C4 INTEGRATION — Answers - Worksheet N

1 $= [\tfrac{1}{4} \times 8 \ln|4x - 3|]_2^7$

$= 2[\ln|4x - 3|]_2^7$

$= 2(\ln 25 - \ln 5)$

$= 2 \ln \tfrac{25}{5}$

$= 2 \ln 5$

$= \ln 5^2$

$= \ln 25$

2 $\frac{dy}{dx} = \frac{x}{\cos y \sin^3 y}$

$\int \cos y \sin^3 y \; dy = \int x \; dx$

$\tfrac{1}{4}\sin^4 y = \tfrac{1}{2}x^2 + c$

$\sin^4 y = 2x^2 + k$

$y = \tfrac{\pi}{4}$ when $x = 1$

$\therefore (\tfrac{1}{\sqrt{2}})^4 = 2 + k$

$\tfrac{1}{4} = 2 + k$

$k = -\tfrac{7}{4}$

$\therefore \sin^4 y = 2x^2 - \tfrac{7}{4}$

3 **a**

x	0	0.5	1	1.5
e^{x^2-1}	0.3679	0.4724	1	3.4903

$\therefore$ integral $\approx \tfrac{1}{2} \times 0.5 \times [0.3679 + 3.4903 + 2(0.4724 + 1)] = 1.70$ (3sf)

b

x	0	0.25	0.5	0.75	1	1.25	1.5
e^{x^2-1}	0.3679	0.3916	0.4724	0.6456	1	1.7551	3.4903

$\therefore$ integral $\approx \tfrac{1}{2} \times 0.25 \times [0.3679 + 3.4903 + 2(0.3916 + 0.4724 + 0.6456 + 1 + 1.7551)]$

$= 1.55$ (3sf)

4 **a** $\frac{3(2-x)}{(1-2x)^2(1+x)} \equiv \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1+x}$

$3(2-x) \equiv A(1-2x)(1+x) + B(1+x) + C(1-2x)^2$

$x = \tfrac{1}{2} \Rightarrow \tfrac{9}{2} = \tfrac{3}{2}B \Rightarrow B = 3$

$x = -1 \Rightarrow 9 = 9C \Rightarrow C = 1$

coeffs $x^2 \Rightarrow 0 = -2A + 4C \Rightarrow A = 2$

$f(x) \equiv \frac{2}{1-2x} + \frac{3}{(1-2x)^2} + \frac{1}{1+x}$

b $= \int_1^2 \left(\frac{2}{1-2x} + \frac{3}{(1-2x)^2} + \frac{1}{1+x}\right) dx$

