4

After completing this chapter you should be able to:

- find the gradient of a curve whose equation is expressed in a parametric form
- differentiate implicit relationships
- differentiate power functions such as a^x
- use the chain rule to connect the rates of change of two variables
- set up simple differential equations from information given in context.

Differentiation

The melting of a snowman can be modelled by a differential equation.

4.1 You can find the gradient of a curve given in parametric coordinates.

When a curve is described by parametric equations

- You differentiate x and y with respect to the parameter t.
- Then you use the chain rule rearranged into the form $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$.

Example 1

Find the gradient at the point P where $t = 2$, on the curve given parametrically by $x = t^3 + t$, $y = t^2 + 1$, $t \in \mathbb{R}$.

$\frac{dx}{dt} = 3t^2 + 1$, $\frac{dy}{dt} = 2t$

First differentiate x and y with respect to the parameter t.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2 + 1}$$

Use the chain rule $\frac{dy}{dx} \times \frac{dx}{dt} = \frac{dy}{dt}$ and rearrange to give $\frac{dy}{dx}$.

When $t = 2$, $\frac{dy}{dx} = \frac{4}{13}$

Substitute $t = 2$ into $\frac{2t}{3t^2 + 1}$.

So the gradient at P is $\frac{4}{13}$.

Example 2

Find the equation of the normal at the point P where $\theta = \frac{\pi}{6}$, on the curve with parametric equations $x = 3 \sin \theta$, $y = 5 \cos \theta$.

$\frac{dx}{d\theta} = 3 \cos \theta$, $\frac{dy}{d\theta} = -5 \sin \theta$

First differentiate x and y with respect to the parameter θ.

$$\therefore \frac{dy}{dx} = \frac{-5 \sin \theta}{3 \cos \theta}$$

At point P, where $\theta = \frac{\pi}{6}$,

$$\frac{dy}{dx} = \frac{-5 \times \frac{1}{2}}{3 \times \frac{\sqrt{3}}{2}} = \frac{-5}{3\sqrt{3}}$$

Use the chain rule, $\frac{dy}{d\theta} \div \frac{dx}{d\theta}$, and substitute $\theta = \frac{\pi}{6}$.

The gradient of the normal at P

is $\frac{3\sqrt{3}}{5}$, and at P, $x = \frac{3}{2}$, $y = \frac{5\sqrt{3}}{2}$.

The normal is perpendicular to the curve, so its gradient is $-\frac{1}{m}$ where m is the gradient of the curve at that point.

The equation of the normal is

$$y - \frac{5\sqrt{3}}{2} = \frac{3\sqrt{3}}{5}\left(x - \frac{3}{2}\right)$$

Use equation for a line in the form $(y - y_1) = m(x - x_1)$.

$$\therefore \quad 5y = 3\sqrt{3}x + 8\sqrt{3}$$

Exercise 4A

1 Find $\frac{dy}{dx}$ for each of the following, leaving your answer in terms of the parameter t:

a $x = 2t,\ y = t^2 - 3t + 2$ **b** $x = 3t^2,\ y = 2t^3$ **c** $x = t + 3t^2,\ y = 4t$

d $x = t^2 - 2,\ y = 3t^5$ **e** $x = \frac{2}{t},\ y = 3t^2 - 2$ **f** $x = \frac{1}{2t - 1},\ y = \frac{t^2}{2t - 1}$

g $x = \frac{2t}{1 + t^2},\ y = \frac{1 - t^2}{1 + t^2}$ **h** $x = t^2e^t,\ y = 2t$ **i** $x = 4\sin 3t,\ y = 3\cos 3t$

j $x = 2 + \sin t,\ y = 3 - 4\cos t$ **k** $x = \sec t,\ y = \tan t$ **l** $x = 2t - \sin 2t,\ y = 1 - \cos 2t$

2 **a** Find the equation of the tangent to the curve with parametric equations $x = 3t - 2\sin t$, $y = t^2 + t\cos t$, at the point P, where $t = \frac{\pi}{2}$.

b Find the equation of the tangent to the curve with parametric equations $x = 9 - t^2$, $y = t^2 + 6t$, at the point P, where $t = 2$.

