After completing this chapter you should be able to:

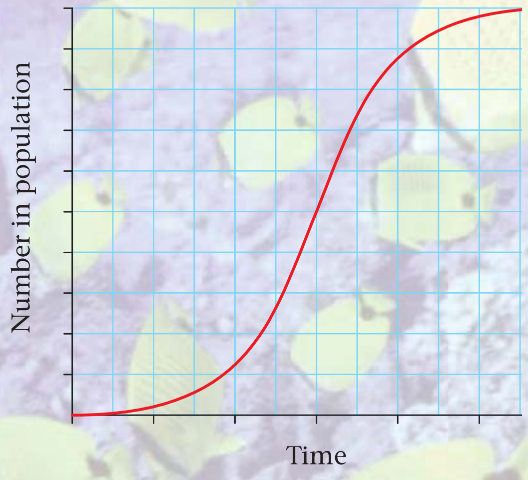
- add and subtract two or more fractions
- convert an expression with linear factors in the denominator into partial fractions
- convert an expression with repeated linear factors in the denominator into partial fractions
- convert an improper fraction into partial fraction form.

1

Partial fractions

Converting a complex fraction into a sum of simpler ones has many uses. You will meet some of these uses later in the book when you study both the binomial expansion and methods of integration.

The method of partial fractions is used to derive the general equation of the logistic function. The initial growth is of this function is approximately exponential, but it slows down as maturity is reached until growth finally ceases. It has applications in many fields including biology and economics.



The logistic function can be used to model the growth of a population of fish in a shoal.

1.1 You can add (or subtract) two or more simple fractions as long as the denominators are the same.

Example 1

Calculate: **a** $\frac{1}{3}+\frac{3}{8}$ **b** $\frac{2}{x+3}-\frac{1}{(x+1)}$

a $\frac{1}{3}+\frac{3}{8}$

The lowest common multiple of 3 and 8 is 24.

$$=\frac{8}{24}+\frac{9}{24}$$

Multiply $\frac{1}{3}$ top and bottom by 8 to express in 24ths.
Multiply $\frac{3}{8}$ top and bottom by 3 to express in 24ths.

$$=\frac{8+9}{24}$$

Add the numerators.

$$=\frac{17}{24}$$

b $\frac{2}{(x+3)}-\frac{1}{(x+1)}$

$$=\frac{2(x+1)}{(x+3)(x+1)}-\frac{1(x+3)}{(x+3)(x+1)}$$

The lowest common multiple is $(x+3)(x+1)$, so change both fractions so that the denominators are $(x+3)(x+1)$.

$$=\frac{2(x+1)-1(x+3)}{(x+3)(x+1)}$$

Subtract the numerators.

$$=\frac{2x+2-1x-3}{(x+3)(x+1)}$$

Expand the brackets.

$$=\frac{x-1}{(x+3)(x+1)}$$

Simplify the numerator.

Exercise 1A

Express each of the following as a single fraction:

1 $\frac{1}{3}+\frac{1}{4}$

2 $\frac{3}{4}-\frac{2}{5}$

3 $\frac{3}{x}-\frac{2}{x+1}$

4 $\frac{2}{(x-1)}+\frac{3}{(x+2)}$

5 $\frac{4}{(2x+1)}+\frac{2}{(x-1)}$

6 $\frac{7}{(x-3)}-\frac{2}{(x+4)}$

7 $\frac{3}{2x}-\frac{6}{(x-1)}$

8 $\frac{3}{x}+\frac{2}{(x+1)}+\frac{1}{(x+2)}$

9 $\frac{4}{3x}-\frac{2}{(x-2)}+\frac{1}{(2x+1)}$

10 $\frac{3}{(x-1)}+\frac{2}{(x+1)}+\frac{4}{(x-3)}$

1.2 You can split a fraction with two linear factors in the denominator into partial fractions.

For example, $\frac{x-1}{(x+1)(x+3)}$ is $\frac{2}{(x+3)}-\frac{1}{(x+1)}$ when split into partial fractions (see Example 1).

- **In general, an expression with two linear terms in the denominator such as $\frac{11}{(x-3)(x+2)}$ can be split into partial fractions of the form $\frac{A}{(x-3)}+\frac{B}{(x+2)}$, where A and B are constants.**

There are two methods of achieving this: by substitution and by equating coefficients.

