After completing this chapter you should be able to

1. ‘cancel down’ algebraic fractions
2. multiply together two or more algebraic fractions
3. divide algebraic fractions
4. add or subtract algebraic fractions
5. convert an improper fraction into a mixed number fraction.

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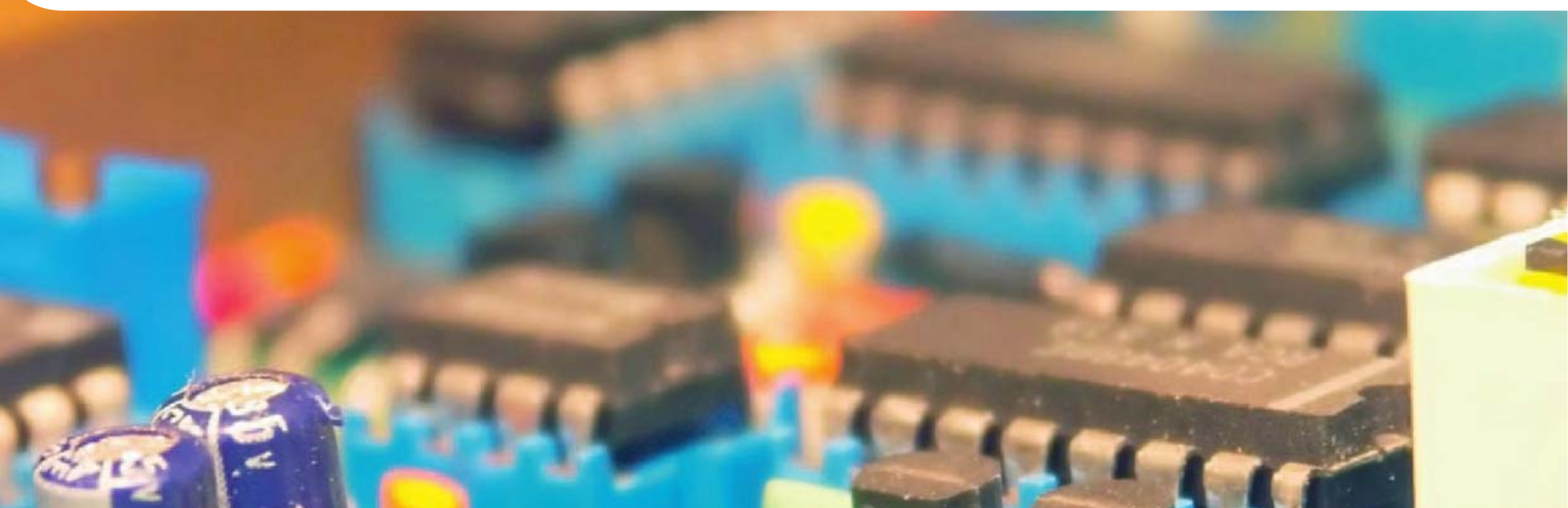
Algebraic fractions



Two resistors in parallel with separate resistances R_1 and R_2 have total resistance R. This can be found by using the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

After completing section you should be able to prove that the formula for R may also be expressed as

$$R = \frac{R_1 R_2}{(R_1 + R_2)}$$

The second formula is much easier to use and can only be found if you know how to manipulate algebraic fractions.

1.1 **You can treat algebraic fractions in exactly the same way as numerical ones. You can cancel them down by finding factors that are common to both the numerator and the denominator.**

Example 1

Find the simplest forms of the fractions

a $\frac{16}{20}$ **b** $\frac{x+3}{2x+6}$ **c** $\frac{x+2}{3x+8}$

a $\frac{16}{20} = \frac{4 \times 4}{5 \times 4}$

$= \frac{4}{5}$

Find a factor that is common to both the numerator and the denominator and cancel down.

b $\frac{x+3}{2x+6} = \frac{1 \times (x+3)}{2 \times (x+3)}$

$= \frac{1}{2}$

Write both the numerator and the denominator as products before cancelling by $(x + 3)$.

c $\frac{x+2}{3x+8}$

This is the simplest form as there are no common factors. Remember you cannot cancel over addition, e.g.

