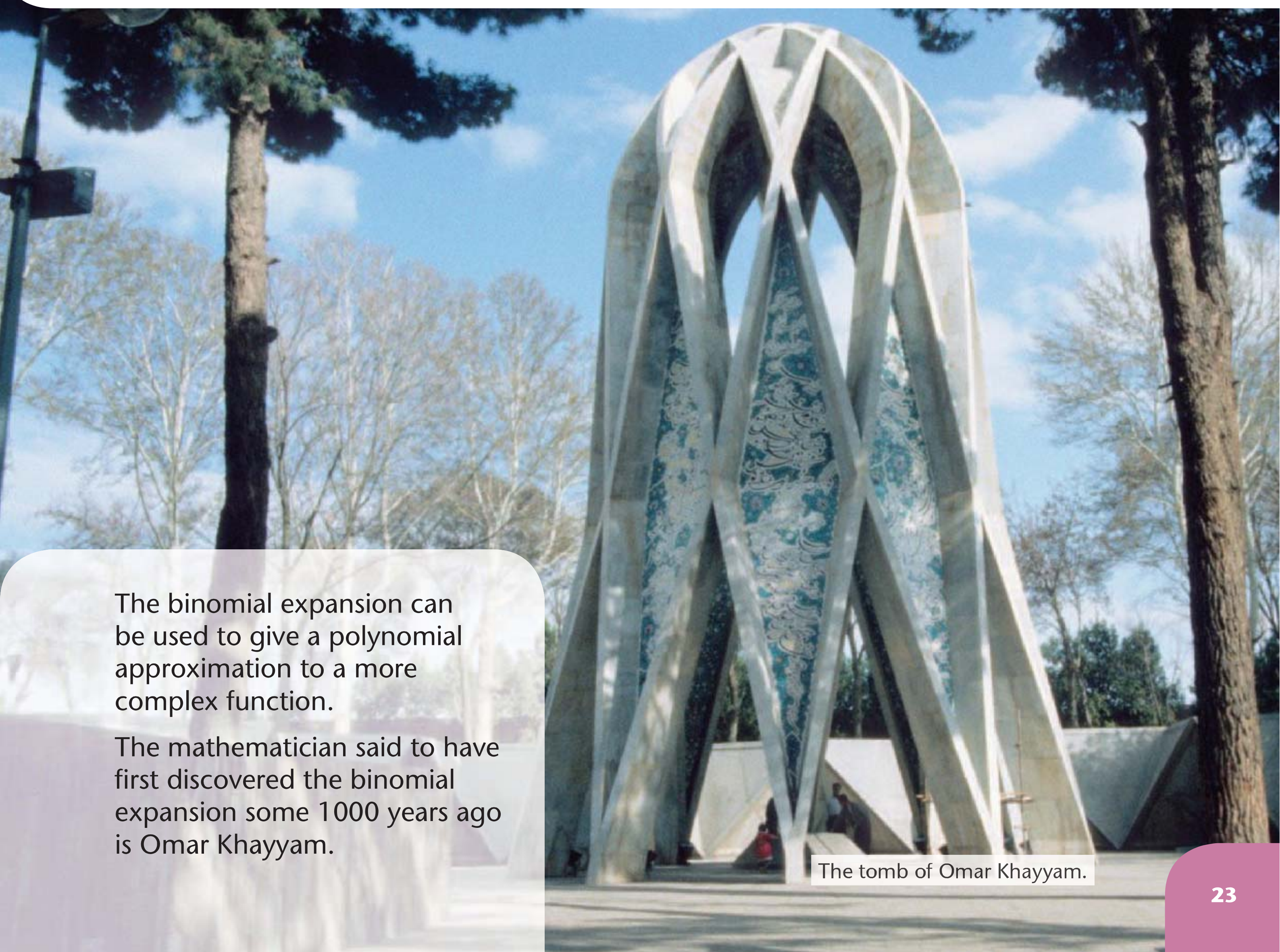
After completing this chapter you should be able to:

- expand $(1 + x)^n$ for any constant n
- expand $(a + bx)^n$ for any constants a, b and n
- determine the range of values of x for which the expansion is valid
- use partial fractions to expand more complex fractional expressions.

3

The binomial expansion



The binomial expansion can be used to give a polynomial approximation to a more complex function.