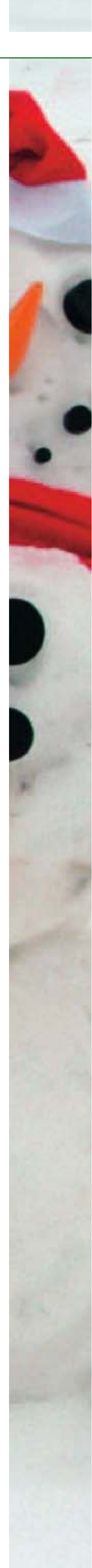
3 **a** Find the equation of the normal to the curve with parametric equations $x = e^t$, $y = e^t + e^{-t}$, at the point P, where $t = 0$.

b Find the equation of the normal to the curve with parametric equations $x = 1 - \cos 2t$, $y = \sin 2t$, at the point P, where $t = \frac{\pi}{6}$.

4 Find the points of zero gradient on the curve with parametric equations

$$x = \frac{t}{1 - t},\ y = \frac{t^2}{1 - t},\ t \neq 1.$$

You do not need to establish whether they are maximum or minimum points.



4.2 You can differentiate relations which are implicit, such as $x^2 + y^2 = 8x$, and $\cos(x + y) = \sin y$.

Differentiate each term in turn using the chain rule and the product rule, as appropriate:

- $\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$ — By the chain rule.
- $\frac{d}{dx}(xy) = x\frac{d}{dx}(y) + y\frac{d}{dx}(x)$

$$= x\frac{dy}{dx} + y \times 1$$

By the product rule.

$$= x\frac{dy}{dx} + y$$

Example 3

Find $\frac{dy}{dx}$ in terms of x and y where $x^3 + x + y^3 + 3y = 6$.

$$3x^2 + 1 + 3y^2\frac{dy}{dx} + 3\frac{dy}{dx} = 0$$

Differentiate the expression term by term with respect to x.

Use the chain rule to differentiate y^3.

$$\frac{dy}{dx}(3y^2 + 3) = -3x^2 - 1$$

$$\frac{dy}{dx} = -\frac{(3x^2 + 1)}{3(1 + y^2)}$$

Then make $\frac{dy}{dx}$ the subject of the formula.

Divide by the coefficient of $\frac{dy}{dx}$ and factorise.

Example 4

Find the value of $\frac{dy}{dx}$ at the point (1, 1)

where $4xy^2 + \frac{6x^2}{y} = 10$.

$$\left(4x \times 2y\frac{dy}{dx} + 4y^2\right) + \frac{12x}{y} - \frac{6x^2}{y^2}\frac{dy}{dx} = 0$$

Differentiate each term with respect to x.

Use the product rule on each term, expressing $\frac{6x^2}{y}$ as $6x^2y^{-1}$.

Substitute $x = 1$, $y = 1$ to give

Find the value of $\frac{dy}{dx}$ at (1, 1) by substituting $x = 1$, $y = 1$.

$$\left(8\frac{dy}{dx} + 4\right) + 12 - 6\frac{dy}{dx} = 0$$

Substitute before rearranging, as this simplifies the working.

i.e. $$16 + 2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -8$$

Finally make $\frac{dy}{dx}$ the subject of the formula.

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Example 5

Find the value of $\frac{dy}{dx}$ at the point (1, 1) where $e^{2x}\ln y = x + y - 2$.

$$e^{2x} \times \frac{1}{y}\frac{dy}{dx} + \ln y \times 2e^{2x} = 1 + \frac{dy}{dx}$$

Differentiate each term with respect to x.

Use the product rule applied to the term on the left hand side of the equation, noting that $\ln y$ differentiates to give $\frac{1}{y}\frac{dy}{dx}$.

Substitute $x = 1$, $y = 1$ to give

$$e^2 \times \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\therefore \quad (e^2 - 1)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^2 - 1}$$

Rearrange to make $\frac{dy}{dx}$ the subject of the formula.