$\frac{{}^1\cancel{x} + \cancel{2}^1}{3\cancel{x} + \cancel{8}_4} = \frac{1+1}{3+4} = \frac{2}{7}$ ✗

■ **When your algebraic expression has fractions in the numerator or denominator, it is sensible to multiply both by the same 'number' to create an equivalent fraction.**

Example 2

Simplify $\frac{\frac{1}{2}x + 1}{\frac{1}{3}x + \frac{2}{3}}$

$\frac{\frac{1}{2}x + 1}{\frac{1}{3}x + \frac{2}{3}} = \frac{(\frac{1}{2}x + 1) \times 6}{(\frac{1}{3}x + \frac{2}{3}) \times 6}$

The LCM of 2 and 3 is 6, so multiply numerator and denominator by 6.

$= \frac{3x+6}{2x+4}$

Factorise numerator and denominator.

$= \frac{3(x+2)}{2(x+2)}$

Cancel any common factors.

$= \frac{3}{2}$

Example 3

Simplify the expressions:

a $\dfrac{x^2 - 1}{x^2 + 4x + 3}$ **b** $\dfrac{x - \frac{1}{x}}{x + 1}$

a $\dfrac{x^2 - 1}{x^2 + 4x + 3} = \dfrac{(x+1)(x-1)}{(x+1)(x+3)}$ — Factorise both numerator and denominator.

$= \dfrac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}(x+3)}$ — Cancel terms that are equal.

$= \dfrac{x-1}{x+3}$

b $\dfrac{\left(x - \frac{1}{x}\right) \times x}{(x+1) \times x} = \dfrac{x^2 - 1}{x(x+1)}$ — Multiply numerator and denominator by x to remove the fraction.

$= \dfrac{(x-1)\cancel{(x+1)}}{x\cancel{(x+1)}}$ — Factorise and cancel common factors of $(x + 1)$.

$= \dfrac{x-1}{x}$

$\left(\text{OR } = \dfrac{x}{x} - \dfrac{1}{x} = 1 - \dfrac{1}{x}\right)$ — By dividing throughout by x.

Exercise 1A

1 Simplify:

a $\dfrac{4x + 4}{x + 1}$ **b** $\dfrac{2x - 1}{6x - 3}$ **c** $\dfrac{x + 4}{x + 2}$

d $\dfrac{x + \frac{1}{2}}{4x + 2}$ **e** $\dfrac{4x + 2y}{6x + 3y}$ **f** $\dfrac{a + 3}{a + 6}$

g $\dfrac{5p - 5q}{10p - 10q}$ **h** $\dfrac{\frac{1}{2}a + b}{2a + 4b}$ **i** $\dfrac{x^2}{x^2 + 3x}$

j $\dfrac{x^2 - 3x}{x^2 - 9}$ **k** $\dfrac{x^2 + 5x + 4}{x^2 + 8x + 16}$ **l** $\dfrac{x^3 - 2x^2}{x^2 - 4}$

m $\dfrac{x^2 - 4}{x^2 + 4}$ **n** $\dfrac{x + 2}{x^2 + 5x + 6}$ **o** $\dfrac{2x^2 - 5x - 3}{2x^2 - 7x - 4}$

p $\dfrac{\frac{1}{2}x^2 + x - 4}{\frac{1}{4}x^2 + \frac{3}{2}x + 2}$ **q** $\dfrac{3x^2 - x - 2}{\frac{1}{2}x + \frac{1}{3}}$ **r** $\dfrac{x^2 - 5x - 6}{\frac{1}{3}x - 2}$

1.2 You multiply fractions together by finding the product of the numerator and dividing by the product of the denominator.

Example 4

Calculate:

a $\frac{1}{2} \times \frac{3}{5}$ **b** $\frac{a}{b} \times \frac{c}{d}$

a $\frac{1}{2} \times \frac{3}{5}$

$= \frac{1 \times 3}{2 \times 5}$

$= \frac{3}{10}$

b $\frac{a}{b} \times \frac{c}{d}$

$= \frac{a \times c}{b \times d}$

$= \frac{ac}{bd}$

Multiply numerators.
Multiply denominators.

