After completing this chapter you should be able to

1. sketch simple transformations of the graph of $y = e^x$
2. sketch simple transformations of the graph $y = \ln x$
3. solve equations involving e^x and $\ln x$
4. know what is meant by the terms exponential growth and decay
5. solve real life examples of exponential growth and decay.

3

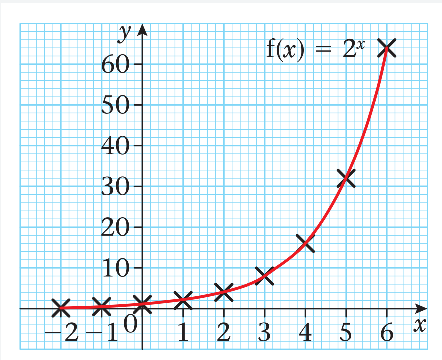
The exponential and log functions

Exponential functions occur naturally in real life. Scientists can model the number of elephants in a herd by using an exponential function.

3.1 **Exponential functions are ones of the form $y = a^x$. Graphs of these functions all pass through (0, 1) because $a^0 = 1$ for any number a.**

Example 1

Sketch the graph of $f(x) = 2^x$ for the domain $x \in \mathbb{R}$. State the range of the function.



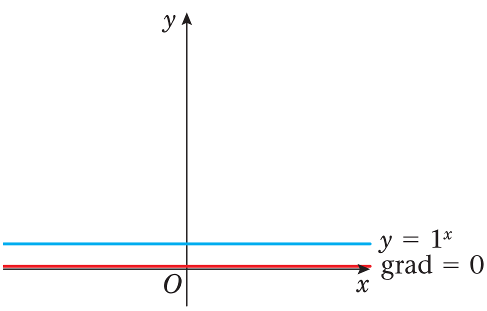
The range of the function is $f(x) > 0$

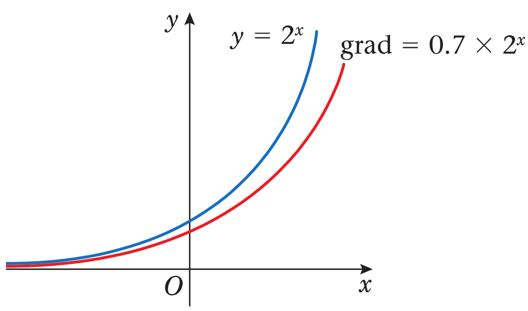
Draw up a table of values.

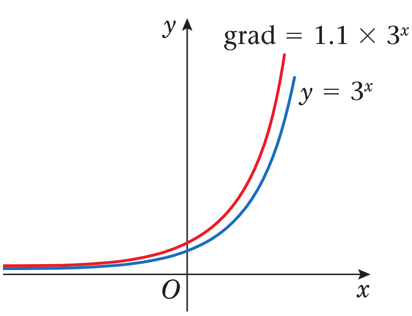
x	−2	−1	0	1	2	3	4	5	6
y	0.25	0.5	1	2	4	8	16	32	64

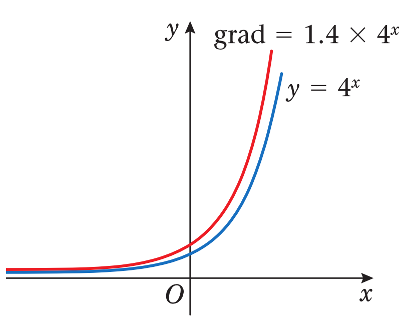
Plot points on a graph.

The gradient functions of these graphs are similar to the functions themselves.









Put these results in a table.

Function	Gradient
$y = 1^x$	grad $= 0 \times 1^x$
$y = 2^x$	grad $= 0.7 \times 2^x$
$y = 3^x$	grad $= 1.1 \times 3^x$
$y = 4^x$	grad $= 1.4 \times 4^x$

You should be able to spot from this table that as a increases for the function $y = a^x$, so does the gradient function.

