C4 INTEGRATION *Answers - Worksheet K*

1 **a** $y = \int (x+2)^3 \; dx$

$y = \frac{1}{4}(x+2)^4 + c$

b $y = \int 4\cos 2x \; dx$

$y = 2\sin 2x + c$

c $x = \int 3e^{2t} + 2 \; dt$

$x = \frac{3}{2}e^{2t} + 2t + c$

d $\frac{dy}{dx} = \frac{1}{2-x}$

$y = \int \frac{1}{2-x} \; dx$

$y = -\ln|2-x| + c$

e $N = \int t\sqrt{t^2+1} \; dt$

$N = \frac{1}{2}\int 2t(t^2+1)^{\frac{1}{2}} \; dt$

$N = \frac{1}{2}\times\frac{2}{3}(t^2+1)^{\frac{3}{2}} + c$

$N = \frac{1}{3}(t^2+1)^{\frac{3}{2}} + c$

f $y = \int xe^x \; dx$

$u = x, \; \frac{du}{dx} = 1; \; \frac{dv}{dx} = e^x, \; v = e^x$

$y = xe^x - \int e^x \; dx$

$y = xe^x - e^x + c$ $\quad [\, y = e^x(x-1) + c \,]$

2 **a** $y = \int e^{-x} \; dx$

$y = -e^{-x} + c$

$y = 3$ when $x = 0$

$\therefore \; 3 = -1 + c$

$c = 4$

$\therefore \; y = 4 - e^{-x}$

b $y = \int \tan^3 t \sec^2 t \; dt$

$y = \frac{1}{4}\tan^4 t + c$

$y = 1$ when $t = \frac{\pi}{3}$

$\therefore \; 1 = \frac{1}{4}(\sqrt{3})^4 + c$

$c = 1 - \frac{9}{4} = -\frac{5}{4}$

$\therefore \; y = \frac{1}{4}\tan^4 t - \frac{5}{4}$ $\quad [\, y = \frac{1}{4}(\tan^4 t - 5) \,]$

c $\frac{du}{dx} = \frac{4x}{x^2-3}$

$u = \int \frac{4x}{x^2-3} \; dx = 2\int \frac{2x}{x^2-3} \; dx$

$u = 2\ln|x^2-3| + c$

$u = 5$ when $x = 2$

$\therefore \; 5 = 0 + c$

$c = 5$

$\therefore \; u = 2\ln|x^2-3| + 5$

d $y = \int 3\cos^2 x \; dx$

$y = \frac{3}{2}\int (1 + \cos 2x) \; dx$

$y = \frac{3}{2}(x + \frac{1}{2}\sin 2x) + c = \frac{3}{4}(2x + \sin 2x) + c$

$y = \pi$ when $x = \frac{\pi}{2}$

$\therefore \; \pi = \frac{3}{4}(\pi + 0) + c$

$c = \frac{\pi}{4}$

$\therefore \; y = \frac{3}{4}(2x + \sin 2x) + \frac{\pi}{4}$

$[\, y = \frac{1}{4}(6x + 3\sin 2x + \pi) \,]$

3 **a** $\frac{x-8}{x^2-x-6} \equiv \frac{A}{x-3} + \frac{B}{x+2}$

$x - 8 \equiv A(x+2) + B(x-3)$

$x = 3 \;\Rightarrow\; -5 = 5A \;\Rightarrow\; A = -1$

$x = -2 \;\Rightarrow\; -10 = -5B \;\Rightarrow\; B = 2$

$\frac{x-8}{x^2-x-6} \equiv \frac{2}{x+2} - \frac{1}{x-3}$

b $\frac{dy}{dx} = \frac{x-8}{x^2-x-6}$

$y = \int \frac{x-8}{x^2-x-6} \; dx = \int (\frac{2}{x+2} - \frac{1}{x-3}) \; dx$