■ **When there are factors common to both the numerator and the denominator cancel down first.**

Example 5

Simplify the following products:

a $\frac{3}{5} \times \frac{5}{9}$ **b** $\frac{a}{b} \times \frac{c}{a}$ **c** $\frac{x+1}{2} \times \frac{3}{x^2 - 1}$

a $\frac{{}^{1}\cancel{3}}{{}_{1}\cancel{5}} \times \frac{\cancel{5}^{1}}{\cancel{9}_{3}} = \frac{1 \times 1}{1 \times 3}$

$= \frac{1}{3}$

Cancel any common factors and multiply numerators and denominators.

b $\frac{{}^{1}\cancel{a}}{b} \times \frac{c}{\cancel{a}_{1}} = \frac{1 \times c}{b \times 1}$

$= \frac{c}{b}$

Cancel any common factors and multiply numerators and denominators.

c $\frac{x+1}{2} \times \frac{3}{x^2 - 1}$

$= \frac{x+1}{2} \times \frac{3}{(x+1)(x-1)}$

$= \frac{{}^{1}\cancel{x+1}}{2} \times \frac{3}{{}_{1}\cancel{(x+1)}(x-1)}$

$= \frac{3}{2(x-1)}$

Factorise $(x^2 - 1)$.

Cancel any common factors and multiply numerators and denominators.

■ **To divide by a fraction multiply by the reciprocal of that fraction.**

Example 6

Divide $\frac{5}{6}$ by $\frac{1}{3}$.

$$\frac{5}{6} \div \frac{1}{3} = \frac{5}{_2\cancel{6}} \times \frac{\cancel{3}^1}{1}$$

Turn divisor upside down and multiply by it. Cancel the common factor 3.

$$= \frac{5 \times 1}{2 \times 1}$$

Multiply numerators and denominators.

$$= \frac{5}{2}$$

Example 7

Simplify:

a $\frac{a}{b} \div \frac{a}{c}$ **b** $\frac{x+2}{x+4} \div \frac{3x+6}{x^2-16}$

a $$\frac{a}{b} \div \frac{a}{c} = \frac{^1\cancel{a}}{b} \times \frac{c}{\cancel{a}_1}$$

Turn divisor upside down and multiply by it. Cancel the common factor a.

$$= \frac{1 \times c}{b \times 1}$$

Multiply numerators and denominators.

$$= \frac{c}{b}$$

b $$\frac{x+2}{x+4} \div \frac{3x+6}{x^2-16}$$

$$= \frac{x+2}{x+4} \times \frac{x^2-16}{3x+6}$$

Turn divisor upside down and multiply by it.

$$= \frac{x+2}{x+4} \times \frac{(x+4)(x-4)}{3(x+2)}$$

Factorise $x^2 - 16$ (difference of two squares).

$$= \frac{\cancel{x+2}^1}{\cancel{x+4}^1} \times \frac{\cancel{(x+4)}^1(x-4)}{3\cancel{(x+2)}^1}$$

Cancel any common factors.

$$= \frac{x-4}{3}$$

Exercise 1B

1 Simplify:

a $\frac{a}{d} \times \frac{a}{c}$ **b** $\frac{a^2}{c} \times \frac{c}{a}$ **c** $\frac{2}{x} \times \frac{x}{4}$

d $\frac{3}{x} \div \frac{6}{x}$ **e** $\frac{4}{xy} \div \frac{x}{y}$ **f** $\frac{2r^2}{5} \div \frac{4}{r^3}$

g $(x+2) \times \dfrac{1}{x^2-4}$

h $\dfrac{1}{a^2+6a+9} \times \dfrac{a^2-9}{2}$

i $\dfrac{x^2-3x}{y^2+y} \times \dfrac{y+1}{x}$

j $\dfrac{y}{y+3} \div \dfrac{y^2}{y^2+4y+3}$

k $\dfrac{x^2}{3} \div \dfrac{2x^3-6x^2}{x^2-3x}$

l $\dfrac{4x^2-25}{4x-10} \div \dfrac{2x+5}{8}$

m $\dfrac{x+3}{x^2+10x+25} \times \dfrac{x^2+5x}{x^2+3x}$

n $\dfrac{3y^2+4y-4}{10} \div \dfrac{3y+6}{15}$

o $\dfrac{x^2+2xy+y^2}{2} \times \dfrac{4}{(x-y)^2}$

1.3 You need to have the same denominator when you add or subtract numeric or algebraic fractions.

Example 8

Add $\frac{1}{3}$ to $\frac{3}{4}$.

$$\frac{1}{3} + \frac{3}{4}$$

$\times \frac{4}{4}$ $\times \frac{3}{3}$

$$= \frac{4}{12} + \frac{9}{12}$$

$$= \frac{13}{12}$$

The lowest common multiple of 3 and 4 is 12.

Remember the lowest common multiple of 3 and 4 is the smallest number into which 3 and 4 both divide.

