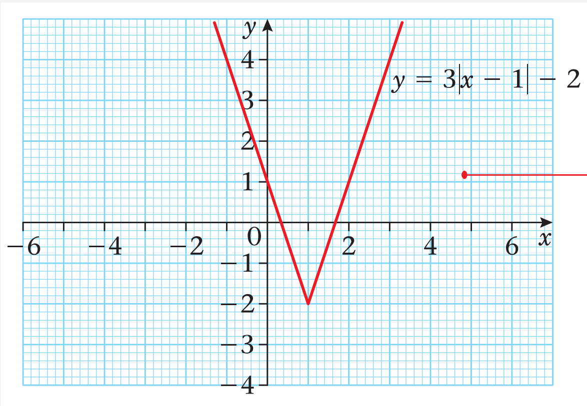
Sketch the graph of $f(x) = |x|$.



Step 1
Horizontal translation of 1.



Step 2
Vertical stretch, scale factor 3.



Step 3
Vertical translation of −2.

Exercise 5D

1 Using combinations of transformations, sketch the graph of each of the following:

a $y = 2x^2 - 4$

b $y = 3(x + 1)^2$

c $y = \frac{3}{x} - 2$

d $y = \frac{3}{x - 2}$

e $y = 5\sin(x + 30°),\ 0 \leqslant x \leqslant 360°$

f $y = \frac{1}{2}e^x + 4$

g $y = |4x| + 1$

h $y = 2x^3 - 3$

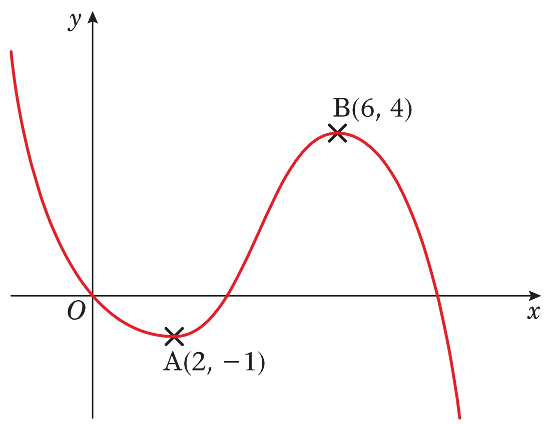
i $y = 3\ln(x - 2),\ x > 2$

j $y = |2e^x - 3|$

5.5 **When you are given a sketch of $y = f(x)$, you need to be able to sketch transformations of the graph, showing coordinates of the points to which given points are mapped.**

Example 15

The diagram shows a sketch of the graph of $y = f(x)$. The curve passes through the origin O, the point A(2, −1) and the point B(6, 4).



Sketch the graph of:

a $y = 2f(x) - 1$

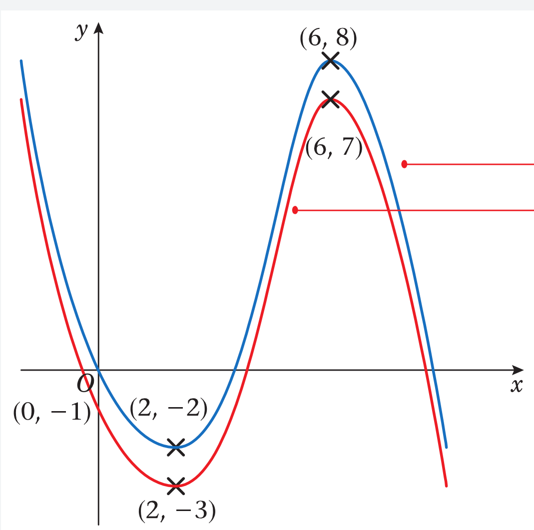
b $y = f(x + 2) + 2$

c $y = \frac{1}{4}f(2x)$

d $y = -f(x - 1)$

In each case, find the coordinates of the images of the points O, A and B.

a $y = 2f(x) - 1$



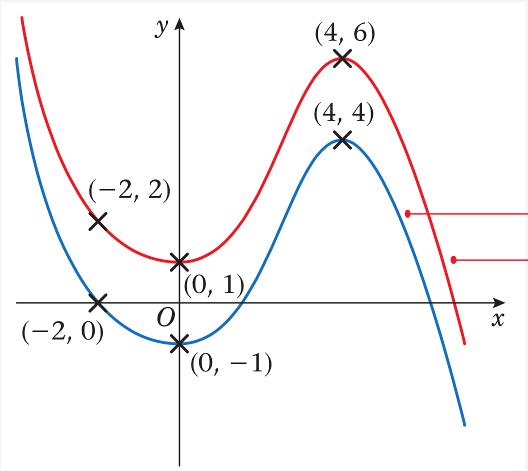
Vertical stretch, scale factor 2.