You should be able to deduce that there is going to be a number between 2 and 3 such that the gradient function would be the same as the function. This number is approximately equal to 2.718 and is represented by the letter 'e'. It is similar to π in the respect that it is an irrational number representing a number that exists in the real world.

■ **The exponential function $y = e^x$ (where $e \approx 2.718$) is therefore the function in which the gradient is identical to the function. For this reason it is often referred to as the exponential function.**

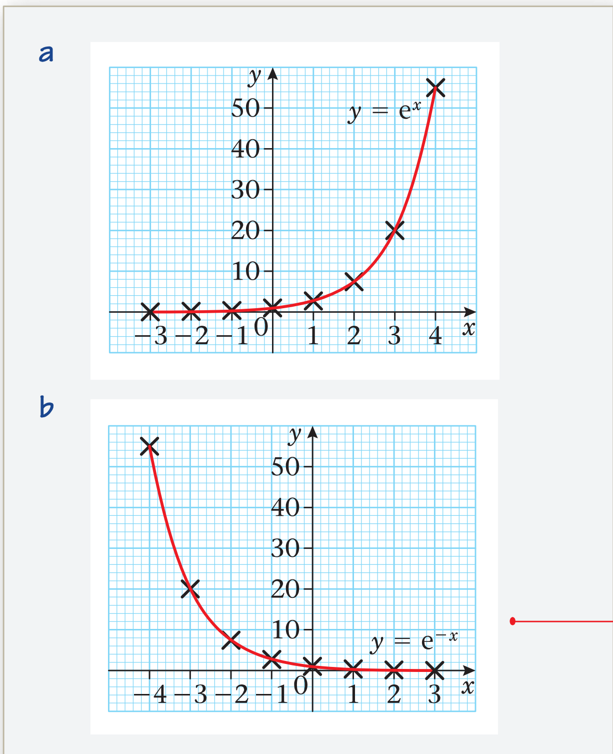
If $y = e^x$ then $\frac{dy}{dx} = e^x$

3.2 All exponential graphs will follow a similar pattern. The standard graph of $y = e^x$ can be used to represent 'exponential' growth, which is how population growth can be modelled in real life.

Example 2

Draw the graphs of:

a $y = e^x$ **b** $y = e^{-x}$



A table of values will show you how rapidly this curve grows.

x	−2	−1	0	1	2	3	4	5
y	0.14	0.37	1	2.7	7.4	20	55	148

With these curves it is worth keeping in mind:

- as $x \to \infty$, $e^x \to \infty$ (it grows very rapidly)
- when $x = 0$, $e^0 = 1$ [(0, 1) lies on the curve]
- as $x \to -\infty$, $e^x \to 0$ (it approaches but never reaches the x-axis).

This curve is similar to the one in part **a** except that its value at $x = 2$ is e^{-2} and its value at $x = -2$ is e^2.

Hence it is a reflection of the curve of part **a** in the y-axis.

The graph in Example 2**b** is often referred to as exponential decay. It is used as a model in many examples from real life including the fall in value of a car as well as the decay in radioactive isotopes.

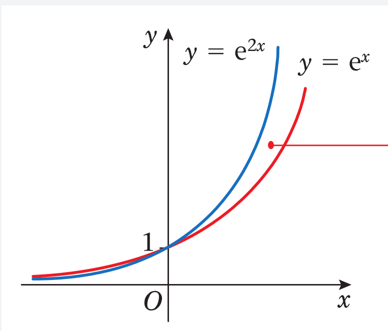
Example 3

Draw graphs of the exponential functions:

a $y = e^{2x}$ **b** $y = 10e^{-x}$ **c** $y = 3 + 4e^{\frac{1}{2}x}$

a $y = e^{2x}$

$= (e^x)^2$

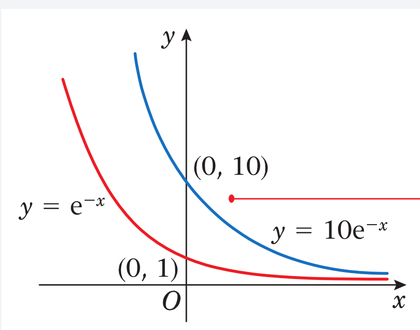


x	−3	0	3
y	0.002	1	403

If you calculate some values it can give you an idea of the shape of the graph.