Example 9

Add $\frac{1}{3}$ to $\frac{5}{12}$.

$$\frac{1}{3} + \frac{5}{12}$$

$\times \frac{4}{4}$

$$= \frac{4}{12} + \frac{5}{12}$$

$$= \frac{9}{12}$$

$$= \frac{3}{4}$$

The lowest common multiple of 3 and 12 is 12.

Cancel down.

Example 10

Simplify the following fractions:

a $\dfrac{a}{x} + b$ **b** $\dfrac{3}{x+1} - \dfrac{4x}{x^2-1}$

a $\dfrac{a}{x} + b = \dfrac{a}{x} + \dfrac{b}{1}$ $\quad\times\dfrac{x}{x}$

$= \dfrac{a}{x} + \dfrac{bx}{x}$

$= \dfrac{a+bx}{x}$

Write b as $\dfrac{b}{1}$.

The LCM is x.

b $\dfrac{3}{x+1} - \dfrac{4x}{x^2-1}$

$= \dfrac{3}{(x+1)} - \dfrac{4x}{(x+1)(x-1)}$

$\times\dfrac{(x-1)}{(x-1)}$

$= \dfrac{3(x-1)}{(x+1)(x-1)} - \dfrac{4x}{(x+1)(x-1)}$

$= \dfrac{3(x-1)-4x}{(x+1)(x-1)}$

$= \dfrac{-x-3}{(x+1)(x-1)}$

Factorise $x^2 - 1$ to $(x+1)(x-1)$.

The LCM of $(x+1)$ and $(x+1)(x-1)$ is $(x+1)(x-1)$.

Simplify the numerator: $3x - 3 - 4x = -x - 3$.

Exercise 1C

1 Simplify:

a $\dfrac{1}{p} + \dfrac{1}{q}$ **b** $\dfrac{a}{b} - 1$ **c** $\dfrac{1}{2x} + \dfrac{1}{x}$

d $\dfrac{3}{x^2} - \dfrac{1}{x}$ **e** $\dfrac{3}{4x} + \dfrac{1}{8x}$ **f** $\dfrac{x}{y} + \dfrac{y}{x}$

g $\dfrac{1}{x+2} - \dfrac{1}{x+1}$ **h** $\dfrac{2}{x+3} - \dfrac{1}{x-2}$ **i** $\frac{1}{3}(x+2) - \frac{1}{2}(x+3)$

j $\dfrac{3x}{(x+4)^2} - \dfrac{1}{(x+4)}$ **k** $\dfrac{1}{2(x+3)} + \dfrac{1}{3(x-1)}$

l $\dfrac{2}{x^2+2x+1} + \dfrac{1}{x+1}$ **m** $\dfrac{3}{x^2+3x+2} - \dfrac{2}{x^2+4x+4}$

n $\dfrac{2}{a^2+6a+9} - \dfrac{3}{a^2+4a+3}$ **o** $\dfrac{2}{y^2-x^2} + \dfrac{3}{y-x}$

p $\dfrac{x+2}{x^2-x-12} - \dfrac{x+1}{x^2+5x+6}$ **q** $\dfrac{3x+1}{(x+2)^3} - \dfrac{2}{(x+2)^2} + \dfrac{4}{(x+2)}$

1.4 **You can divide two algebraic fractions. As with ordinary numbers, an improper fraction has a larger numerator than denominator, e.g.**

$$\frac{12}{5} = 2\frac{2}{5}$$

Improper fraction — **Mixed number**

An improper algebraic fraction is one whose numerator has a degree equal to or larger than the denominator, e.g.

$$\frac{x^2 + 5x + 8}{x - 2} \quad \text{and} \quad \frac{x^3 - 8x + 5}{x^2 + 2x + 1}$$

are both improper fractions because the numerator has a larger degree than the denominator. You can change them into mixed number fractions either by long division or by using the remainder theorem.

Example 11

Divide $x^3 + x^2 - 7$ by $x - 3$ by using long division.

$$\begin{array}{r|l} & x^2 + 4x + 12 \\ x - 3 & x^3 + x^2 + 0x - 7 \\ & x^3 - 3x^2 \\ & \quad 4x^2 + 0x \\ & \quad 4x^2 - 12x \\ & \qquad 12x - 7 \\ & \qquad 12x - 36 \\ & \qquad\quad 29 \end{array}$$

Include all coefficients, including ‘0’.

Divide x into x^3: it divides in x^2 times.

Multiply the divisor by x^2 then subtract.