■ **In an implicit equation:**

- **Note that when f(y) is differentiated with respect to x it becomes $f'(y)\frac{dy}{dx}$.**
- **A product term such as f(x).g(y) is differentiated by the product rule and becomes $f(x).g'(y)\frac{dy}{dx} + g(y).f'(x)$.**

Exercise 4B

1 Find an expression in terms of x and y for $\frac{dy}{dx}$, given that:

a $x^2 + y^3 = 2$ **b** $x^2 + 5y^2 = 14$ **c** $x^2 + 6x - 8y + 5y^2 = 13$

d $y^3 + 3x^2y - 4x = 0$ **e** $3y^2 - 2y + 2xy = x^3$ **f** $x = \frac{2y}{x^2 - y}$

g $(x - y)^4 = x + y + 5$ **h** $e^x y = xe^y$ **i** $\sqrt{(xy)} + x + y^2 = 0$

2 Find the equation of the tangent to the curve with implicit equation $x^2 + 3xy^2 - y^3 = 9$ at the point (2, 1).

3 Find the equation of the normal to the curve with implicit equation $(x + y)^3 = x^2 + y$ at the point (1, 0).

4 Find the coordinates of the points of zero gradient on the curve with implicit equation $x^2 + 4y^2 - 6x - 16y + 21 = 0$.

4.3 You can differentiate the general power function a^x, where a is constant.

This function describes growth and decay and its derivative gives a measure of the rate of change of this growth or decay.

Example 6

Differentiate $y = a^x$, where a is a constant.

Since $y = a^x$

$\ln y = \ln a^x$

$\therefore \quad \ln y = x \ln a$

$\therefore \quad \dfrac{1}{y}\dfrac{dy}{dx} = \ln a$

$\therefore \quad \dfrac{dy}{dx} = y \ln a$

$= a^x \ln a$

Take logs of both sides, then use properties of logs to express $\ln a^x$ as $x \ln a$.

Use implicit differentiation to differentiate $\ln y$.

Replace y by a^x.

■ **If $y = a^x$, then $\dfrac{dy}{dx} = a^x \ln a$**

You should learn this result.

(In particular, if $y = e^x$, then $\dfrac{dy}{dx} = e^x \ln e = e^x$, as you know from the C3 book.)

Exercise 4C

1 Find $\dfrac{dy}{dx}$ for each of the following:

a $y = 3^x$ **b** $y = \left(\frac{1}{2}\right)^x$ **c** $y = xa^x$ **d** $y = \dfrac{2^x}{x}$

2 Find the equation of the tangent to the curve $y = 2^x + 2^{-x}$ at the point $(2, 4\frac{1}{4})$.

3 A particular radioactive isotope has an activity R millicuries at time t days given by the equation $R = 200(0.9)^t$. Find the value of $\dfrac{dR}{dt}$, when $t = 8$.

4 The population of Cambridge was 37 000 in 1900 and was about 109 000 in 2000. Find an equation of the form $P = P_0k^t$ to model this data, where t is measured as years since 1900. Evaluate $\dfrac{dP}{dt}$ in the year 2000. What does this value represent?

4.4 You can relate one rate of change to another.

You can use the chain rule once, or several times, to connect the rates of change in a question involving more than two variables.

Example 7

Given that the area of a circle A cm^2 is related to its radius r cm by the formula $A = \pi r^2$, and that the rate of change of its radius in cm s^{-1} is given by $\frac{dr}{dt} = 5$, find $\frac{dA}{dt}$ when $r = 3$.

$$A = \pi r^2$$

$$\therefore \quad \frac{dA}{dr} = 2\pi r$$

As A is a function of r, find $\frac{dA}{dr}$.

$$\text{Using } \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

You should use the chain rule, giving the derivative which you need to find in terms of known derivatives.

$$\frac{dA}{dt} = 2\pi r \times 5$$

$$= 30\pi, \text{ when } r = 3.$$

Example 8

The volume of a hemisphere V cm^3 is related to its radius r cm by the formula $V = \frac{2}{3}\pi r^3$ and the total surface area S cm^2 is given by the formula $S = \pi r^2 + 2\pi r^2 = 3\pi r^2$. Given that the rate of increase of volume, in cm^3 s^{-1}, $\frac{dV}{dt} = 6$, find the rate of increase of surface area area $\frac{dS}{dt}$.

$$V = \tfrac{2}{3}\pi r^3 \text{ and } S = 3\pi r^2$$

This is area of circular base plus area of curved surface.

$$\frac{dV}{dr} = 2\pi r^2 \text{ and } \frac{dS}{dr} = 6\pi r$$

As V and S are functions of r, find $\frac{dV}{dr}$ and $\frac{dS}{dr}$.

$$\text{Now } \frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$$

$$= 6\pi r \times \frac{1}{2\pi r^2} \times 6$$

Use an extended chain rule together with the property that $\frac{dr}{dV} = 1 \div \frac{dV}{dr}$.

