C4 INTEGRATION *Answers - Worksheet J*

1 **a** $\frac{dy}{dx} = 2x$, grad = 2

$\therefore$ grad of normal = $-\frac{1}{2}$

$\therefore\ y - 2 = -\frac{1}{2}(x - 1)$

$[\ y = \frac{5}{2} - \frac{1}{2}x\ ]$

b $y = 0\ \therefore x = 5$

$\therefore$ (5, 0)

c volume $0 \le x \le 1$

$= \pi\int_0^1 (x^2 + 1)^2\ dx$

$= \pi\int_0^1 (x^4 + 2x^2 + 1)\ dx$

$= \pi[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x]_0^1$

$= \pi[(\frac{1}{5} + \frac{2}{3} + 1) - (0)] = \frac{28}{15}\pi$

volume $1 < x \le 5$ = volume of cone

$= \frac{1}{3} \times \pi \times 2^2 \times 4 = \frac{16}{3}\pi$

total volume

$= \frac{28}{15}\pi + \frac{16}{3}\pi$

$= \frac{36}{5}\pi$

2 **a** $= \int_1^e (4x + \frac{9}{x})\ dx$

$= [2x^2 + 9\ln|x|]_1^e$

$= (2e^2 + 9) - (2 + 0)$

$= 2e^2 + 7$

b $= \pi\int_1^e (4x + \frac{9}{x})^2\ dx$

$= \pi\int_1^e (16x^2 + 72 + 81x^{-2})\ dx$

$= \pi[\frac{16}{3}x^3 + 72x - 81x^{-1}]_1^e$

$= \pi[(\frac{16}{3}e^3 + 72e - 81e^{-1}) - (\frac{16}{3} + 72 - 81)]$

= 869 (3sf)

3 **a** $= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \text{cosec}^2 x\ dx$

$= \pi[-\cot x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= -\pi(\frac{1}{\sqrt{3}} - \sqrt{3})$

$= \pi(\sqrt{3} - \frac{1}{3}\sqrt{3})$

$= \frac{2}{3}\pi\sqrt{3}$

b $= \pi\int_1^4 (\sqrt{\frac{x+3}{x+2}})^2\ dx$

$= \pi\int_1^4 \frac{x+3}{x+2}\ dx$

$= \pi\int_1^4 \frac{(x+2)+1}{x+2}\ dx$

$= \pi\int_1^4 (1 + \frac{1}{x+2})\ dx$

$= \pi[x + \ln|x+2|]_1^4$

$= \pi[(4 + \ln 6) - (1 + \ln 3)]$

$= \pi(3 + \ln 2)$

c $= \pi\int_0^{\frac{\pi}{4}} (1 + \cos 2x)^2\ dx$

$= \pi\int_0^{\frac{\pi}{4}} (1 + 2\cos 2x + \cos^2 2x)\ dx$

$= \pi\int_0^{\frac{\pi}{4}} (\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x)\ dx$

$= \pi[\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x]_0^{\frac{\pi}{4}}$

$= \pi[(\frac{3}{8}\pi + 1 + 0) - (0)]$

$= \frac{1}{8}\pi(3\pi + 8)$

d $= \pi\int_1^2 (x^{\frac{1}{2}} e^{2-x})^2\ dx$

$= \pi\int_1^2 xe^{4-2x}\ dx$

$u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = e^{4-2x},\ v = -\frac{1}{2}e^{4-2x}$

