1 **a** when $y = 3$, $x = \frac{1}{3}$

area $0 \le x \le \frac{1}{3} = 3 \times \frac{1}{3} = 1$

area $\frac{1}{3} < x \le 3 = \int_{\frac{1}{3}}^{3} \frac{1}{x} \, dx$

$= [\ln |x|]_{\frac{1}{3}}^{3}$

$= \ln 3 - \ln \frac{1}{3} = \ln \frac{3}{\frac{1}{3}} = \ln 9$

total area $= 1 + \ln 9$

b volume $0 \le x \le \frac{1}{3}$ = volume of cylinder

$= \pi \times 3^2 \times \frac{1}{3} = 3\pi$

volume $\frac{1}{3} < x \le 3 = \pi \int_{\frac{1}{3}}^{3} (\frac{1}{x})^2 \, dx$

$= \pi \int_{\frac{1}{3}}^{3} x^{-2} \, dx = \pi[-x^{-1}]_{\frac{1}{3}}^{3}$

$= \pi[-\frac{1}{3} - (-3)] = \frac{8}{3}\pi$

total volume $= 3\pi + \frac{8}{3}\pi = \frac{17}{3}\pi$

2 **a**

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$x \sec(\frac{1}{3}x)$	0	0.507	1.058	1.709	2.545	3.718	5.552	8.901	17.004

i $\approx \frac{1}{2} \times 2 \times [0 + 17.004 + 2(2.545)] = 22.1$ (3sf)

ii $\approx \frac{1}{2} \times 1 \times [0 + 17.004 + 2(1.058 + 2.545 + 5.552)] = 17.7$ (3sf)

iii $\approx \frac{1}{2} \times 0.5 \times [0 + 17.004 + 2(0.507 + 1.058 + 1.709 + 2.545 + 3.718 + 5.552 + 8.901)] = 16.2$ (3sf)

b e.g. halving the interval width leads to the estimate reducing by 4.4, then 1.5
further halving might lead to reductions of approximately 0.5 and 0.2 so $I \approx 15.5$

3 **a** $-\frac{d\theta}{dt} = k\theta$

$\int \frac{1}{\theta} \, d\theta = \int -k \, dt$

$\ln |\theta| = -kt + c$

$\theta = e^{-kt + c} = e^c \times e^{-kt}$

$\therefore \theta = A e^{-kt}$

$t = 0, \theta = 25 - 10 = 15$

$\therefore A = 15$

$\therefore \theta = 15 e^{-kt}$

b $t = 30, \theta = 20 - 10 = 10$

$\therefore 10 = 15 e^{-30k}$

$k = -\frac{1}{30} \ln \frac{2}{3} = 0.0135$ (3sf)

c $\theta = 15 - 10 = 5$

$\therefore 5 = 15 e^{-0.01352t}$

$t = -\frac{1}{0.01352} \ln \frac{1}{3}$

$= 81.3$ minutes (3sf)

4 **a** $\sin^4 x \equiv (\sin^2 x)^2$

$\equiv [\frac{1}{2}(1 - \cos 2x)]^2$

$\equiv \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$

$\equiv \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}[\frac{1}{2}(1 + \cos 4x)]$

$\equiv \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$

$\therefore p = \frac{3}{8}, q = -\frac{1}{2}, r = \frac{1}{8}$

b $= \int_{0}^{\frac{\pi}{2}} (\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x) \, dx$

$= [\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x]_{0}^{\frac{\pi}{2}}$

$= (\frac{3}{16}\pi - 0 + 0) - (0 - 0 + 0)$

$= \frac{3}{16}\pi$

5   **a** $\int y^{-3} \, dy = \int x \, dx$

$-\frac{1}{2}y^{-2} = \frac{1}{2}x^2 + c$

$y^{-2} = k - x^2$

$y^2 = \dfrac{1}{k - x^2}$

  b $y = \frac{1}{2}$ when $x = 1$

$\therefore \quad \frac{1}{4} = \dfrac{1}{k-1}$

$k = 5$

$\therefore \quad y^2 = \dfrac{1}{5 - x^2}$

6   **a** $x = e^u \; \therefore \; u = \ln x, \; \dfrac{dx}{du} = e^u$

$\int \dfrac{2 + \ln x}{x^2} \, dx = \int \dfrac{2 + u}{(e^u)^2} \times e^u \, du$