$$= \frac{18}{r}$$

Exercise 4D

1 Given that $V = \frac{1}{3}\pi r^3$ and that $\frac{dV}{dt} = 8$, find $\frac{dr}{dt}$ when $r = 3$.

2 Given that $A = \frac{1}{4}\pi r^2$ and that $\frac{dr}{dt} = 6$, find $\frac{dA}{dt}$ when $r = 2$.

3 Given that $y = xe^x$ and that $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when $x = 2$.

4 Given that $r = 1 + 3\cos\theta$ and that $\frac{d\theta}{dt} = 3$, find $\frac{dr}{dt}$ when $\theta = \frac{\pi}{6}$.

4.5 You can set up a differential equation from information given in context.

Differential equations arise from many problems in mechanics, physics, chemistry, biology and economics. As their name suggests, these equations involve differential coefficients and so equations of the form

$$\frac{dy}{dx} = 3y, \quad \frac{ds}{dt} = 2 + 6t, \quad \frac{d^2y}{dt^2} = -25y \quad \frac{dP}{dt} = 10 - 4P$$

are differential equations (x, y, t, s and P are variables).

In the C4 book you will consider only first order differential equations, which involve first derivatives only.

■ **You can set up simple differential equations from information given in context. This may involve using connected rates of change, or ideas of proportion.**

Example 9

In the decay of radioactive particles the rate at which particles decay is proportional to the number of particles remaining. Write down an equation for the rate of change of the number of particles.

Let N be the number of particles and let t be time. The rate of change of the number of particles $\frac{dN}{dt}$ decays at a rate proportional to N.

i.e. $\frac{dN}{dt} = -kN$, where k is a positive constant.

The minus sign arises because the number of particles is decreasing.

Note that this is a proportional problem.

$\frac{dN}{dt} \propto N \rightarrow \frac{dN}{dt} = kN$

k is the constant of proportion.

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Example 10

A population is growing at a rate which is proportional to the size of the population at a given time. Write down an equation for the rate of growth of the population.

Let P be the population and t be the time.

The rate of change of the population $\frac{dP}{dt}$

grows at a rate proportional to P.

i.e. $\frac{dP}{dt} = kP$, where k is a positive constant.

The population is increasing, so there is no minus sign.

k is the constant of proportion.

Example 11

Newton's Law of Cooling states that the rate of loss of temperature of a body is proportional to the excess temperature of the body over its surroundings. Write an equation that expresses this law.

Let the temperature of the body be θ degrees and the time be t seconds.

The rate of change of the temperature

$\frac{d\theta}{dt}$ decreases at a rate proportional to

$(\theta - \theta_0)$, where θ_0 is the temperature of the surroundings.

i.e. $\frac{d\theta}{dt} = -k(\theta - \theta_0)$, where k is a positive constant.

$(\theta - \theta_0)$ is the difference between the temperature of the body and that of its surroundings.

The minus sign arises because the temperature is decreasing. The question mentions loss of temperature.

Example 12

The head of a snowman of radius R cm loses volume by evaporation at a rate proportional to its surface area. Assuming that the head is spherical, that the volume of a sphere is $\frac{4}{3}\pi R^3$ cm^3 and that the surface is $4\pi R^2$ cm^2, write down a differential equation for the rate of change of radius of the snowman's head.

The first sentence tells you

that $\frac{dV}{dt} = -kA$, where V cm^3 is the

volume, t seconds is time, k is a positive constant and A cm^2 is the surface area referred to in the question.

Since $V = \frac{4}{3}\pi R^3$

$$\frac{dV}{dR} = 4\pi R^2$$

$$\therefore \quad \frac{dV}{dt} = \frac{dV}{dR} \times \frac{dR}{dt} = 4\pi R^2 \times \frac{dR}{dt}$$

But as $\frac{dV}{dt} = -kA$

$$4\pi R^2 \times \frac{dR}{dt} = -k \times 4\pi R^2$$

$$\therefore \quad \frac{dR}{dt} = -k$$

The question asks for a differential equation in terms of R, and so you need to use the expressions for V and A in terms of R.

The chain rule is used here because this is a related rate of change.

Divide both sides by the common factor $4\pi R^2$.

This gives the rate of change of radius as required.