Divide x into $4x^2$: it divides in $4x$ times.

Multiply the divisor by $4x$ then subtract.

Divide x into $12x$: it divides in 12 times.

Multiply the divisor by 12 then subtract.

Therefore $\frac{x^3 + x^2 - 7}{x - 3}$

$\equiv x^2 + 4x + 12$ remainder 29

How many whole times $(x - 3)$ divides into $x^3 + x^2 - 7$.

$= x^2 + 4x + 12 + \frac{29}{x - 3}$

The remainder is $\frac{29}{x-3}$ because you are dividing by $(x - 3)$.

- **The remainder theorem:**
  Any polynomial F(x) can be put in the form

  F(x) ≡ Q(x) × divisor + remainder

  where Q(x) is the quotient and is how many times the divisor divides into the function.

This is a more general version of the remainder theorem you met in Book C2. It allows you to divide by a quadratic expression.

Example 12

Divide $x^3 + x^2 - 7$ by $x - 3$ by using the remainder theorem.

Set up the identity

$F(x) \equiv Q(x) \times \text{divisor} + \text{remainder}$

$x^3 + x^2 - 7 \equiv (Ax^2 + Bx + C)(x - 3) + D$

and solve to find the constants A, B, C and D.

Let $x = 3$

$$27 + 9 - 7 = (9A + 3B + C) \times 0 + D$$
$$D = 29$$

Let $x = 0$

$$0 + 0 - 7 = (A \times 0 + B \times 0 + C) \times (0 - 3) + D$$
$$-7 = -3C + D$$
$$-7 = -3C + 29$$
$$-3C = -36$$
$$C = 12$$

Compare coefficients in x^3 $\quad 1 = A$

Compare coefficients in x^2 $\quad 1 = -3A + B$

$$1 = -3 + B$$
$$B = 4$$

Therefore

$$x^3 + x^2 - 7 \equiv (1x^2 + 4x + 12)(x - 3) + 29$$

$$\Rightarrow \frac{x^3 + x^2 - 7}{x - 3} \equiv x^2 + 4x + 12 + \frac{29}{x - 3}$$

As the divisor is a linear expression and $F(x)$ is a cubic polynomial then $Q(x)$ must be a quadratic and the remainder must be a constant.

This is true for all x, so you can substitute into the RHS and LHS to work out the values of A, B, C and D.

Put $x = 3$ into both sides to give an equation in D only.

Substitute $D = 29$ and $x = 0$ to give an equation in C only.

Because this is an equivalence relation you can compare coefficients of terms in x^3 and x^2 on the RHS and LHS of the equation

In x^3: LHS $= x^3$
RHS $= Ax^3$
In x^2: LHS $= x^2$
RHS $= (-3A + B)x^2$

Substitute $A = 1$.

$\div (x - 3)$

Example 13

Divide $x^4 + x^3 + x - 10$ by $x^2 + 2x - 3$.

Method 1. Using long division.

$$\begin{array}{r|l} & x^2 - x + 5 \\ \hline x^2 + 2x - 3 & x^4 + x^3 + 0x^2 + x - 10 \\ & \underline{x^4 + 2x^3 - 3x^2} \\ & \quad -x^3 + 3x^2 + x \\ & \quad \underline{-x^3 - 2x^2 + 3x} \\ & \qquad 5x^2 - 2x - 10 \\ & \qquad \underline{5x^2 + 10x - 15} \\ & \qquad\quad -12x + 5 \end{array}$$

All coefficients need to be included.

x^2 goes into x^4, x^2 times. Multiply x^2 by $(x^2 + 2x - 3)$ and subtract.

x^2 goes into $-x^3$, $-x$ times. Multiply $-x$ by $(x^2 + 2x - 3)$ and subtract.

x^2 goes into $5x^2$, 5 times. Multiply 5 by $(x^2 + 2x - 3)$ and subtract.

Since the degree of $(-12x + 5)$ is smaller than $(x^2 + 2x - 3)$ this is the remainder.

Therefore

$$\frac{x^4 + x^3 + x - 10}{x^2 + 2x - 3}$$

$$= x^2 - x + 5 \text{ remainder } -12x + 5$$

$$= x^2 - x + 5 + \frac{-12x + 5}{x^2 + 2x - 3}$$

Write the remainder as a 'fraction'.

Method 2. Using the remainder theorem.