The mathematician said to have first discovered the binomial expansion some 1000 years ago is Omar Khayyam.

The tomb of Omar Khayyam.

3.1 The binomial expansion is

$$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \ldots + {}^nC_r x^r$$

When n is a positive integer, this expansion is finite and exact. This is not generally the case when n is negative or a fraction.

Example 1

Use the binomial expansion to find **a** $(1+x)^4$ **b** $(1-2x)^3$

a $(1+x)^4$

$$= 1 + 4x + \frac{4 \times 3x^2}{2!} + \frac{4 \times 3 \times 2x^3}{3!} + \frac{4 \times 3 \times 2 \times 1x^4}{4!} + \frac{4 \times 3 \times 2 \times 1 \times 0x^5}{5!}$$

Replace n by 4 in the formula.

$$= 1 + 4x + \frac{4 \times 3x^2}{2} + \frac{4 \times 3 \times 2x^3}{6} + \frac{4 \times 3 \times 2 \times 1x^4}{24} + \frac{4 \times 3 \times 2 \times 1 \times 0x^5}{120}$$

Simplify coefficients.

$$= 1 + 4x + 6x^2 + 4x^3 + 1x^4 + 0x^5$$

All terms after this will also have zero as a coefficient.

$$= 1 + 4x + 6x^2 + 4x^3 + x^4$$

b $(1-2x)^3$

$= 1 + 3 \times (-2x)$

$+ \dfrac{3 \times 2 \times (-2x)^2}{2!}$

$+ \dfrac{3 \times 2 \times 1 \times (-2x)^3}{3!}$

$+ \dfrac{3 \times 2 \times 1 \times 0 \times (-2x)^4}{4!}$

$= 1 - 6x$

$+ \dfrac{3 \times 2 \times 4x^2}{2}$

$+ \dfrac{3 \times 2 \times 1 \times -8x^3}{6}$

$+ \dfrac{3 \times 2 \times 1 \times 0 \times 16x^4}{24}$

$= 1 - 6x + 12x^2 - 8x^3 + 0x^4$

$= 1 - 6x + 12x^2 - 8x^3$

Replace n by 3 and x by $-2x$.

Simplify coefficients.

All terms after this will also have zero as a coefficient.

Example 2

Use the binomial expansion to find the first four terms of **a** $\dfrac{1}{(1+x)}$ **b** $\sqrt{(1-3x)}$

a $\dfrac{1}{(1+x)} = (1+x)^{-1}$

$= 1 + (-1)(x)$

$+ \dfrac{(-1)(-2)(x)^2}{2!}$

$+ \dfrac{(-1)(-2)(-3)(x)^3}{3!} + \ldots$

$= 1 - 1x + 1x^2 - 1x^3 + \ldots$

$= 1 - x + x^2 - x^3 + \ldots$

Write in index form.

Replace n by -1 in the expansion.

As n is not a positive integer, no coefficient will ever be equal to zero.
The expansion is **infinite**, and convergent when $|x| < 1$.

b $\sqrt{(1-3x)} = (1-3x)^{\frac{1}{2}}$

Write in index form.

$= 1 + (\tfrac{1}{2})(-3x)$

Replace n by $\frac{1}{2}$ and x by $-3x$.

$+ \dfrac{(\frac{1}{2})(\frac{1}{2}-1)(-3x)^2}{2!}$

Be careful to write this as $(-3x)^2$, not $-3x^2$.

$+ \dfrac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)(-3x)^3}{3!} + \ldots$

$= 1 - \dfrac{3x}{2} + \dfrac{(\frac{1}{2})(-\frac{1}{2})9x^2}{2}$

Simplify terms.

$+ \dfrac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-27x^3)}{6} + \ldots$

$= 1 - \dfrac{3x}{2} - \dfrac{9x^2}{8} - \dfrac{27x^3}{16} + \ldots$

Because n is not a positive integer, no coefficient will ever be equal to zero. The expansion is **infinite**, and convergent when $|x| < \frac{1}{3}$ because $|3x| < 1$

Example 3

Find the binomial expansions of **a** $(1-x)^{\frac{1}{3}}$, **b** $\dfrac{1}{(1+4x)^2}$, up to and including the term in x^3.

State the range of values of x for which the expansions are valid.

a $(1-x)^{\frac{1}{3}}$

$= 1 + (\tfrac{1}{3})(-x)$

Replace n by $\frac{1}{3}$, x by $(-x)$.