$y = 2\ln|x+2| - \ln|x-3| + c$

$y = \ln 9$ when $x = 1$

$\therefore \; \ln 9 = 2\ln 3 - \ln 2 + c$

$c = \ln 2 \quad (\ln 9 = \ln 3^2 = 2\ln 3)$

$\therefore \; y = 2\ln|x+2| - \ln|x-3| + \ln 2$

when $x = 2$, $y = 2\ln 4 - 0 + \ln 2 = \ln(4^2 \times 2)$

$= \ln 32$

4 **a** $\int \frac{1}{2y+3}\ dy = \int dx$

$\frac{1}{2}\ln|2y+3| = x + c$

$[\ y = \frac{1}{2}(ke^{2x} - 3)\ ]$

b $\int \operatorname{cosec}^2 2y\ dy = \int dx$

$-\frac{1}{2}\cot 2y = x + c$

$[\ \cot 2y = k - 2x\ ]$

c $\int \frac{1}{y}\ dy = \int x\ dx$

$\ln|y| = \frac{1}{2}x^2 + c$

$[\ y = k e^{\frac{1}{2}x^2}\ ]$

d $\int \frac{1}{y}\ dy = \int \frac{1}{x+1}\ dx$

$\ln|y| = \ln|x+1| + c$

$[\ y = k(x+1)\ ]$

e $\int y\ dy = \int (x^2 - 2)\ dx$

$\frac{1}{2}y^2 = \frac{1}{3}x^3 - 2x + c$

$[\ y^2 = \frac{2}{3}x^3 - 4x + k\ ]$

f $\int \sec^2 y\ dy = \int 2\cos x\ dx$

$\tan y = 2\sin x + c$

g $\int e^{3-y}\ dy = \int x^{-\frac{1}{2}}\ dx$

$-e^{3-y} = 2x^{\frac{1}{2}} + c$

$[\ y = 3 - \ln(k - 2\sqrt{x})\ ]$

h $y\frac{dy}{dx} = x(y^2 + 3)$

$\int \frac{y}{y^2+3}\ dy = \int x\ dx$

$\frac{1}{2}\int \frac{2y}{y^2+3}\ dy = \int x\ dx$

$\frac{1}{2}\ln|y^2+3| = \frac{1}{2}x^2 + c$

$[\ y^2 = k e^{x^2} - 3\ ]$

i $\int \frac{1}{y}\ dy = \int x\sin x\ dx$

$u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = \sin x,\ v = -\cos x$

$\ln|y| = -x\cos x + \int \cos x\ dx$

$\ln|y| = \sin x - x\cos x + c$

$[\ y = ke^{\sin x - x\cos x}\ ]$

j $\frac{dy}{dx} = \frac{e^{2x}}{e^y}$

$\int e^y\ dy = \int e^{2x}\ dx$

$e^y = \frac{1}{2}e^{2x} + c$

$[\ y = \ln(\frac{1}{2}e^{2x} + c)\ ]$

k $\int \frac{y-3}{y(y-1)}\ dy = \int x\ dx$

$\frac{y-3}{y(y-1)} \equiv \frac{A}{y} + \frac{B}{y-1}$

$y - 3 \equiv A(y-1) + By$

$y = 0 \Rightarrow A = 3,\ y = 1 \Rightarrow B = -2$

$\int (\frac{3}{y} - \frac{2}{y-1})\ dy = \int x\ dx$

$3\ln|y| - 2\ln|y-1| = \frac{1}{2}x^2 + c$

l $\int y^{-2}\ dy = \int \ln x\ dx$

$u = \ln x,\ \frac{du}{dx} = \frac{1}{x};\ \frac{dv}{dx} = 1,\ v = x$

$-y^{-1} = x\ln x - \int dx$

$-y^{-1} = x\ln x - x + c$

$[\ y = \frac{1}{x - x\ln x + k}\ ]$

5 **a** $\int 2y \, dy = \int x \, dx$

$y^2 = \frac{1}{2}x^2 + c$

$y = 3$ when $x = 4$

$\therefore\ 9 = 8 + c$

$c = 1$

$\therefore\ y^2 = \frac{1}{2}x^2 + 1$