$\qquad\qquad\qquad = \int (2 + u)e^{-u} \, du$

  b $x = e^u$

$x = 1 \implies u = 0$

$x = e \implies u = 1$

$\therefore \; \int_1^e \dfrac{2 + \ln x}{x^2} \, dx = \int_0^1 (2 + u)e^{-u} \, du$

$v = 2 + u, \; \dfrac{dv}{du} = 1; \; \dfrac{dw}{du} = e^{-u}, \; w = -e^{-u}$

$= [-(2 + u)e^{-u}]_0^1 + \int_0^1 e^{-u} \, du$

$= [-(2 + u)e^{-u} - e^{-u}]_0^1$

$= (-3e^{-1} - e^{-1}) - (-2 - 1)$

$= 3 - 4e^{-1}$

7   **a** $x = 0 \implies \cos 2t = 0 \implies t = \frac{\pi}{4}$

$y = 0 \implies \tan t = 0 \implies t = 0$

$x = \cos 2t \; \therefore \; \dfrac{dx}{dt} = -2 \sin 2t$

shaded area $= \displaystyle\int_{\frac{\pi}{4}}^{0} \tan t \times (-2 \sin 2t) \, dt$

$= \displaystyle\int_0^{\frac{\pi}{4}} \dfrac{\sin t}{\cos t} \times 4 \sin t \cos t \, dt$

$= \displaystyle\int_0^{\frac{\pi}{4}} 4 \sin^2 t \, dt$

  b $= \displaystyle\int_0^{\frac{\pi}{4}} (2 - 2 \cos 2t) \, dt$

$= [2t - \sin 2t]_0^{\frac{\pi}{4}}$

$= (\frac{\pi}{2} - 1) - (0 - 0)$

$= \frac{\pi}{2} - 1$

  c   **i** $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

     ii $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$y^2 = \tan^2 t = \dfrac{\sin^2 t}{\cos^2 t} = \dfrac{\frac{1}{2}(1 - \cos 2t)}{\frac{1}{2}(1 + \cos 2t)}$

$\therefore \; y^2 = \dfrac{1 - x}{1 + x}$

8   **a** $\dfrac{6 - 2x^2}{(x+1)^2(x+3)} \equiv \dfrac{A}{(x+1)^2} + \dfrac{B}{x+1} + \dfrac{C}{x+3}$

$6 - 2x^2 \equiv A(x + 3) + B(x + 1)(x + 3) + C(x + 1)^2$

$x = -1 \quad \implies \quad 4 = 2A \qquad \implies \quad A = 2$

$x = -3 \quad \implies \quad -12 = 4C \quad \implies \quad C = -3$

coeffs $x^2 \implies \; -2 = B + C \implies B = 1$

  b $x = 0 \implies y = 2$

$f(x) = 2(x + 1)^{-2} + (x + 1)^{-1} - 3(x + 3)^{-1}$

$f'(x) = -4(x + 1)^{-3} - (x + 1)^{-2} + 3(x + 3)^{-2}$

$\therefore \; f'(0) = -4 - 1 + \frac{1}{3} = -\frac{14}{3}$

$\therefore \; y - 2 = -\frac{14}{3}(x - 0)$

$3y - 6 = -14x$

$14x + 3y = 6$

  c $= \displaystyle\int_0^1 \left(\dfrac{2}{(x+1)^2} + \dfrac{1}{x+1} - \dfrac{3}{x+3} \right) \, dx$

$= [-2(x + 1)^{-1} + \ln |x + 1| - 3 \ln |x + 3|]_0^1$

$= (-1 + \ln 2 - 3 \ln 4) - (-2 + 0 - 3 \ln 3)$

$= 1 - 5 \ln 2 + 3 \ln 3$