$+ \dfrac{(\frac{1}{3})(\frac{1}{3}-1)(-x)^2}{2!}$

Simplify brackets.

$+ \dfrac{(\frac{1}{3})(\frac{1}{3}-1)(\frac{1}{3}-2)(-x)^3}{3!} + \ldots$

$= 1 + (\tfrac{1}{3})(-x) + \dfrac{(\frac{1}{3})(-\frac{2}{3})(-x)^2}{2}$

Simplify coefficients.

$+ \dfrac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})(-x)^3}{6} + \ldots$

$= 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 + \ldots$

Terms in expansion are $(-x)$, $(-x)^2$, $(-x)^3$.

Expansion is valid as long as $|-x| < 1$

$\Rightarrow |x| < 1$

b $\dfrac{1}{(1+4x)^2}$ — Write in index form.

$= (1+4x)^{-2}$ — Replace n by -2, x by '$4x$'.

$= 1 + (-2)(4x) + \dfrac{(-2)(-2-1)(4x)^2}{2!} + \dfrac{(-2)(-2-1)(-2-2)(4x)^3}{3!} + \ldots$ — Simplify brackets.

$= 1 + (-2)(4x) + \dfrac{(-2)(-3)16x^2}{2} + \dfrac{(-2)(-3)(-4)64x^3}{6} + \ldots$ — Simplify coefficients.

$= 1 - 8x + 48x^2 - 256x^3 + \ldots$ — Terms in expansion are $(4x)$, $(4x)^2$, $(4x)^3$.

Expansion is valid as long as $|4x| < 1$

$\Rightarrow |x| < \frac{1}{4}$.

Example 4

Find the expansion of $\sqrt{(1-2x)}$ up to and including the term in x^3. By substituting in $x = 0.01$, find a suitable decimal approximation to $\sqrt{2}$.

$\sqrt{(1-2x)} = (1-2x)^{\frac{1}{2}}$ — Write in index form.

Replace n by $\frac{1}{2}$, x by $(-2x)$.

$= 1 + (\frac{1}{2})(-2x) + \dfrac{(\frac{1}{2})(\frac{1}{2}-1)(-2x)^2}{2!} + \dfrac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)(-2x)^3}{3!} + \ldots$ — Simplify brackets.

$= 1 + (\frac{1}{2})(-2x) + \dfrac{(\frac{1}{2})(-\frac{1}{2})(4x^2)}{2} + \dfrac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-8x^3)}{6} + \ldots$ — Simplify coefficients.

$= 1 - x - \dfrac{x^2}{2} - \dfrac{x^3}{2} + \ldots$ — Terms in expansion are $(-2x)$, $(-2x)^2$, $(-2x)^3$.

Expansion is valid if $|2x| < 1$

$\Rightarrow |x| < \frac{1}{2}$.

$$\sqrt{(1-2\times 0.01)} \approx 1 - 0.01 - \frac{(0.01)^2}{2} - \frac{(0.01)^3}{2}$$

Substitute $x = 0.01$ into both sides of expansion. This is valid as $|x| < \frac{1}{2}$.

$$\sqrt{0.98} \approx 1 - 0.01 - 0.00005 - 0.0000005$$

Simplify both sides.
Note that the terms are getting smaller.

$$\sqrt{\frac{98}{100}} \approx 0.9899495$$

Write 0.98 as $\frac{98}{100}$.

$$\sqrt{\frac{49\times 2}{100}} \approx 0.9899495$$

Use rules of surds.

$$\frac{7\sqrt{2}}{10} \approx 0.9899495$$

$\times 10$, $\div 7$

$$\sqrt{2} \approx \frac{0.9899495 \times 10}{7}$$

Simplify.

$$\sqrt{2} \approx 1.414213571$$

Exercise 3A

1 Find the binomial expansion of the following up to and including the terms in x^3. State the range values of x for which these expansions are valid.

a $(1+2x)^3$ **b** $\frac{1}{1-x}$ **c** $\sqrt{(1+x)}$ **d** $\frac{1}{(1+2x)^3}$

e $\sqrt[3]{(1-3x)}$ **f** $(1-10x)^{\frac{3}{2}}$ **g** $\left(1+\frac{x}{4}\right)^{-4}$ **h** $\frac{1}{(1+2x^2)}$

2 By first writing $\frac{(1+x)}{(1-2x)}$ as $(1+x)(1-2x)^{-1}$ show that the cubic approximation to $\frac{(1+x)}{(1-2x)}$ is $1 + 3x + 6x^2 + 12x^3$. State the range of values of x for which this expansion is valid.