$= [-\ln|1 - 2x| + \tfrac{3}{2}(1 - 2x)^{-1} + \ln|1 + x|]_1^2$

$= (-\ln 3 - \tfrac{1}{2} + \ln 3) - (0 - \tfrac{3}{2} + \ln 2)$

$= 1 - \ln 2$

5 **a** $\frac{dN}{dt} = kN$

$\int \frac{1}{N} \; dN = \int k \; dt$

$\ln|N| = kt + c$

$N = e^{kt + c} = e^c \times e^{kt}$

$\therefore N = Ae^{kt}$

b $t = 0, N = 200 \quad \therefore A = 200$

$t = 2, N = 3000 \quad \therefore 3000 = 200e^{2k}$

$\therefore k = \tfrac{1}{2}\ln 15 = 1.354$

$\therefore N = 200e^{1.354t}$

$\therefore 10\,000 = 200e^{1.354t}$

$t = \tfrac{1}{1.354}\ln 50 = 2.889$ hours

= 2 hours 53 minutes

c 5 per second = 18 000 per hour

$\frac{dN}{dt} = 200 \times 0.1354e^{1.354t}$

$\therefore 18\,000 = 270.8e^{1.354t}$

$t = \tfrac{1}{1.354}\ln\tfrac{18000}{270.8} = 3.099$ hours

= 3 hours 6 minutes

6 **a** $= \int_0^4 (2x+1)^{-\frac{1}{2}}\ dx$

$= [\,\frac{1}{2} \times 2(2x+1)^{\frac{1}{2}}\,]_0^4$

$= [\,(2x+1)^{\frac{1}{2}}\,]_0^4$

$= 3 - 1 = 2$

b $= \pi \int_0^4 (\frac{1}{\sqrt{2x+1}})^2\ dx$

$= \pi \int_0^4 \frac{1}{2x+1}\ dx$

$= \pi[\,\frac{1}{2}\ln|2x+1|\,]_0^4$

$= \frac{1}{2}\pi(\ln 9 - 0) = \pi \ln 9^{\frac{1}{2}}$

$= \pi \ln 3$

7 $u^2 = x + 3 \ \therefore\ x = u^2 - 3,\ \frac{dx}{du} = 2u$

$x = 0 \Rightarrow u = \sqrt{3}$

$x = 1 \Rightarrow u = 2$

$\int_0^1 x\sqrt{x+3}\ dx = \int_{\sqrt{3}}^2 (u^2 - 3)u \times 2u\ du$

$= \int_{\sqrt{3}}^2 (2u^4 - 6u^2)\ du$

$= [\,\frac{2}{5}u^5 - 2u^3]_{\sqrt{3}}^2$

$= (\frac{64}{5} - 16) - (\frac{2}{5} \times 9\sqrt{3} - 2 \times 3\sqrt{3})$

$= -\frac{16}{5} - (-\frac{12}{5}\sqrt{3})$

$= \frac{4}{5}(3\sqrt{3} - 4) \qquad [\,k = \frac{4}{5}\,]$

8 **a** $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$

$\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$

adding

$2 \sin A \cos B \equiv \sin(A + B) + \sin(A - B)$

b $y = 0 \Rightarrow \sin 4t = 0 \Rightarrow t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$

curve in 1st quadrant: $0 \le t \le \frac{\pi}{4}$

$x = 2\sin 2t \ \therefore\ \frac{dx}{dt} = 4\cos 2t$

area in 1st quadrant

$= \int_0^{\frac{\pi}{4}} \sin 4t \times 4\cos 2t\ dt$

total area = 4 × area in 1st quadrant

$= \int_0^{\frac{\pi}{4}} 16 \sin 4t \cos 2t\ dt$

c $= 8\int_0^{\frac{\pi}{4}} (\sin 6t + \sin 2t)\ dt$

$= 8[-\frac{1}{6}\cos 6t - \frac{1}{2}\cos 2t]_0^{\frac{\pi}{4}}$

$= 8[(0 - 0) - (-\frac{1}{6} - \frac{1}{2})] = 5\frac{1}{3}$

9 **a** $\frac{x^2 - 22}{(x+2)(x-4)} \equiv A + \frac{B}{x+2} + \frac{C}{x-4}$

$x^2 - 22 \equiv A(x+2)(x-4) + B(x-4) + C(x+2)$

$x = -2 \Rightarrow -18 = -6B \Rightarrow B = 3$

$x = 4 \Rightarrow -6 = 6C \Rightarrow C = -1$

coeffs $x^2 \Rightarrow A = 1$

b $= \int_0^2 (1 + \frac{3}{x+2} - \frac{1}{x-4})\ dx$

$= [x + 3\ln|x+2| - \ln|x-4|\,]_0^2$

$= (2 + 3\ln 4 - \ln 2) - (0 + 3\ln 2 - \ln 4)$

$= 2 + 6\ln 2 - \ln 2 - 3\ln 2 + 2\ln 2$

$= 2 + 4\ln 2$

$= 2 + \ln 16$

10 **a** $= \frac{1}{2}\int (1 - \cos 2x)\ dx = \frac{1}{2}(x - \frac{1}{2}\sin 2x) + c = \frac{1}{4}(2x - \sin 2x) + c$

b $u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = \sin^2 x,\ v = \frac{1}{2}x - \frac{1}{4}\sin 2x$

$\int x\sin^2 x\ dx = x(\frac{1}{2}x - \frac{1}{4}\sin 2x) - \int (\frac{1}{2}x - \frac{1}{4}\sin 2x)\ dx$

$= \frac{1}{2}x^2 - \frac{1}{4}x\sin 2x - (\frac{1}{4}x^2 + \frac{1}{8}\cos 2x) + c$

$= \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + c$

$= \frac{1}{8}(2x^2 - 2x\sin 2x - \cos 2x) + c$

c $= \pi\int_0^{\pi} (x^{\frac{1}{2}}\sin x)^2\ dx = \pi\int_0^{\pi} x\sin^2 x\ dx$

$= \frac{1}{8}\pi[2x^2 - 2x\sin 2x - \cos 2x]_0^{\pi}$

$= \frac{1}{8}\pi[(2\pi^2 - 0 - 1) - (0 - 0 - 1)] = \frac{1}{4}\pi^3$