The y values of $y = e^{2x}$ are the 'square' of the y values of $y = e^x$.

b $y = 10e^{-x}$

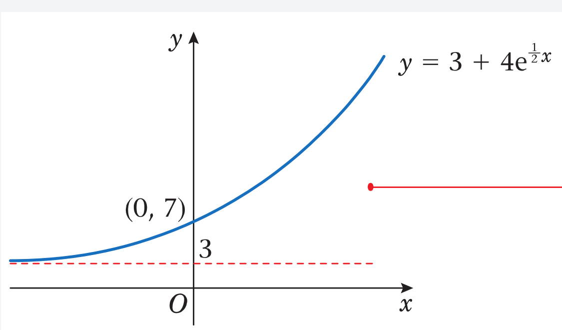


x	−3	0	3
y	201	10	0.5

Calculating some y values helps you sketch the curve.

The y values of $y = 10e^{-x}$ are 10 times bigger than the y values of $y = e^{-x}$.

c $y = 3 + 4e^{\frac{1}{2}x}$



x	−3	0	3
y	3.9	7	21

Since $e^{\frac{1}{2}x} > 0$

$3 + 4e^{\frac{1}{2}x} > 3$.

Range of function is $y > 3$.

Example 4

The price of a used car can be represented by the formula

$$P = 16\,000\,e^{-\frac{t}{10}}$$

where P is the price in £'s and t is the age in years from new.

Calculate:

a the new price

b the value at 5 years old

c what the model suggests about the eventual value of the car.

Use this to sketch the graph of P against t.

a Substitute $t = 0$ into $P = 16\,000\,e^{-\frac{0}{10}}$
$= 16\,000 \times 1$

The new price is £16 000.

The new price is when $t = 0$.
Remember $e^0 = 1$.

b Substitute $t = 5$ into $P = 16\,000\,e^{-\frac{5}{10}}$
$= 16\,000\,e^{-\frac{1}{2}}$
$= £9704.49$

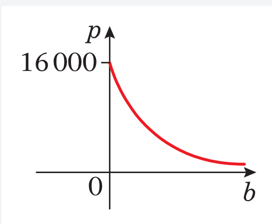
The price after 5 years is £9704.49.

Its price at 5 years old is when $t = 5$.

c As $t \to \infty$, $e^{-\frac{t}{10}} \to 0$
Therefore $P \to 16\,000 \times 0 = 0$.

The eventual value is zero.

For the eventual value, let $t \to \infty$.



Use the values from parts **a**, **b** and **c** to sketch the graph.

Exercise 3A

1 Sketch the graphs of

a $y = e^x + 1$ **b** $y = 4e^{-2x}$ **c** $y = 2e^x - 3$

d $y = 4 - e^x$ **e** $y = 6 + 10e^{\frac{1}{2}x}$ **f** $y = 100e^{-x} + 10$

2 The value of a car varies according to the formula

$$V = 20\,000\,e^{-\frac{t}{12}}$$

where V is the value in £'s and t is its age in years from new.

a State its value when new.

b Find its value (to the nearest £) after 4 years.

c Sketch the graph of V against t.

3 The population of a country is increasing according to the formula

$$P = 20 + 10\,e^{\frac{t}{50}}$$

where P is the population in thousands and t is the time in years after the year 2000.

a State the population in the year 2000.

b Use the model to predict the population in the year 2020.

c Sketch the graph of P against t for the years 2000 to 2100.

4 The number of people infected with a disease varies according to the formula

$$N = 300 - 100\,e^{-0.5t}$$

where N is the number of people infected with the disease and t is the time in years after detection.

a How many people were first diagnosed with the disease?

b What is the long term prediction of how this disease will spread?

c Graph N against t.