3 Find the binomial expansion of $\sqrt{(1+3x)}$ in ascending powers of x up to and including the term in x^3. By substituting $x = 0.01$ in the expansion, find an approximation to $\sqrt{103}$. By comparing it with the exact value, comment on the accuracy of your approximation.

4 In the expansion of $(1+ax)^{-\frac{1}{2}}$ the coefficient of x^2 is 24. Find possible values of the constant a and the corresponding term in x^3.

5 Show that if x is small, the expression $\sqrt{\left(\frac{1+x}{1-x}\right)}$ is approximated by $1 + x + \frac{1}{2}x^2$.

6 Find the first four terms in the expansion of $(1-3x)^{\frac{3}{2}}$. By substituting in a suitable value of x, find an approximation to $97^{\frac{3}{2}}$.

3.2 You can use the binomial expansion of $(1 + x)^n$ to expand $(a + bx)^n$ for any constants a and b by simply taking out a as a factor.

Example 5

Find the first four terms in the binomial expansion of **a** $\sqrt{(4 + x)}$ **b** $\frac{1}{(2 + 3x)^2}$.
State the range in values of x for which these expansions are valid.

a $\sqrt{(4 + x)} = (4 + x)^{\frac{1}{2}}$ — Write in index form.

$= \left[4\left(1 + \frac{x}{4}\right)\right]^{\frac{1}{2}}$ — Take out a factor of 4.

$= 4^{\frac{1}{2}}\left(1 + \frac{x}{4}\right)^{\frac{1}{2}}$

$= 2\left(1 + \frac{x}{4}\right)^{\frac{1}{2}}$ — Write $4^{\frac{1}{2}}$ as 2.

$$= 2\left[1 + \left(\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\left(\frac{x}{4}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)\left(\frac{x}{4}\right)^3}{3!} + \ldots\right]$$

Expand $\left(1 + \frac{x}{4}\right)^{\frac{1}{2}}$ using the binomial expansion with $n = \frac{1}{2}$ and $x = \frac{x}{4}$.

$$= 2\left[1 + \left(\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{x^2}{16}\right)}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x^3}{64}\right)}{6} + \ldots\right]$$

Simplify coefficients.

$$= 2\left[1 + \frac{x}{8} - \frac{x^2}{128} + \frac{x^3}{1024} + \ldots\right]$$

Multiply by the 2.

$$= 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + \ldots$$

Expansion is valid if $\left|\frac{x}{4}\right| < 1$

$\Rightarrow |x| < 4.$

Terms in expansion are $\left(\frac{x}{4}\right)$, $\left(\frac{x}{4}\right)^2$, $\left(\frac{x}{4}\right)^3$.

b $\dfrac{1}{(2+3x)^2} = (2+3x)^{-2}$

Write in index form.

$= \left[2\left(1+\dfrac{3x}{2}\right)\right]^{-2}$

Take out a factor of 2.

$= 2^{-2}\left(1+\dfrac{3x}{2}\right)^{-2}$

Write $2^{-2} = \dfrac{1}{2^2} = \dfrac{1}{4}$

$= \dfrac{1}{4}\left(1+\dfrac{3x}{2}\right)^{-2}$

$$= \frac{1}{4}\left[1 + (-2)\left(\frac{3x}{2}\right) + \frac{(-2)(-2-1)\left(\frac{3x}{2}\right)^2}{2!} + \frac{(-2)(-2-1)(-2-2)\left(\frac{3x}{2}\right)^3}{3!} + \ldots\right]$$

Expand $\left(1+\dfrac{3x}{2}\right)^{-2}$ using the binomial expansion with $n = -2$ and $x = \dfrac{3x}{2}$.

$$= \frac{1}{4}\left[1 + (-2)\left(\frac{3x}{2}\right) + \frac{(-2)(-3)\left(\frac{9x^2}{4}\right)}{2} + \frac{(-2)(-3)(-4)\left(\frac{27x^3}{8}\right)}{6} + \ldots\right]$$

Simplify coefficients.

$$= \frac{1}{4}\left[1 - 3x + \frac{27x^2}{4} - \frac{27x^3}{2} + \ldots\right]$$

Multiply by the $\frac{1}{4}$.

$$= \frac{1}{4} - \frac{3}{4}x + \frac{27x^2}{16} - \frac{27x^3}{8} + \ldots$$

Expansion is valid if $\left|\dfrac{3x}{2}\right| < 1$

$\Rightarrow \quad |x| < \dfrac{2}{3}$.