$= \pi\{[-\frac{1}{2}xe^{4-2x}]_1^2 + \int_1^2 \frac{1}{2}e^{4-2x}\ dx\}$

$= \pi[-\frac{1}{2}xe^{4-2x} - \frac{1}{4}e^{4-2x}]_1^2$

$= \pi[(-1 - \frac{1}{4}) - (-\frac{1}{2}e^2 - \frac{1}{4}e^2)]$

$= \frac{1}{4}\pi(3e^2 - 5)$

4 volume $= \pi\int_0^1 (x e^{-\frac{1}{2}x})^2 \, dx$

$= \pi\int_0^1 x^2 e^{-x} \, dx$

$u = x^2, \ \frac{du}{dx} = 2x; \ \frac{dv}{dx} = e^{-x}, \ v = -e^{-x}$

$\int x^2 e^{-x} \, dx = -x^2 e^{-x} + \int 2x e^{-x} \, dx$

$u = 2x, \ \frac{du}{dx} = 2; \ \frac{dv}{dx} = e^{-x}, \ v = -e^{-x}$

$\int x^2 e^{-x} \, dx = -x^2 e^{-x} - 2x e^{-x} + \int 2e^{-x} \, dx$

$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + c$

volume $= \pi[-e^{-x}(x^2 + 2x + 2)]_0^1$

$= \pi[-e^{-1}(1 + 2 + 2)] - [-1(2)]$

$= \pi(2 - 5e^{-1})$

5 **a** $= \int_0^{\frac{\pi}{2}} (2\sin x + \cos x) \, dx$

$= [-2\cos x + \sin x]_0^{\frac{\pi}{2}}$

$= (0 + 1) - (-2 + 0)$

$= 3$

b $= \pi\int_0^{\frac{\pi}{2}} (2\sin x + \cos x)^2 \, dx$

$= \pi\int_0^{\frac{\pi}{2}} (4\sin^2 x + 4\sin x\cos x + \cos^2 x) \, dx$

$= \pi\int_0^{\frac{\pi}{2}} (2 - 2\cos 2x + 2\sin 2x + \frac{1}{2} + \frac{1}{2}\cos 2x) \, dx$

$= \pi\int_0^{\frac{\pi}{2}} (\frac{5}{2} - \frac{3}{2}\cos 2x + 2\sin 2x) \, dx$

$= \pi[\frac{5}{2}x - \frac{3}{4}\sin 2x - \cos 2x]_0^{\frac{\pi}{2}}$

$= \pi[(\frac{5}{4}\pi - 0 + 1) - (0 - 0 - 1)]$

$= \frac{1}{4}\pi(5\pi + 8)$

6 **a** $x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{4}$

b $x = \tan\theta \ \therefore \ \frac{dx}{d\theta} = \sec^2\theta$

$\therefore$ volume $= \pi\int_0^{\frac{\pi}{4}} (\sin 2\theta)^2 \times \sec^2\theta \, d\theta$

$= \pi\int_0^{\frac{\pi}{4}} (4\sin^2\theta\cos^2\theta \times \frac{1}{\cos^2\theta}) \, d\theta$

$= 4\pi\int_0^{\frac{\pi}{4}} \sin^2\theta \, d\theta$

c $= 4\pi\int_0^{\frac{\pi}{4}} (\frac{1}{2} - \frac{1}{2}\cos 2\theta) \, d\theta$

$= 4\pi[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta]_0^{\frac{\pi}{4}}$

$= 4\pi[(\frac{1}{8}\pi - \frac{1}{4}) - (0)]$

$= \frac{1}{2}\pi(\pi - 2)$

7 **a** $y = 0 \Rightarrow t = 0, -1$

$x = 0 \Rightarrow t = \pm 1$

$t \geq 0 \ \therefore \ t = 0, 1$

b $x = t^2 - 1 \ \therefore \ \frac{dx}{dt} = 2t$

$\therefore$ volume $= \pi\int_0^1 [t(t + 1)]^2 \times 2t \, dt$

$= 2\pi\int_0^1 t^3(t^2 + 2t + 1) \, dt$

$= 2\pi\int_0^1 (t^5 + 2t^4 + t^3) \, dt$

$= 2\pi[\frac{1}{6}t^6 + \frac{2}{5}t^5 + \frac{1}{4}t^4]_0^1$

$= 2\pi[(\frac{1}{6} + \frac{2}{5} + \frac{1}{4}) - (0)]$

$= \frac{49}{30}\pi$