5 The value of an investment varies according to the formula

$$V = A\,e^{\frac{t}{12}}$$

where V is the value of the investment in £'s, A is a constant to be found and t is the time in years after the investment was made.

a If the investment was worth £8000 after 3 years find A to the nearest £.

b Find the value of the investment after 10 years.

c By what factor will be the original investment have increased by after 20 years?

3.3

To study the exponential function further, it becomes necessary to introduce its inverse function. From Chapter 2 you should know that inverse functions perform the 'opposite' operation to a function, in exactly the same way as '+4' and '−4' and 'x^2' and '$\sqrt{x}$' are inverse operations.

- **the inverse to e^x is $\log_e x$ (often written $\ln x$).**

Example 5

Solve the equations

a $e^x = 3$ **b** $\ln x = 4$

a When $e^x = 3$

$x = \ln 3$

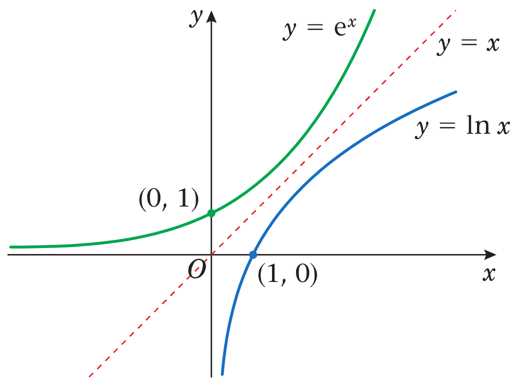
b When $\ln x = 4$

$x = e^4$

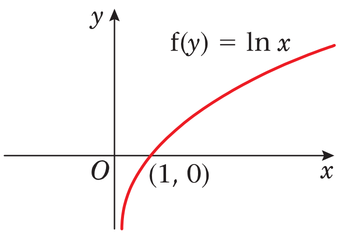
The key to solving any equation is knowing the inverse operation.
When $x^2 = 10$, $x = \sqrt{10}$.

The inverse of e^x is $\ln x$ and vice versa.

Using your knowledge of inverse functions, the graph of $\ln x$ will be a reflection of e^x in the line $y = x$.



The function $f(x) = \ln x$ therefore has a domain of $\{x \in \mathbb{R}, x > 0\}$ and a range of $\{f(x) \in \mathbb{R}\}$.



The important points about the graph $y = \ln x$ are:

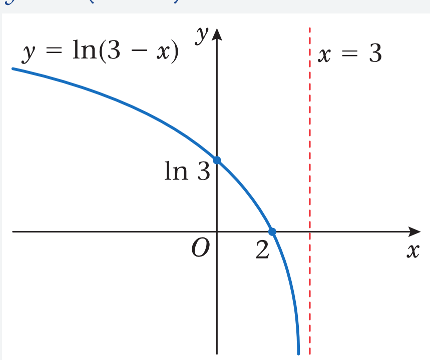
- as $x \to 0$, $y \to -\infty$
- $\ln x$ does not exist for negative numbers
- when $x = 1$, $y = 0$
- as $x \to \infty$, $y \to \infty$ (slowly).

Example 6

Sketch the graphs of

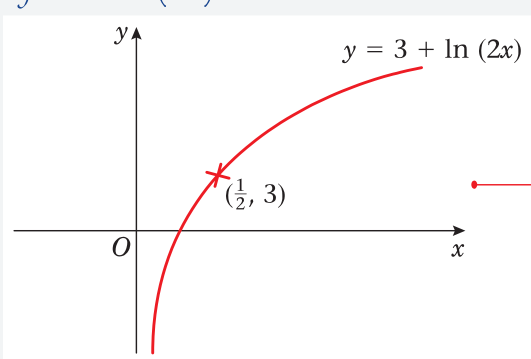
a $y = \ln(3 - x)$ **b** $y = 3 + \ln(2x)$

a $y = \ln(3 - x)$



When $x \to 3$, $y \to -\infty$.
y does not exist for values of x bigger than 3.
When $x = 2$, $y = \ln(3 - 2) = \ln 1 = 0$.
As $x \to -\infty$, $y \to \infty$ (slowly).

b $y = 3 + \ln(2x)$



When $x \to 0$, $y \to -\infty$.
When $x = \frac{1}{2}$, $y = 3 + \ln 1 = 3$.
As $x \to \infty$, $y \to \infty$ (slowly).