Terms in expansion are $\left(\dfrac{3x}{2}\right), \left(\dfrac{3x}{2}\right)^2, \left(\dfrac{3x}{2}\right)^3$.

Exercise 3B

1 Find the binomial expansions of the following in ascending powers of x as far as the term in x^3. State the range of values of x for which the expansions are valid.

a $\sqrt{(4+2x)}$ **b** $\dfrac{1}{2+x}$ **c** $\dfrac{1}{(4-x)^2}$ **d** $\sqrt{(9+x)}$

e $\dfrac{1}{\sqrt{(2+x)}}$ **f** $\dfrac{5}{3+2x}$ **g** $\dfrac{1+x}{2+x}$ **h** $\sqrt{\left(\dfrac{2+x}{1-x}\right)}$

2 Prove that if x is sufficiently small, $\dfrac{3+2x-x^2}{4-x}$ may be approximated by $\frac{3}{4}+\frac{11}{16}x-\frac{5}{64}x^2$. What does 'sufficiently small' mean in this question?

3 Find the first four terms in the expansion of $\sqrt{(4-x)}$. By substituting $x=\frac{1}{9}$ find a fraction that is an approximation to $\sqrt{35}$. By comparing this to the exact value, state the degree of accuracy of your approximation.

4 The expansion of $(a+bx)^{-2}$ may be approximated by $\frac{1}{4}+\frac{1}{4}x+cx^2$. Find the values of the constants a, b and c.

3.3 You can use partial fractions to simplify the expansions of many more difficult expressions.

Example 6

a Express $\dfrac{4-5x}{(1+x)(2-x)}$ as partial fractions.

b Hence show that the cubic approximation of $\dfrac{4-5x}{(1+x)(2-x)}$ is $2-\dfrac{7x}{2}+\dfrac{11}{4}x^2-\dfrac{25}{8}x^3$.

c State the range of values of x for which the expansion is valid.

a $\dfrac{4-5x}{(1+x)(2-x)} \equiv \dfrac{A}{(1+x)}+\dfrac{B}{(2-x)}$

The denominators must be $(1+x)$ and $(2-x)$.

$\equiv \dfrac{A(2-x)+B(1+x)}{(1+x)(2-x)}$

Add the fractions.

$4-5x \equiv A(2-x)+B(1+x)$

Set the numerators equal.

Substitute $x=2$

$4-10 = A\times 0 + B\times 3$

$-6 = 3B$

$B=-2$

Set $x=2$ to find B.

Substitute $x=-1$

$4+5 = A\times 3 + B\times 0$

$9=3A$

$A=3$

Set $x=-1$ to find A.

so $\dfrac{4-5x}{(1+x)(2-x)} = \dfrac{3}{1+x}-\dfrac{2}{2-x}$

b $\dfrac{4-5x}{(1+x)(2-x)} = \dfrac{3}{(1+x)} - \dfrac{2}{(2-x)}$

$= 3(1+x)^{-1} - 2(2-x)^{-1}$

Write in index form.

The expansion of $3(1+x)^{-1}$

$= 3\left[1 + (-1)(x) + (-1)(-2)\dfrac{(x)^2}{2!} + (-1)(-2)(-3)\dfrac{(x)^3}{3!} + \ldots\right]$

$= 3[1 - x + x^2 - x^3 + \ldots]$

$= 3 - 3x + 3x^2 - 3x^3 + \ldots$

Expand $3(1+x)^{-1}$ using the binomial expansion with $n = -1$.

The expansion of $2(2-x)^{-1}$

Take out a factor of 2.