The last example included four variables, V, A, R and t. You used the chain rule to connect the rate of change. This is a connected rate of change problem (see Section 4.4).

Exercise 4E

1 In a study of the water loss of picked leaves the mass M grams of a single leaf was measured at times t days after the leaf was picked. It was found that the rate of loss of mass was proportional to the mass M of the leaf.
Write down a differential equation for the rate of change of mass of the leaf.

2 A curve C has equation $y = f(x)$, $y > 0$. At any point P on the curve, the gradient of C is proportional to the product of the x and the y coordinates of P. The point A with coordinates (4, 2) is on C and the gradient of C at A is $\frac{1}{2}$.

Show that $\dfrac{dy}{dx} = \dfrac{xy}{16}$.

3 Liquid is pouring into a container at a constant rate of $30\,\text{cm}^3\,\text{s}^{-1}$. At time t seconds liquid is leaking from the container at a rate of $\frac{2}{15}V\,\text{cm}^3\,\text{s}^{-1}$, where $V\,\text{cm}^3$ is the volume of liquid in the container at that time.

Show that $-15\dfrac{dV}{dt} = 2V - 450$

4 An electrically charged body loses its charge Q coulombs at a rate, measured in coulombs per second, proportional to the charge Q.
Write down a differential equation in terms of Q and t where t is the time in seconds since the body started to lose its charge.

5 The ice on a pond has a thickness x mm at a time t hours after the start of freezing. The rate of increase of x is inversely proportional to the square of x.
Write down a differential equation in terms of x and t.

6 In another pond the amount of pondweed (P) grows at a rate proportional to the amount of pondweed already present in the pond. Pondweed is also removed by fish eating it at a constant rate of Q per unit of time.
Write down a differential equation relating P and t, where t is the time which has elapsed since the start of the observation.

7 A circular patch of oil on the surface of some water has radius r and the radius increases over time at a rate inversely proportional to the radius.
Write down a differential equation relating r and t, where t is the time which has elapsed since the start of the observation.

8 A metal bar is heated to a certain temperature, then allowed to cool down and it is noted that, at time t, the rate of loss of temperature is proportional to the difference in temperature between the metal bar, θ, and the temperature of its surroundings θ_0.
Write down a differential equation relating θ and t.

The next three questions involve connected rates of change.

9 Fluid flows out of a cylindrical tank with constant cross section. At time t minutes, $t > 0$, the volume of fluid remaining in the tank is $V\,\mathrm{m}^3$. The rate at which the fluid flows in $\mathrm{m}^3\,\mathrm{min}^{-1}$ is proportional to the square root of V.
Show that the depth h metres of fluid in the tank satisfies the differential equation
$\frac{\mathrm{d}h}{\mathrm{d}t} = -k\sqrt{h}$, where k is a positive constant.

10 At time t seconds the surface area of a cube is $A\,\mathrm{cm}^2$ and the volume is $V\,\mathrm{cm}^3$.
The surface area of the cube is expanding at a constant rate of $2\,\mathrm{cm}^2\,\mathrm{s}^{-1}$.
Show that $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{2}V^{\frac{1}{3}}$.

11 An inverted conical funnel is full of salt. The salt is allowed to leave by a small hole in the vertex. It leaves at a constant rate of $6\,\mathrm{cm}^3\,\mathrm{s}^{-1}$.
Given that the angle of the cone between the slanting edge and the vertical is 30°, show that the volume of the salt is $\frac{1}{9}\pi h^3$, where h is the height of salt at time t seconds.
Show that the rate of change of the height of the salt in the funnel is inversely proportional to h^2. Write down the differential equation relating h and t.

Mixed exercise 4F

1 The curve C is given by the equations

$$x = 4t - 3,\ y = \frac{8}{t^2},\ t > 0$$

where t is a parameter.

At A, $t = 2$. The line l is the normal to C at A.

a Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. **b** Hence find an equation of l. *E*

2 The curve C is given by the equations $x = 2t$, $y = t^2$, where t is a parameter. Find an equation of the normal to C at the point P on C where $t = 3$. *E*

3 The curve C has parametric equations

$$x = t^3,\ y = t^2,\ t > 0$$

Find an equation of the tangent to C at A (1, 1). *E*

4 A curve C is given by the equations

$$x = 2\cos t + \sin 2t,\ y = \cos t - 2\sin 2t,\ 0 < t < \pi$$

where t is a parameter.

a Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in terms of t.

b Find the value of $\frac{dy}{dx}$ at the point P on C where $t = \frac{\pi}{4}$.

c Find an equation of the normal to the curve at P. *E*

5 A curve is given by $x = 2t + 3$, $y = t^3 - 4t$, where t is a parameter. The point A has parameter $t = -1$ and the line l is the tangent to C at A. The line l also cuts the curve at B.