Example 7

Solve the equations:

a $e^{2x+3} = 4$ **b** $2 \ln x + 1 = 5$

These questions are solved by changing the subject of the formula and using the fact that $\ln x$ and e^x are inverse functions.

a

$$e^{2x+3} = 4$$
$$2x + 3 = \ln 4$$
$$2x = \ln 4 - 3$$
$$x = \frac{\ln 4 - 3}{2}$$
$$= \frac{\ln 4}{2} - \frac{3}{2}$$
$$= \ln 2 - \tfrac{3}{2}$$

The inverse to e^x is $\ln x$.

Sometimes questions ask you to put an answer in a particular form. Note that $\frac{\ln 4}{2} = \frac{1}{2}\ln 4 = \ln 4^{\frac{1}{2}} = \ln 2$

b

$$2 \ln x + 1 = 5$$
$$2 \ln x = 4$$
$$\ln x = 2$$
$$x = e^2$$

Isolate $\ln x$.

The inverse of $\ln x$ is e^x.

Example 8

The number of elephants in a herd can be represented by the equation

$$N = 150 - 80\,e^{-\frac{t}{40}}$$

where N is the number of elephants in the herd and t is the time in years after the year 2003.

Calculate:

a the number of elephants in the herd in 2003

b the number of elephants in the herd in 2007

c the year when the population will grow to above 100

d the long term population of the herd as predicted by the model.

Use all of the above information to sketch a graph of N against t for the model.

$$N = 150 - 80\,e^{-\frac{t}{40}}$$

a In the year 2003, $N = 150 - 80\,e^{-\frac{0}{40}}$

$$= 150 - 80 \times 1$$
$$= 70$$

For 2003 substitute $t = 0$.

$e^0 = 1$

b In the year 2007,

$$N = 150 - 80\,e^{-\frac{4}{40}}$$
$$= 150 - 72.4$$
$$= 78 \quad \text{(to nearest elephant)}$$

For 2007 substitute $t = 4$.

c If population is 100, then

$100 = 150 - 80\,e^{-\frac{t}{40}}$

$80\,e^{-\frac{t}{40}} = 50$

$e^{-\frac{t}{40}} = \frac{50}{80}$

$-\frac{t}{40} = \ln\frac{50}{80}$

$t = 18.8$ years

Therefore the population will be over 100 by the year 2022.

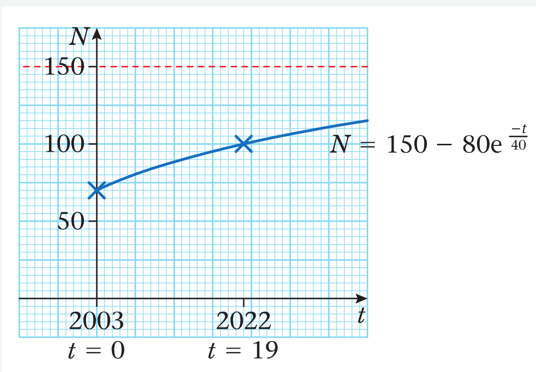
> Substitute $N = 100$.
> Isolate $e^{-\frac{t}{40}}$.
> The inverse of e^x is $\ln x$.
> Add 18.8 years to 2003 and round up answer.

d As $t \to \infty$, $e^{-\frac{t}{40}} \to 0$ and therefore

$N \to 150 - 80 \times 0 = 150$

The long term population as predicted by the model is 150.

> Long term suggests as $t \to \infty$.



> Use all the above information to sketch the graph.

