C4 DIFFERENTIATION — Answers - Worksheet E

1 $6x + 1 \times y + x \times \frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

$6x + y = \frac{dy}{dx}(2y - x)$

$\frac{dy}{dx} = \frac{6x + y}{2y - x}$

2 **a** $\frac{dx}{d\theta} = -a \sin \theta$, $\frac{dy}{d\theta} = a(\cos \theta - 1)$

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a(\cos\theta - 1)}{-a\sin\theta} = \frac{1-\cos\theta}{\sin\theta}$

$= \frac{1-(1-2\sin^2\frac{\theta}{2})}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$

b $x = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

$\therefore y = a(1 - \frac{\pi}{2})$, grad = 1

$\therefore y = x + a(1 - \frac{\pi}{2})$

3 **a** $\frac{dx}{d\theta} = -\sin\theta$, $\frac{dy}{d\theta} = \cos 2\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{\cos 2\theta}{-\sin\theta}$

$= -\text{cosec}\,\theta \cos 2\theta$

b $x = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

c $\theta = \frac{\pi}{2}$, grad $= -1 \times (-1) = 1$

$\theta = \frac{3\pi}{2}$, grad $= 1 \times (-1) = -1$

product of gradients $= 1 \times (-1) = -1$

$\therefore$ tangents are perpendicular

d $y = \frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$

$y^2 = \sin^2\theta\cos^2\theta = \cos^2\theta(1 - \cos^2\theta)$

$\therefore y^2 = x^2(1 - x^2)$

4 **a** $2x - 4 \times y - 4x \times \frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$2x - 4y = \frac{dy}{dx}(4x - 2y)$

$\frac{dy}{dx} = \frac{2x - 4y}{4x - 2y} = \frac{x - 2y}{2x - y}$

b grad = 3

$\therefore y - 10 = 3(x - 2)$ [$y = 3x + 4$]

c $\frac{x - 2y}{2x - y} = 3$

$x - 2y = 3(2x - y)$

$y = 5x$, sub. into eqn of curve

$x^2 - 4x(5x) + (5x)^2 = 24$

$x^2 = 4$

$x = 2$ (at P) or -2 $\therefore (-2, -10)$

5 **a** $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 2t - 1$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2t - 1}{2t}$

$\therefore \frac{2t - 1}{2t} = 0$

$t = \frac{1}{2}$

$\therefore (\frac{9}{4}, -\frac{1}{4})$

b $x = 3 \Rightarrow t^2 + 2 = 3 \Rightarrow t = \pm 1$

$y = 2 \Rightarrow t^2 - t = 2$

$t^2 - t - 2 = 0$

$(t - 2)(t + 1) = 0$

$t = -1$ or 2

$\therefore$ at (3, 2), $t = -1$

$\therefore$ grad $= \frac{3}{2}$

$\therefore y - 2 = \frac{3}{2}(x - 3)$

$2y - 4 = 3x - 9$

$3x - 2y = 5$

6 $3x^2 - 3 + 1 \times y + x \times \frac{dy}{dx} - 4y\frac{dy}{dx} = 0$

$3x^2 - 3 + y = \frac{dy}{dx}(4y - x)$

$\frac{dy}{dx} = \frac{3x^2 - 3 + y}{4y - x}$

grad $= \frac{1}{3}$

$\therefore$ grad of normal $= -3$

$\therefore y - 1 = -3(x - 1)$

$y = 4 - 3x$

7 **a** $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}, \ \frac{dV}{dt} = 80$

$\frac{dV}{dh} = 40\pi \times 0.1e^{0.1h} = 4\pi e^{0.1h}$

$h = 4, \ \frac{dV}{dh} = 4\pi e^{0.4}$

$\therefore \ 80 = 4\pi e^{0.4} \times \frac{dh}{dt}, \ \frac{dh}{dt} = 4.27$

$\therefore$ depth increasing at 4.27 cm s^{-1} (3sf)

b after 5 seconds, $V = 5 \times 80 = 400$

$\therefore \ 400 = 40\pi(e^{0.1h} - 1)$

$h = 10 \ln(\frac{10}{\pi} + 1) = 14.31$

$\therefore \ \frac{dV}{dh} = 4\pi e^{1.431}$

$\therefore \ 80 = 4\pi e^{1.431} \times \frac{dh}{dt}, \ \frac{dh}{dt} = 1.52$

$\therefore$ depth increasing at 1.52 cm s^{-1} (3sf)

8 **a** $\frac{dx}{dt} = \frac{1\times(1+t) - t\times 1}{(1+t)^2} = \frac{1}{(1+t)^2}$

$\frac{dy}{dt} = \frac{1\times(1-t) - t\times(-1)}{(1-t)^2} = \frac{1}{(1-t)^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{1}{(1-t)^2} \div \frac{1}{(1+t)^2}$

$= \frac{(1+t)^2}{(1-t)^2} = \left(\frac{1+t}{1-t}\right)^2$

b $t = \frac{1}{2} \ \therefore \ x = \frac{1}{3}, \ y = 1$

grad = 9 $\therefore$ grad of normal = $-\frac{1}{9}$

$\therefore \ y - 1 = -\frac{1}{9}(x - \frac{1}{3})$

$27y - 27 = -3x + 1$

$3x + 27y = 28$

c $\frac{3t}{1+t} + \frac{27t}{1-t} = 28$

$3t(1 - t) + 27t(1 + t) = 28(1 - t^2)$

$26t^2 + 15t - 14 = 0$

$(13t + 14)(2t - 1) = 0$

$t = \frac{1}{2}$ (at P) or $-\frac{14}{13}$

$\therefore \ t = -\frac{14}{13}$ at Q

9 $2 + 2x \times y + x^2 \times \frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

$2 + 2xy = \frac{dy}{dx}(2y - x^2)$

$\frac{dy}{dx} = \frac{2+2xy}{2y-x^2}$

$\therefore \ \frac{2+2xy}{2y-x^2} = 0, \quad 2 + 2xy = 0$

$xy = -1, \quad y = -\frac{1}{x}$

sub. $2x + x^2(-\frac{1}{x}) - (-\frac{1}{x})^2 = 0$

$2x - x - \frac{1}{x^2} = 0$

$x = \frac{1}{x^2}, \quad x^3 = 1$

$x = 1 \ \therefore \ (1, -1)$

10 **a** $\frac{dx}{d\theta} = a \sec\theta \tan\theta, \ \frac{dy}{d\theta} = 2a \sec^2\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{2a\sec^2\theta}{a\sec\theta\tan\theta} = 2 \operatorname{cosec}\theta$

b $\theta = \frac{\pi}{4}, \ x = \sqrt{2}\,a, \ y = 2a$

grad = $2\sqrt{2}$

$\therefore$ grad of normal = $-\frac{1}{2\sqrt{2}}$

$\therefore \ y - 2a = -\frac{1}{2\sqrt{2}}(x - \sqrt{2}\,a)$

$2\sqrt{2}\,y - 4\sqrt{2}\,a = -x + \sqrt{2}\,a$

$x + 2\sqrt{2}\,y = 5\sqrt{2}\,a$

c $y^2 = 4a^2\tan^2\theta = 4a^2(\sec^2\theta - 1)$

$\sec\theta = \frac{x}{a}$

$\therefore \ y^2 = 4a^2[(\frac{x}{a})^2 - 1]$

$y^2 = 4(x^2 - a^2)$