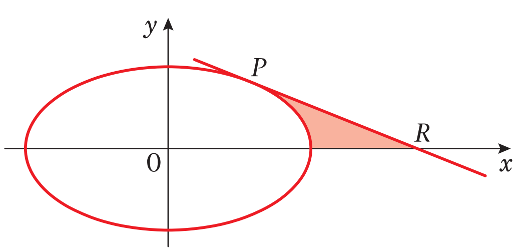
a Show that an equation for l is $2y + x = 7$.

b Find the value of t at B. *E*

6 A Pancho car has value £V at time t years. A model for V assumes that the rate of decrease of V at time t is proportional to V. Form an appropriate differential equation for V. *E*

7 The curve shown has parametric equations

$$x = 5\cos\theta,\ y = 4\sin\theta,\ 0 \leqslant \theta < 2\pi$$



a Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$.

b Find an equation of the tangent to the curve at the point P.

c Find the coordinates of the point R where this tangent meets the x-axis. *E*

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8 The curve C has parametric equations

$$x = 4\cos 2t,\ y = 3\sin t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$$

A is the point $(2, 1\frac{1}{2})$, and lies on C.

a Find the value of t at the point A.

b Find $\frac{dy}{dx}$ in terms of t.

c Show that an equation of the normal to C at A is $6y - 16x + 23 = 0$.

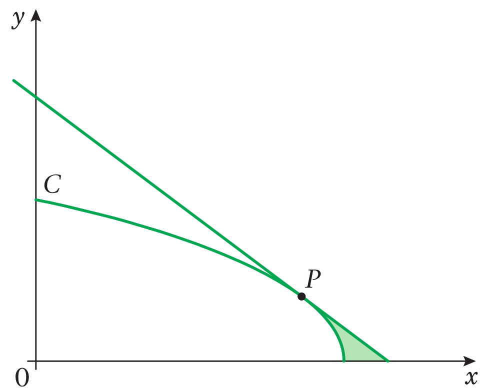
The normal at A cuts C again at the point B.

d Find the y-coordinate of the point B. *E*

9 The diagram shows the curve C with parametric equations

$$x = a\sin^2 t,\ y = a\cos t,\ 0 \leqslant t \leqslant \tfrac{1}{2}\pi$$

where a is a positive constant. The point P lies on C and has coordinates $(\frac{3}{4}a, \frac{1}{2}a)$.



a Find $\frac{dy}{dx}$, giving your answer in terms of t.

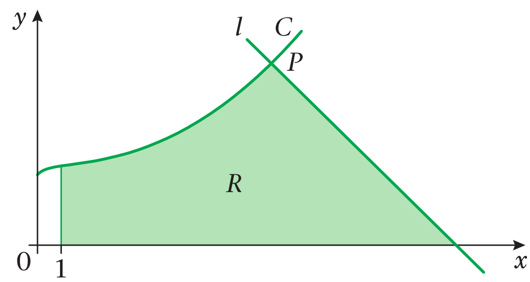
b Find an equation of the tangent at P to C. *E*

10 This graph shows part of the curve C with parametric equations

$$x = (t+1)^2,\ y = \tfrac{1}{2}t^3 + 3,\ t \geqslant -1$$

P is the point on the curve where $t = 2$. The line l is the normal to C at P.

Find the equation of l.