Vertical stretch, scale factor 2, then a vertical translation of −1.

$y = 2f(x) - 1$ is shown in red in the diagram.

The images of O, A and B are (0, −1), (2, −3) and (6, 7) respectively.

b $y = f(x + 2) + 2$



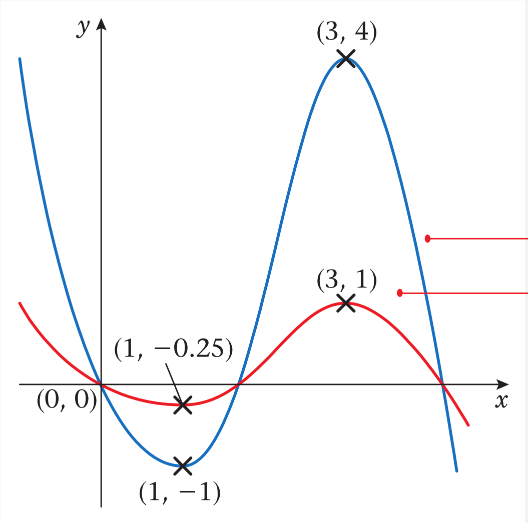
Horizontal translation of −2.

Horizontal translation of −2, then a vertical translation of 2.

$y = f(x + 2) + 2$ is shown in red in the diagram.

The images of O, A and B are (−2, 2), (0, 1) and (4, 6) respectively.

c $y = \frac{1}{4}f(2x)$



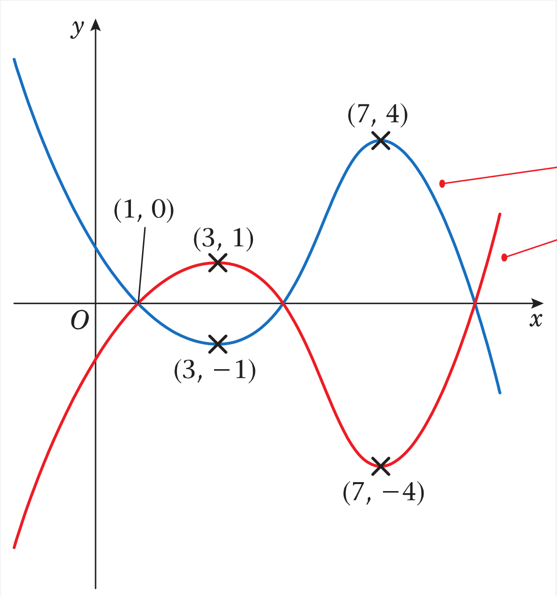
Horizontal stretch, scale factor $\frac{1}{2}$.

Horizontal stretch, scale factor $\frac{1}{2}$, then a vertical stretch, scale factor $\frac{1}{4}$.

$y = \frac{1}{4}f(2x)$ is shown in red in the diagram.

The images of O, A and B are (0, 0), (1, −0.25) and (3, 1) respectively.

d $y = -f(x - 1)$



Horizontal translation of 1.

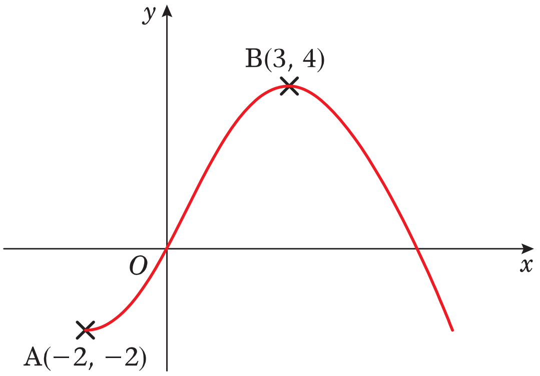
Horizontal translation of 1, then a vertical stretch, scale factor −1.