$= 2\left[2\left(1 - \dfrac{x}{2}\right)\right]^{-1}$

$= 2 \times 2^{-1}\left(1 - \dfrac{x}{2}\right)^{-1}$

$= 1 \times \left[1 + (-1)\left(-\dfrac{x}{2}\right) + \dfrac{(-1)(-2)\left(-\dfrac{x}{2}\right)^2}{2!} + \dfrac{(-1)(-2)(-3)\left(-\dfrac{x}{2}\right)^3}{3!} + \ldots\right]$

$= 1 \times \left[1 + \dfrac{x}{2} + \dfrac{x^2}{4} + \dfrac{x^3}{8} + \ldots\right]$

$= 1 + \dfrac{x}{2} + \dfrac{x^2}{4} + \dfrac{x^3}{8}$

Expand $\left(1 - \dfrac{x}{2}\right)^{-1}$ using the binomial expansion with $n = -1$ and $x = \dfrac{x}{2}$.

Hence $\dfrac{4-5x}{(1+x)(2-x)}$

$= 3(1+x)^{-1} - 2(2-x)^{-1}$

'Add' both expressions.

$= (3 - 3x + 3x^2 - 3x^3)$

$- \left(1 + \dfrac{x}{2} + \dfrac{x^2}{4} + \dfrac{x^3}{8}\right)$

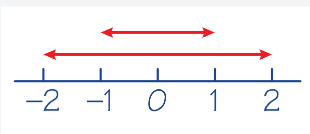
$= 2 - \dfrac{7}{2}x + \dfrac{11}{4}x^2 - \dfrac{25}{8}x^3$

c $\dfrac{3}{(1+x)}$ is valid if $|x| < 1$

Terms are x, x^2, x^3.

$\dfrac{2}{(2-x)}$ is valid if $\left|\dfrac{x}{2}\right| < 1 \Rightarrow |x| < 2$

Terms are $\dfrac{x}{2}$, $\left(\dfrac{x}{2}\right)^2$, $\left(\dfrac{x}{2}\right)^3$.



Look for values of x that satisfy both expressions.

Both expressions are valid provided $|x| < 1$.

Exercise 3C

1 **a** Express $\dfrac{8x+4}{(1-x)(2+x)}$ as partial fractions.

b Hence or otherwise expand $\dfrac{8x+4}{(1-x)(2+x)}$ in ascending powers of x as far as the term in x^2.

c State the set of values of x for which the expansion is valid.

2 **a** Express $\dfrac{-2x}{(2+x)^2}$ as a partial fraction.

b Hence prove that $\dfrac{-2x}{(2+x)^2}$ can be expressed in the form $0 - \dfrac{1}{2}x + Bx^2 + Cx^3$ where constants B and C are to be determined.

c State the set of values of x for which the expansion is valid.

3 **a** Express $\dfrac{6+7x+5x^2}{(1+x)(1-x)(2+x)}$ as a partial fraction.

b Hence or otherwise expand $\dfrac{6+7x+5x^2}{(1+x)(1-x)(2+x)}$ in ascending powers of x as far as the term in x^3.

c State the set of values of x for which the expansion is valid.

Mixed exercise 3D

1 Find binomial expansions of the following in ascending powers of x as far as the term in x^3. State the set of values of x for which the expansion is valid.

a $(1 - 4x)^3$ **b** $\sqrt{(16 + x)}$ **c** $\frac{1}{(1 - 2x)}$ **d** $\frac{4}{2 + 3x}$

e $\frac{4}{\sqrt{(4 - x)}}$ **f** $\frac{1 + x}{1 + 3x}$ **g** $\left(\frac{1 + x}{1 - x}\right)^2$ **h** $\frac{x - 3}{(1 - x)(1 - 2x)}$

2 Find the first four terms of the expansion in ascending powers of x of:

$(1 - \frac{1}{2}x)^{\frac{1}{2}}$, $|x| < 2$

and simplify each coefficient. *E*

(adapted)

3 Show that if x is sufficiently small then $\frac{3}{\sqrt{(4 + x)}}$ can be approximated by $\frac{3}{2} - \frac{3}{16}x + \frac{9}{256}x^2$.