$F(x) \equiv Q(x) \times$ divisor + remainder

$x^4 + x^3 + x - 10$

$\equiv (Ax^2 + Bx + C)(x^2 + 2x - 3) + Dx + E$

Now solve for A, B, C, D and E.

As the divisor is a quadratic expression and $F(x)$ has a power of 4 then $Q(x)$ must be a quadratic and the remainder must be a linear expression.

As $x^2 + 2x - 3 \equiv (x + 3)(x - 1)$ we should start by substituting $x = 1$, $x = -3$ then $x = 0$.

Exercise 1D

1 Express the following improper fractions in 'mixed' number form by:
i using long division **ii** using the remainder theorem

a $\dfrac{x^3 + 2x^2 + 3x - 4}{x - 1}$ **b** $\dfrac{2x^3 + 3x^2 - 4x + 5}{x + 3}$ **c** $\dfrac{x^3 - 8}{x - 2}$

d $\dfrac{2x^2 + 4x + 5}{x^2 - 1}$ **e** $\dfrac{8x^3 + 2x^2 + 5}{2x^2 + 2}$ **f** $\dfrac{4x^3 - 5x^2 + 3x - 14}{x^2 + 2x - 1}$

g $\dfrac{x^4 + 3x^2 - 4}{x^2 + 1}$ **h** $\dfrac{x^4 - 1}{x + 1}$ **i** $\dfrac{2x^4 + 3x^3 - 2x^2 + 4x - 6}{x^2 + x - 2}$

2 Find the value of the constants A, B, C, D and E in the following identity:

$$3x^4 - 4x^3 - 8x^2 + 16x - 2 \equiv (Ax^2 + Bx + C)(x^2 - 3) + Dx + E$$

Mixed exercise 1E

1 Simplify the following fractions:

a $\dfrac{ab}{c} \times \dfrac{c^2}{a^2}$ **b** $\dfrac{x^2 + 2x + 1}{4x + 4}$ **c** $\dfrac{x^2 + x}{2} \div \dfrac{x + 1}{4}$

d $\dfrac{x + \frac{1}{x} - 2}{x - 1}$ **e** $\dfrac{a + 4}{a + 8}$ **f** $\dfrac{b^2 + 4b - 5}{b^2 + 2b - 3}$

2 Simplify:

a $\dfrac{x}{4} + \dfrac{x}{3}$ **b** $\dfrac{4}{y} - \dfrac{3}{2y}$ **c** $\dfrac{x + 1}{2} - \dfrac{x - 2}{3}$

d $\dfrac{x^2 - 5x - 6}{x - 1}$ **e** $\dfrac{x^3 + 7x - 1}{x + 2}$ **f** $\dfrac{x^4 + 3}{x^2 + 1}$

3 Find the value of the constants A, B, C and D in the following identity:

$$x^3 - 6x^2 + 11x + 2 \equiv (x - 2)(Ax^2 + Bx + C) + D$$

4 $f(x) = x + \dfrac{3}{x-1} - \dfrac{12}{x^2 + 2x - 3}$ $\{x \in \mathbb{R}, x > 1\}$

Show that $f(x) = \dfrac{x^2 + 3x + 3}{x + 3}$

E

5 Show that $\dfrac{x^4 + 2}{x^2 - 1} \equiv x^2 + B + \dfrac{C}{x^2 - 1}$ for constants B and C, which should be found.

6 Show that $\dfrac{4x^3 - 6x^2 + 8x - 5}{2x + 1}$ can be put in the form $Ax^2 + Bx + C + \dfrac{D}{2x + 1}$. Find the values of the constants A, B, C and D.

Summary of key points

1 Algebraic fractions can be simplified by cancelling down. To do this the numerators and denominators must be fully factorised first.

2 If the numerator and denominator contain fractions then you can multiply both by the same number (the lowest common multiple) to create an equivalent fraction.

3 To multiply fractions, you simply multiply the numerators and multiply the denominators. If possible cancel down first.

4 To divide two fractions, multiply the first fraction by the reciprocal of the second fraction.

5 To add (or subtract) fractions each fraction must have the same denominator. This is done by finding the **lowest** common multiple of the denominators.

6 When the numerator has the same or higher degree than the denominator, you can divide the terms to produce a ‘mixed’ number fraction. This can be done either by using long division or by using the remainder theorem:

$$F(x) \equiv Q(x) \times \text{divisor} + \text{remainder}$$

where $Q(x)$ is the quotient and is how many times the divisor divides into the function.