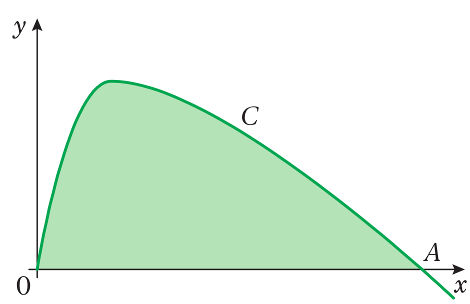


E

11 The diagram shows part of the curve C with parametric equations

$$x = t^2,\ y = \sin 2t,\ t \geqslant 0$$

The point A is an intersection of C with the x-axis.



a Find, in terms of π, the x-coordinate of A.

b Find $\frac{dy}{dx}$ in terms of t, $t > 0$.

c Show that an equation of the tangent to C at A is $4x + 2\pi y = \pi^2$. *E*

12 Find the gradient of the curve with equation

$$5x^2 + 5y^2 - 6xy = 13$$

at the point (1, 2). *E*

13 Given that $e^{2x} + e^{2y} = xy$, find $\dfrac{dy}{dx}$ in terms of x and y. *E*

14 Find the coordinates of the turning points on the curve $y^3 + 3xy^2 - x^3 = 3$. *E*

15 Given that $y(x + y) = 3$, evaluate $\dfrac{dy}{dx}$ when $y = 1$. *E*

16 **a** If $(1 + x)(2 + y) = x^2 + y^2$, find $\dfrac{dy}{dx}$ in terms of x and y.

b Find the gradient of the curve $(1 + x)(2 + y) = x^2 + y^2$ at each of the two points where the curve meets the y-axis.

c Show also that there are two points at which the tangents to this curve are parallel to the y-axis. *E*

17 A curve has equation $7x^2 + 48xy - 7y^2 + 75 = 0$. A and B are two distinct points on the curve and at each of these points the gradient of the curve is equal to $\frac{2}{11}$. Use implicit differentiation to show that $x + 2y = 0$ at the points A and B. *E*

18 Given that $y = x^x$, $x > 0$, $y > 0$, by taking logarithms show that

$$\frac{dy}{dx} = x^x(1 + \ln x)$$

E

19 **a** Given that $x = 2^t$, by using logarithms prove that

$$\frac{dx}{dt} = 2^t \ln 2$$

A curve C has parametric equations $x = 2^t$, $y = 3t^2$. The tangent to C at the point with coordinates (2, 3) cuts the x-axis at the point P.

b Find $\dfrac{dy}{dx}$ in terms of t.

c Calculate the x-coordinate of P, giving your answer to 3 decimal places. *E*

20 **a** Given that $a^x \equiv e^{kx}$, where a and k are constants, $a > 0$ and $x \in \mathbb{R}$, prove that $k = \ln a$.

b Hence, using the derivative of e^{kx}, prove that when $y = 2^x$

$$\frac{dy}{dx} = 2^x \ln 2.$$

c Hence deduce that the gradient of the curve with equation $y = 2^x$ at the point (2, 4) is ln 16. *E*

21 A population P is growing at the rate of 9% each year and at time t years may be approximated by the formula

$$P = P_0(1.09)^t, \ t \geqslant 0$$

where P is regarded as a continuous function of t and P_0 is the starting population at time $t = 0$.

a Find an expression for t in terms of P and P_0.

b Find the time T years when the population has doubled from its value at $t = 0$, giving your answer to 3 significant figures.

c Find, as a multiple of P_0, the rate of change of population $\frac{\mathrm{d}P}{\mathrm{d}t}$ at time $t = T$. *E*

Summary of key points

1 When a relation is described by parametric equations:

- You differentiate x and y with respect to the parameter t.
- Then you use the chain rule rearranged into the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$.

2 When a relation is described by an implicit equation:

- Differentiate each term in turn, using the chain rule and product and quotient rules as appropriate.
- $\frac{\mathrm{d}}{\mathrm{d}x}(y^n) = ny^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}$ By the chain rule.
- $\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x\frac{\mathrm{d}}{\mathrm{d}x}(y) + y\frac{\mathrm{d}}{\mathrm{d}x}(x) = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$ By the product rule.

3 In an implicit equation:

- Note that when $\mathrm{f}(y)$ is differentiated with respect to x it becomes $\mathrm{f}'(y)\frac{\mathrm{d}y}{\mathrm{d}x}$.
- A product term such as $\mathrm{f}(x).\mathrm{g}(y)$ is differentiated by the product rule and becomes $\mathrm{f}(x).\mathrm{g}'(y)\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{g}(y).\mathrm{f}'(x)$.

4 You can differentiate the function $\mathrm{f}(x) = a^x$:

- If $y = a^x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = a^x \ln a$

5 You can use the chain rule once, or several times, to connect the rates of change in a question involving more than two variables.

6 You can set up simple differential equations from information given in context. This may involve using connected rates of change, or ideas of proportion.