A 'vertical stretch with scale factor −1' is equivalent to a reflection in the x-axis.

$y = -f(x - 1)$ is shown in red in the diagram.
The images of O, A and B are (1, 0), (3, 1) and (7, −4) respectively.

Exercise 5E

1 The diagram shows a sketch of the graph of $y = f(x)$.
The curve passes through the origin O, the point A(−2, −2) and the point B(3, 4).



Sketch the graph of:

a $y = 3f(x) + 2$

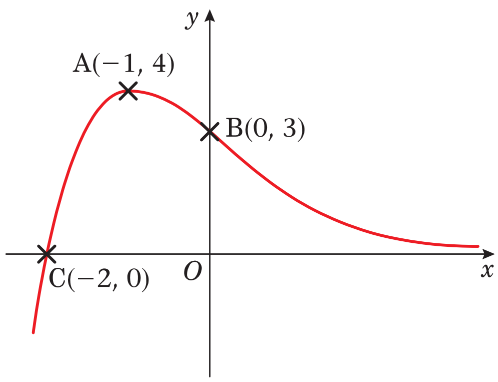
b $y = f(x - 2) - 5$

c $y = \frac{1}{2}f(x + 1)$

d $y = -f(2x)$

In each case, find the coordinates of the images of the points O, A and B.

2 The diagram shows a sketch of the graph of $y = f(x)$. The curve has a maximum at the point A$(-1, 4)$ and crosses the axes at the points B$(0, 3)$ and C$(-2, 0)$.

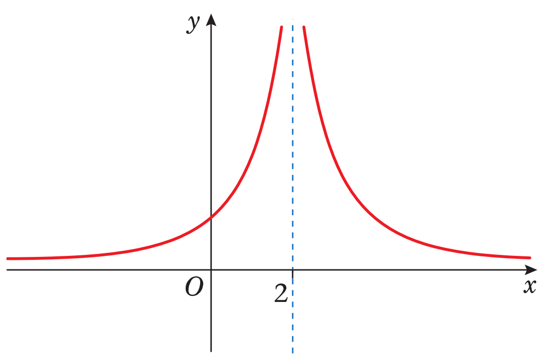


Sketch the graph of:

a $y = 3f(x - 2)$ **b** $y = \frac{1}{2}f(\frac{1}{2}x)$ **c** $y = -f(x) + 4$ **d** $y = -2f(x + 1)$

For each graph, find, where possible, the coordinates of the maximum or minimum and the coordinates of the intersection points with the axes.

3 The diagram shows a sketch of the graph of $y = f(x)$. The lines $x = 2$ and $y = 0$ (the x-axis) are asymptotes to the curve.



Sketch the graph of:

a $y = 3f(x) - 1$ **b** $y = f(x + 2) + 4$ **c** $y = -f(2x)$

For each part, state the equations of the asymptotes.

Mixed exercise 5F

1 **a** Using the same scales and the same axes, sketch the graphs of $y = |2x|$ and $y = |x - a|$, where $a > 0$.

b Write down the coordinates of the points where the graph of $y = |x - a|$ meets the axes.

c Show that the point with coordinates $(-a, 2a)$ lies on both graphs.

d Find the coordinates, in terms of a, of a second point which lies on both graphs. *E*

2 **a** Sketch, on a single diagram, the graphs of $y = a^2 - x^2$ and $y = |x + a|$, where a is a constant and $a > 1$.

b Write down the coordinates of the points where the graph of $y = a^2 - x^2$ cuts the coordinate axes.

c Given that the two graphs intersect at $x = 4$, calculate the value of a. *E*

3 **a** On the same axes, sketch the graphs of $y = 2 - x$ and $y = 2|x + 1|$.

b Hence, or otherwise, find the values of x for which $2 - x = 2|x + 1|$. *E*

4 Functions f and g are defined by

$$f : x \to 4 - x \quad \{x \in \mathbb{R}\}$$
$$g : x \to 3x^2 \quad \{x \in \mathbb{R}\}$$

a Find the range of g.

b Solve $gf(x) = 48$.