4 Given that $|x| < 4$, find, in ascending powers of x up to and including the term in x^3, the series expansion of:

a $(4 - x)^{\frac{1}{2}}$ **b** $(4 - x)^{\frac{1}{2}}(1 + 2x)$ *E*

(adapted)

5 **a** Find the first four terms of the expansion, in ascending powers of x, of $(2 + 3x)^{-1}$, $|x| < \frac{2}{3}$

b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of:

$\frac{1 + x}{2 + 3x}$, $|x| < \frac{2}{3}$ *E*

6 Find, in ascending powers of x up to and including the term in x^3, the series expansion of $(4 + x)^{-\frac{1}{2}}$, giving your coefficients in their simplest form. *E*

7 $f(x) = (1 + 3x)^{-1}$, $|x| < \frac{1}{3}$.

a Expand $f(x)$ in ascending powers of x up to and including the term in x^3.

b Hence show that, for small x:

$$\frac{1 + x}{1 + 3x} \approx 1 - 2x + 6x^2 - 18x^3.$$

c Taking a suitable value for x, which should be stated, use the series expansion in part **b** to find an approximate value for $\frac{101}{103}$, giving your answer to 5 decimal places. *E*

8 Obtain the first four non-zero terms in the expansion, in ascending powers of x, of the function $f(x)$ where $f(x) = \frac{1}{\sqrt{(1 + 3x^2)}}$, $3x^2 < 1$. *E*

9 Give the binomial expansion of $(1 + x)^{\frac{1}{2}}$ up to and including the term in x^3. By substituting $x = \frac{1}{4}$, find the fraction that is an approximation to $\sqrt{5}$.

10 When $(1 + ax)^n$ is expanded as a series in ascending powers of x, the coefficients of x and x^2 are -6 and 27 respectively.

a Find the values of a and n.

b Find the coefficient of x^3.

c State the values of x for which the expansion is valid.

E (adapted)

11 **a** Express $\dfrac{9x^2 + 26x + 20}{(1 + x)(2 + x)^2}$ as a partial fraction.

b Hence or otherwise show that the expansion of $\dfrac{9x^2 + 26x + 20}{(1 + x)(2 + x)^2}$ in ascending powers of x can be approximated to $5 - \dfrac{7x}{2} + Bx^2 + Cx^3$ where B and C are constants to be found.

c State the set of values of x for which this expansion is valid.

Summary of key points

1 The binomial expansion $(1 + x)^n = 1 + nx + \dfrac{n(n-1)x^2}{2!} + \dfrac{n(n-1)(n-2)x^3}{3!} + \ldots$ can be used to give an exact expression if n is a positive integer, or an approximate expression for any other rational number.

- $(1 + 2x)^3 = 1 + 3(2x) + 3 \times 2\dfrac{(2x)^2}{2!} + 3 \times 2 \times 1 \times \dfrac{(2x)^3}{3!} + 3 \times 2 \times 1 \times 0 \times \dfrac{(2x)^4}{4!}$

  $= 1 + 6x + 12x^2 + 8x^3$ (Expansion is *finite* and *exact*.)

- $\sqrt{(1 - x)} = (1 - x)^{\frac{1}{2}} = 1 + \left(\dfrac{1}{2}\right)(-x) + \left(\dfrac{1}{2}\right)\left(-\dfrac{1}{2}\right)\dfrac{(-x)^2}{2!} + \left(\dfrac{1}{2}\right)\left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right)\dfrac{(-x)^3}{3!} + \ldots$

  $= 1 - \dfrac{1}{2}x - \dfrac{1}{8}x^2 - \dfrac{1}{16}x^3 + \ldots$

  (Expansion is *infinite* and *approximate*.)

2 The expansion $(1 + x)^n = 1 + nx + n(n-1)\dfrac{x^2}{2!} + n(n-1)(n-2)\dfrac{x^3}{3!} + \ldots$, where n is negative or a fraction, is only valid if $|x| < 1$.

3 You can adapt the binomial expansion to include expressions of the form $(a + bx)^n$ by simply taking out a common factor of a:

e.g. $$\frac{1}{(3 + 4x)} = (3 + 4x)^{-1} = \left[3\left(1 + \frac{4x}{3}\right)\right]^{-1}$$

$$= 3^{-1}\left(1 + \frac{4x}{3}\right)^{-1}$$

4 You can use knowledge of partial fractions to expand more difficult expressions, e.g.

$$\frac{7 + x}{(3 - x)(2 + x)} = \frac{2}{(3 - x)} + \frac{1}{(2 + x)}$$

$$= 2(3 - x)^{-1} + (2 + x)^{-1}$$

$$= \frac{2}{3}\left(1 - \frac{x}{3}\right)^{-1} + \frac{1}{2}\left(1 + \frac{x}{2}\right)^{-1}$$