C3 DIFFERENTIATION — Answers - Worksheet D

1 $x = \frac{1}{2} \quad \therefore y = \frac{1}{4}$

$\frac{dy}{dx} = 2x + \frac{1}{4x-1} \times 4 = 2x + \frac{4}{4x-1}$

grad = 1 + 4 = 5

$\therefore\ y - \frac{1}{4} = 5(x - \frac{1}{2})$

$[\ y = 5x - \frac{9}{4}\ ]$

2 **a** $\sqrt{8 - e^{2x}} = 2$

$8 - e^{2x} = 4$

$x = \frac{1}{2}\ln 4 = \ln 2$

b $\frac{dy}{dx} = \frac{1}{2}(8 - e^{2x})^{-\frac{1}{2}} \times (-2e^{2x})$

$= \frac{-e^{2x}}{\sqrt{8 - e^{2x}}}$

grad = –2

$\therefore\ y - 2 = -2(x - \ln 2)$

$2x + y = 2 + 2\ln 2$

$2x + y = 2 + \ln 2^2$

$2x + y = 2 + \ln 4$

3 **a** $\frac{dy}{dx} = 2 + \frac{1}{4-2x} \times (-2) = 2 - \frac{1}{2-x}$

$\frac{d^2y}{dx^2} = (2-x)^{-2} \times (-1) = \frac{-1}{(2-x)^2}$

b SP: $2 - \frac{1}{2-x} = 0$

$2 - x = \frac{1}{2}$

$x = \frac{3}{2} \quad \therefore (\frac{3}{2}, 4)$

c $x = \frac{3}{2},\ \frac{d^2y}{dx^2} = -4 \quad \therefore$ maximum

4 **a** $\frac{dy}{dx} = -3(2x+1)^{-2} \times 2 = \frac{-6}{(2x+1)^2}$

$x = 1$, grad $= -\frac{2}{3}$, $\therefore$ grad of normal $= \frac{3}{2}$

$\therefore\ y - 1 = \frac{3}{2}(x - 1)$

$[\ y = \frac{3}{2}x - \frac{1}{2}\ ]$

b at Q $\quad \frac{3x-1}{2} = \frac{3}{2x+1}$

$(3x-1)(2x+1) = 6$

$6x^2 + x - 7 = 0$

$(6x+7)(x-1) = 0$

$x = 1$ (at P) or $-\frac{7}{6}$

$\therefore\ Q\,(-\frac{7}{6}, -\frac{9}{4})$

5 **a** $t = 0, N = 20 \quad \therefore a = 20$

$t = 8, N = 60 \quad \therefore 60 = 20e^{8k}$

$k = \frac{1}{8}\ln 3 = 0.137$ (3sf)

b $N = 20e^{0.1373t}$

$t = 12, N = 104$ (3sf)

c $\frac{dN}{dt} = 20 \times 0.1373e^{0.1373t} = 2.747e^{0.1373t}$

$t = 12,\ \frac{dN}{dt} = 14.3$

$\therefore$ N increasing at 14.3 per second (3sf)

6 **a** $= 3(5 - 2x^2)^2 \times (-4x)$

$= -12x(5 - 2x^2)^2$

b SP: $-12x(5 - 2x^2)^2 = 0$

$x = 0$ or $x^2 = \frac{5}{2}$

$x = 0, \pm\frac{1}{2}\sqrt{10}$

$\therefore\ (-\frac{1}{2}\sqrt{10}, 0), (0, 125), (\frac{1}{2}\sqrt{10}, 0)$

c $x = \frac{3}{2},\ y = \frac{1}{8}$

grad $= -18 \times \frac{1}{4} = -\frac{9}{2}$

$\therefore\ y - \frac{1}{8} = -\frac{9}{2}(x - \frac{3}{2})$

$8y - 1 = -36x + 54$

$36x + 8y - 55 = 0$

7 **a** $\frac{dy}{dx} = 4 - e^{2x}$

SP: $4 - e^{2x} = 0$

$x = \frac{1}{2}\ln 4 = \ln 2$

$\therefore$ $(\ln 2, 4\ln 2 - 2)$

b $\frac{d^2y}{dx^2} = -2e^{2x}$

$x = \ln 2$: $\frac{d^2y}{dx^2} = -8$ $\therefore$ maximum

8 **a** $f'(x) = \frac{3}{x} - 2$

b grad of curve = 4

$\therefore$ $\frac{3}{x} - 2 = 4$

$x = \frac{1}{2}$

c SP: $\frac{3}{x} - 2 = 0$

$x = \frac{3}{2}$ $\therefore$ $(\frac{3}{2}, 3\ln\frac{15}{2} - 3)$

d $x \geq \frac{3}{2}$

9 **a** $\frac{dy}{dx} = \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \times 2x = \frac{x}{\sqrt{x^2+3}}$

at A, grad $= -\frac{1}{2}$

$\therefore$ $y - 2 = -\frac{1}{2}(x + 1)$

$[\, y = \frac{3}{2} - \frac{1}{2}x \,]$

b at B, grad $= \frac{1}{2}$

$\therefore$ grad of normal $= -2$

$\therefore$ $y - 2 = -2(x - 1)$

$[\, y = 4 - 2x \,]$

c $\frac{3}{2} - \frac{1}{2}x = 4 - 2x$

$x = \frac{5}{3}$

10 **a** 80°C

b 20°C, as $t \to \infty$, $T \to 20$

c $30 = 20 + 60e^{-25k}$

$e^{-25k} = \frac{30-20}{60} = \frac{1}{6}$

$k = \frac{-1}{25}\ln\frac{1}{6} = 0.0717$ (3sf)

d $T = 20 + 60e^{-0.07167t}$

$\frac{dT}{dt} = 60 \times (-0.07167)e^{-0.07167t}$

$= -4.300e^{-0.07167t}$

$t = 40$, $\frac{dT}{dt} = -0.245$

$\therefore$ temp. decreasing at 0.245°C min^{-1} (3sf)

11 **a** $f'(x) = 2x - 7 + \frac{4}{x} = 0$

$2x^2 - 7x + 4 = 0$

$x = \frac{7 \pm \sqrt{49-32}}{4} = \frac{7 \pm \sqrt{17}}{4}$

$x = 0.72, 2.78$

b $x = 2$ $\therefore$ $y = -10$, grad $= -1$

$\therefore$ $y + 10 = -(x - 2)$

$[\, y = -x - 8 \,]$

12 **a** $\frac{dy}{dx} = 2x + 8(x - 1)^{-2}$

SP: $2x + \frac{8}{(x-1)^2} = 0$

$2x(x - 1)^2 + 8 = 0$

$2x(x^2 - 2x + 1) + 8 = 0$

$2x^3 - 4x^2 + 2x + 8 = 0$

$x^3 - 2x^2 + x + 4 = 0$

b let $f(x) = x^3 - 2x^2 + x + 4$

$f(1) = 4$, $f(2) = 6$, $f(-1) = 0$

$\therefore$ $(x + 1)$ is a factor

$\therefore$ $(x + 1)(x^2 - 3x + 4) = 0$

$x = -1$ or $x^2 - 3x + 4 = 0$

$b^2 - 4ac = 9 - 16 = -7$

$b^2 - 4ac < 0$ $\therefore$ no real roots

$\therefore$ exactly one SP

$(-1, 5)$

c $\frac{d^2y}{dx^2} = 2 - 16(x - 1)^{-3}$

when $x = -1$, $\frac{d^2y}{dx^2} = 4$

$\frac{d^2y}{dx^2} > 0$ $\therefore$ minimum