c Sketch the graph of $y = |f(x)|$ and hence find the values of x for which $|f(x)| = 2$. *E*

5 The function f is defined by $f : x \to |2x - a| \quad \{x \in \mathbb{R}\}$, where a is a positive constant.

a Sketch the graph of $y = f(x)$, showing the coordinates of the points where the graph cuts the axes.

b On a separate diagram, sketch the graph of $y = f(2x)$, showing the coordinates of the points where the graph cuts the axes.

c Given that a solution of the equation $f(x) = \frac{1}{2}x$ is $x = 4$, find the two possible values of a. *E*

6 **a** Sketch the graph of $y = |x - 2a|$, where a is a positive constant. Show the coordinates of the points where the graph meets the axes.

b Using algebra solve, for x in terms of a, $|x - 2a| = \frac{1}{3}x$.

c On a separate diagram, sketch the graph of $y = a - |x - 2a|$, where a is a positive constant. Show the coordinates of the points where the graph cuts the axes. *E*

7 **a** Sketch the graph of $y = |2x + a|$, $a > 0$, showing the coordinates of the points where the graph meets the coordinate axes.

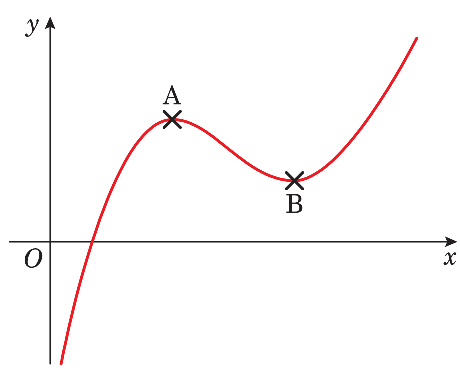
b On the same axes, sketch the graph of $y = \dfrac{1}{x}$.

c Explain how your graphs show that there is only one solution of the equation $x|2x + a| - 1 = 0$.

d Find, using algebra, the value of x for which $x|2x + a| - 1 = 0$. *E*

8 The diagram shows part of the curve with equation $y = f(x)$, where

$$f(x) = x^2 - 7x + 5\ln x + 8 \quad x > 0$$



The points A and B are the stationary points of the curve.

a Using calculus and showing your working, find the coordinates of the points A and B.

b Sketch the curve with equation $y = -3f(x - 2)$.

c Find the coordinates of the stationary points of the curve with equation $y = -3f(x - 2)$. State, without proof, which point is a maximum and which point is a minimum. *E*

Summary of key points

1 The modulus of a number a, written as $|a|$, is its **positive** numerical value.

- For $|a| \geqslant 0$, $|a| = a$.
- For $|a| < 0$, $|a| = -a$.

2 To sketch the graph of $y = |\mathrm{f}(x)|$:

- Sketch the graph of $y = \mathrm{f}(x)$.
- Reflect in the x-axis any parts where $\mathrm{f}(x) < 0$ (parts below the x-axis).
- Delete the parts below the x-axis.

3 To sketch the graph of $y = \mathrm{f}(|x|)$:

- Sketch the graph of $y = \mathrm{f}(x)$ for $x \geqslant 0$.
- Reflect this in the y-axis.

4 To solve an equation of the type $|\mathrm{f}(x)| = \mathrm{g}(x)$ or $|\mathrm{f}(x)| = |\mathrm{g}(x)|$:

- Use a sketch to locate the roots.
- Solve algebraically, using $-\mathrm{f}(x)$ for reflected parts of $y = \mathrm{f}(x)$ and $-\mathrm{g}(x)$ for reflected parts of $y = \mathrm{g}(x)$.

5 Basic types of transformation are

$\mathrm{f}(x + a)$	a horizontal translation of $-a$
$\mathrm{f}(x) + a$	a vertical translation of $+a$
$\mathrm{f}(ax)$	a horizontal stretch of scale factor $\frac{1}{a}$
$a\mathrm{f}(x)$	a vertical stretch of scale factor a

These may be combined to give, for example $b\mathrm{f}(x + a)$, which is a horizontal translation of $-a$ followed by a vertical stretch of scale factor b.

6 For combinations of transformations, the graph can be built up 'one step at a time', starting from a basic or given curve.