Exercise 3B

1 Solve the following equations giving exact solutions:

a $e^x = 5$ **b** $\ln x = 4$ **c** $e^{2x} = 7$

d $\ln\frac{x}{2} = 4$ **e** $e^{x-1} = 8$ **f** $\ln(2x + 1) = 5$

g $e^{-x} = 10$ **h** $\ln(2 - x) = 4$ **i** $2e^{4x} - 3 = 8$

2 Solve the following giving your solution in terms of ln 2:

a $e^{3x} = 8$ **b** $e^{-2x} = 4$ **c** $e^{2x+1} = 0.5$

3 Sketch the following graphs stating any asymptotes and intersections with axes:

a $y = \ln(x + 1)$ **b** $y = 2\ln x$ **c** $y = \ln(2x)$

d $y = (\ln x)^2$ **e** $y = \ln(4 - x)$ **f** $y = 3 + \ln(x + 2)$

4 The price of a new car varies according to the formula

$$P = 15\,000\,e^{-\frac{t}{10}}$$

where P is the price in £'s and t is the age in years from new.

a State its new value.

b Calculate its value after 5 years (to the nearest £).

c Find its age when its price falls below £5000.

d Sketch the graph showing how the price varies over time. Is this a good model?

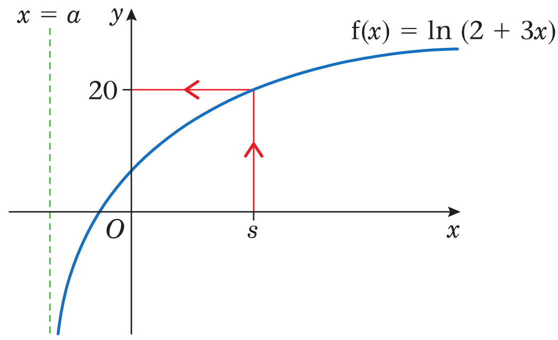
5 The graph opposite is of the function $f(x) = \ln(2 + 3x)$ $\{x \in \mathbb{R}, x > a\}$.

a State the value of a.

b Find the value of s for which $f(s) = 20$.

c Find the function $f^{-1}(x)$ stating its domain.

d Sketch the graphs $f(x)$ and $f^{-1}(x)$ on the same axes stating the relationship between them.



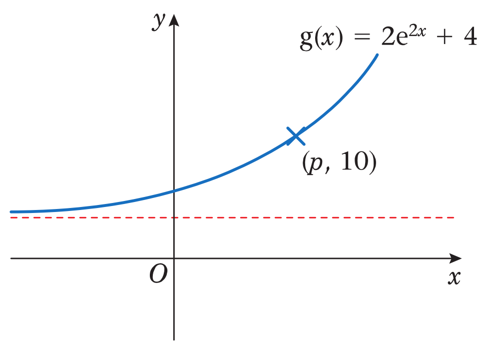
6 The graph opposite is of the function $g(x) = 2e^{2x} + 4$ $\{x \in \mathbb{R}\}$.

a Find the range of the function.

b Find the value of p to 2 significant figures.

c Find $g^{-1}(x)$ stating its domain.

d Sketch $g(x)$ and $g^{-1}(x)$ on the same set of axes stating the relationship between them.



7 The number of bacteria in a culture grows according to the following equation:

$$N = 100 + 50\,e^{\frac{t}{30}}$$

where N is the number of bacteria present and t is the time in days from the start of the experiment.

a State the number of bacteria present at the start of the experiment.

b State the number after 10 days.

c State the day on which the number first reaches 1 000 000.

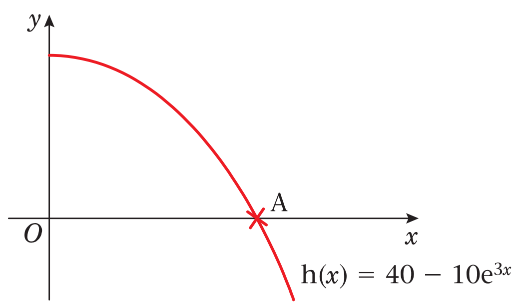
d Sketch the graph showing how N varies with t.

8 The graph opposite shows the function $h(x) = 40 - 10\,e^{3x}$ $\{x > 0, x \in \mathbb{R}\}$.

a State the range of the function.

b Find the exact coordinates of A in terms of ln 2.

c Find $h^{-1}(x)$ stating its domain.