b $\int (y + 1)^{-3} \, dy = \int dx$

$-\frac{1}{2}(y + 1)^{-2} = x + c$

$(y + 1)^{-2} = k - 2x$

$y = 0$ when $x = 2$

$\therefore\ 1 = k - 4$

$k = 5$

$\therefore\ (y + 1)^{-2} = 5 - 2x$

$[\ (y + 1)^2 = \frac{1}{5 - 2x}\ ]$

c $\int \frac{1}{y} \, dy = \int \cot^2 x \, dx$

$\int \frac{1}{y} \, dy = \int (\operatorname{cosec}^2 x - 1) \, dx$

$\ln|y| = -\cot x - x + c$

$y = 1$ when $x = \frac{\pi}{2}$

$\therefore\ 0 = 0 - \frac{\pi}{2} + c$

$c = \frac{\pi}{2}$

$\therefore\ \ln|y| = \frac{\pi}{2} - \cot x - x$

d $\int \frac{1}{y+2} \, dy = \int \frac{1}{x-1} \, dx$

$\ln|y + 2| = \ln|x - 1| + c$

$y = 6$ when $x = 3$

$\therefore\ \ln 8 = \ln 2 + c$

$c = \ln 4$

$\therefore\ \ln|y + 2| = \ln|x - 1| + \ln 4$

$[\ y + 2 = 4(x - 1) \Rightarrow y = 4x - 6\ ]$

e $\int \cot y \, dy = \int x^2 \, dx$

$\int \frac{\cos y}{\sin y} \, dy = \int x^2 \, dx$

$\ln|\sin y| = \frac{1}{3}x^3 + c$

$y = \frac{\pi}{6}$ when $x = 0$

$\therefore\ \ln \frac{1}{2} = 0 + c$

$c = -\ln 2$

$\therefore\ \ln|\sin y| = \frac{1}{3}x^3 - \ln 2$

$[\ 2\sin y = e^{\frac{1}{3}x^3}\ ]$

f $\frac{dy}{dx} = \frac{\sqrt{y}}{\sqrt{x+3}}$

$\int y^{-\frac{1}{2}} \, dy = \int (x+3)^{-\frac{1}{2}} \, dx$

$2y^{\frac{1}{2}} = 2(x+3)^{\frac{1}{2}} + c$

$\sqrt{y} = \sqrt{x+3} + k$

$y = 16$ when $x = 1$

$\therefore\ 4 = 2 + k$

$k = 2$

$\therefore\ \sqrt{y} = \sqrt{x+3} + 2$

$[\ y = (\sqrt{x+3} + 2)^2\ ]$

g $\int \sin y \, dy = \int x e^{-x} \, dx$

$u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = e^{-x},\ v = -e^{-x}$

$-\cos y = -xe^{-x} + \int e^{-x} \, dx$

$-\cos y = -xe^{-x} - e^{-x} + c$

$\cos y = (x + 1)e^{-x} + k$

$y = \pi$ when $x = -1$

$\therefore\ -1 = 0 + k$

$k = -1$

$\therefore\ \cos y = (x + 1)e^{-x} - 1$

h $\int \frac{\sin y}{1 + \cos y} \, dy = \int \frac{1}{2}x^{-2} \, dx$

$-\int \frac{-\sin y}{1 + \cos y} \, dy = \int \frac{1}{2}x^{-2} \, dx$

$-\ln|1 + \cos y| = -\frac{1}{2}x^{-1} + c$

$\ln|1 + \cos y| = \frac{1}{2}x^{-1} + k$

$y = \frac{\pi}{3}$ when $x = 1$

$\therefore\ \ln \frac{3}{2} = \frac{1}{2} + k$

$k = \ln \frac{3}{2} - \frac{1}{2}$

$\therefore\ \ln|1 + \cos y| = \frac{1}{2}x^{-1} + \ln \frac{3}{2} - \frac{1}{2}$

$[\ (1 + \cos y)^2 = \frac{9}{4}e^{\frac{1-x}{x}}\ ]$