Mixed exercise 3C

1 Sketch the following functions stating any asymptotes and intersections with axes:

a $y = e^x + 3$ **b** $y = \ln(-x)$ **c** $y = \ln(x + 2)$

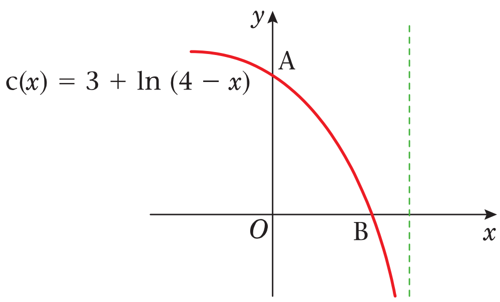
d $y = 3e^{-2x} + 4$ **e** $y = e^{x+2}$ **f** $y = 4 - \ln x$

2 Solve the following equations, giving exact solutions:

a $\ln(2x - 5) = 8$ **b** $e^{4x} = 5$ **c** $24 - e^{-2x} = 10$

d $\ln x + \ln(x - 3) = 0$ **e** $e^x + e^{-x} = 2$ **f** $\ln 2 + \ln x = 4$

3 The function $c(x) = 3 + \ln(4 - x)$ is shown below.



a State the exact coordinates of point A.

b Calculate the exact coordinates of point B.

c Find the inverse function $c^{-1}(x)$ stating its domain.

d Sketch $c(x)$ and $c^{-1}(x)$ on the same set of axes stating the relationship between them.

4 The price of a computer system can be modelled by the formula

$$P = 100 + 850e^{-\frac{t}{2}}$$

where P is the price of the system in £s and t is the age of the computer in years after being purchased.

a Calculate the new price of the system.

b Calculate its price after 3 years.

c When will it be worth less than £200?

d Find its price as $t \to \infty$.

e Sketch the graph showing P against t.

Comment on the appropriateness of this model.

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5 The function f is defined by

$$f : x \to \ln(5x - 2) \ \{x \in \mathbb{R}, x > \tfrac{2}{5}\}.$$

a Find an expression for $f^{-1}(x)$.

b Write down the domain of $f^{-1}(x)$.

c Solve, giving your answer to 3 decimal places,

$$\ln(5x - 2) = 2.$$

E

6 The functions f and g are given by

$$f : x \to 3x - 1 \ \{x \in \mathbb{R}\}$$
$$g : x \to e^{\frac{x}{2}} \ \{x \in \mathbb{R}\}$$

a Find the value of fg(4), giving your answer to 2 decimal places.

b Express the inverse function $f^{-1}(x)$ in the form $f^{-1} : x \to \dots$.

c Using the same axes, sketch the graphs of the functions f and gf. Write on your sketch the value of each function at $x = 0$.

d Find the values of x for which $f^{-1}(x) = \dfrac{5}{f(x)}$.

E

7 The points P and Q lie on the curve with equation $y = e^{\frac{1}{2}x}$. The x-coordinates of P and Q are ln 4 and ln 16 respectively.

a Find an equation for the line PQ.

b Show that this line passes through the origin O.

c Calculate the length, to 3 significant figures, of the line segment PQ.

E

8 The functions f and g are defined over the set of real numbers by

$$f : x \to 3x - 5$$
$$g : x \to e^{-2x}$$

a State the range of $g(x)$.

b Sketch the graphs of the inverse functions f^{-1} and g^{-1} and write on your sketches the coordinates of any points at which a graph meets the coordinate axes.

c State, giving a reason, the number of roots of the equation

$$f^{-1}(x) = g^{-1}(x).$$

d Evaluate $fg(-\tfrac{1}{3})$, giving your answer to 2 decimal places.

9 The function f is defined by $f : x \to e^x + k$, $x \in \mathbb{R}$ and k is a positive constant.

a State the range of $f(x)$.

b Find $f(\ln k)$, simplifying your answer.

c Find f^{-1}, the inverse function of f, in the form $f^{-1} : x \to \dots$, stating its domain.

d On the same axes, sketch the curves with equations $y = f(x)$ and $y = f^{-1}(x)$, giving the coordinates of all points where the graphs cut the axes.

E

10 The function f is given by

$$f : x \to \ln(4 - 2x) \quad \{x \in \mathbb{R}, x < 2\}$$

a Find an expression for $f^{-1}(x)$.

b Sketch the curve with equation $y = f^{-1}(x)$, showing the coordinates of the points where the curve meets the axes.

c State the range of $f^{-1}(x)$.

The function g is given by

$$g : x \to e^x \quad \{x \in \mathbb{R}\}$$

d Find the value of gf(0.5). *E*

11 The function f(x) is defined by

$$f(x) = 3x^3 - 4x^2 - 5x + 2$$

a Show that $(x + 1)$ is a factor of f(x).

b Factorise f(x) completely.

c Solve, giving your answers to 2 decimal places, the equation

$$3[\ln(2x)]^3 - 4[\ln(2x)]^2 - 5\ln(2x) + 2 = 0 \quad x > 0$$ *E*

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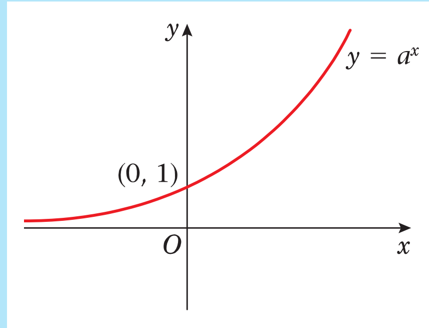
Summary of key points

1 Exponential functions are ones of the form $y = a^x$. They all pass through the point (0, 1).

The domain is all the real numbers. The range is $f(x) > 0$.

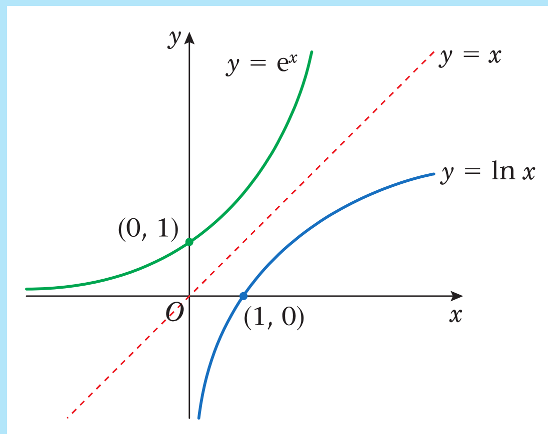


2 The exponential function $y = e^x$ (where $e \approx 2.718$) is a special function whose gradient is identical to the function.

3 The inverse function to e^x is $\ln x$.

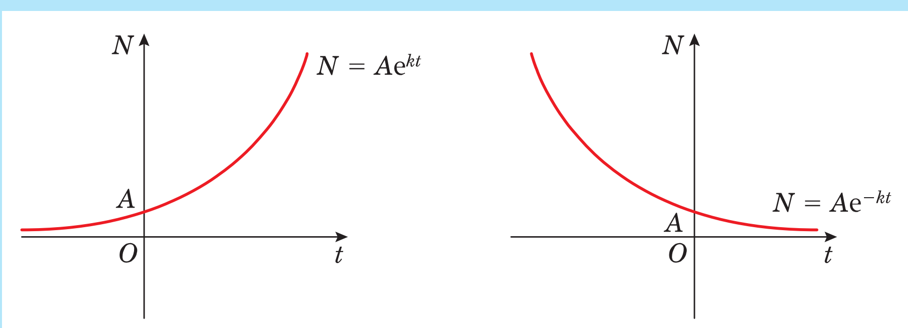
4 The natural log function is a reflection of $y = e^x$ in the line $y = x$. It passes through the point (1, 0).

The domain is the positive numbers. The range is all the real numbers.



5 To solve an equation using $\ln x$ or e^x you must change the subject of the formula and use the fact that they are inverses of each other.

6 Growth and decay models are based around the exponential equations



where A and k are positive numbers.