Example 2

Split $\frac{6x-2}{(x-3)(x+1)}$ into partial fractions by **a** substitution **b** equating coefficients.

a

$$\frac{6x-2}{(x-3)(x+1)}\equiv\frac{A}{(x-3)}+\frac{B}{(x+1)}$$

$$\equiv\frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

$$6x-2\equiv A(x+1)+B(x-3)$$

$$6\times3-2=A(3+1)+B(3-3)$$

$$16=4A$$

$$A=4$$

$$6\times-1-2=A(-1+1)+B(-1-3)$$

$$-8=-4B$$

$$B=2$$

$$\therefore\ \frac{6x-2}{(x-3)(x+1)}\equiv\frac{4}{(x-3)}+\frac{2}{(x+1)}$$

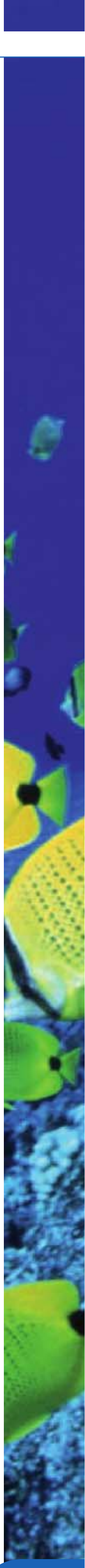
Set $\frac{6x-2}{(x-3)(x+1)}$ identical to $\frac{A}{(x-3)}+\frac{B}{(x+1)}$.

Add the two fractions.

Because this is an equivalence relation set the numerators equal to each other.

To find A substitute $x = 3$.

To find B substitute $x = -1$.



b $$\frac{6x-2}{(x-3)(x+1)} \equiv \frac{A}{(x-3)} + \frac{B}{(x+1)}$$

$$\equiv \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

$$6x-2 \equiv A(x+1)+B(x-3)$$

$$\equiv Ax + A + Bx - 3B$$

$$\equiv (A+B)x + (A-3B)$$

Equate coefficients of x:

$$6 = A + B \quad ①$$

Equate constant terms:

$$-2 = A - 3B \quad ②$$

① − ②:

$$8 = 4B$$

$$\Rightarrow \quad B = 2$$

Substitute $B = 2$ in ① $\Rightarrow 6 = A + 2$

$$A = 4$$

Set $\frac{6x-2}{(x-3)(x+1)}$ identical to $\frac{A}{(x-3)} + \frac{B}{(x+1)}$.

Add the two fractions.

Because this is an equivalence relation set the numerators equal to each other.

Expand the brackets.

Collect like terms.

You want $(A+B)x + A - 3B = 6x - 2$. Hence coefficient of x is 6, and constant term is -2.

Solve simultaneously.

Exercise 1B

1 Express the following as partial fractions:

a $\frac{6x-2}{(x-2)(x+3)}$

b $\frac{2x+11}{(x+1)(x+4)}$

c $\frac{-7x-12}{2x(x-4)}$

d $\frac{2x-13}{(2x+1)(x-3)}$

e $\frac{6x+6}{x^2-9}$

f $\frac{7-3x}{x^2-3x-4}$

g $\frac{8-x}{x^2+4x}$

h $\frac{2x-14}{x^2+2x-15}$

2 Show that $\frac{-2x-5}{(4+x)(2-x)}$ can be written in the form $\frac{A}{(4+x)} + \frac{B}{(2-x)}$ where A and B are constants to be found.

1.3 You can also split fractions that have more than two linear factors in the denominator into partial fractions.

■ An expression with three or more linear terms in the denominator such as $\frac{4}{(x+1)(x-3)(x+4)}$ can be split into $\frac{A}{(x+1)} + \frac{B}{(x-3)} + \frac{C}{(x+4)}$, and so on if there are more terms.

Example 3

Express $\frac{6x^2 + 5x - 2}{x(x-1)(2x+1)}$ in partial fractions.

$$\text{Let } \frac{6x^2+5x-2}{x(x-1)(2x+1)} \equiv \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(2x+1)}$$

The denominators must be x, $(x - 1)$ and $(2x + 1)$.

$$\equiv \frac{A(x-1)(2x+1) + Bx(2x+1) + Cx(x-1)}{x(x-1)(2x+1)}$$

Add the fractions.

$$\therefore \quad 6x^2 + 5x - 2 \equiv A(x-1)(2x+1) + Bx(2x+1) + Cx(x-1)$$

Set the numerators equal.

Let $x = 1$

$$6 + 5 - 2 = 0 + B \times 1 \times 3 + 0$$
$$9 = 3B$$
$$B = 3$$

Let $x = 0$

$$0 + 0 - 2 = A \times -1 \times 1 + 0 + 0$$
$$-2 = -1A$$
$$A = 2$$

Let $x = -\frac{1}{2}$

$$\frac{6}{4} - \frac{5}{2} - 2 = 0 + 0 + C \times -\frac{1}{2} \times -\frac{3}{2}$$
$$-3 = \frac{3}{4}C$$
$$C = -4$$

Proceed by substitution OR by equating coefficients.

$$\text{So } \frac{6x^2+5x-2}{x(x-1)(2x+1)} \equiv \frac{2}{x} + \frac{3}{x-1} - \frac{4}{2x+1}$$

Exercise 1C

1 Express the following as partial fractions:

a $\frac{2x^2 - 12x - 26}{(x+1)(x-2)(x+5)}$ **b** $\frac{-10x^2 - 8x + 2}{x(2x+1)(3x-2)}$ **c** $\frac{-5x^2 - 19x - 32}{(x+1)(x+2)(x-5)}$

2 By firstly factorising the denominator, express the following as partial fractions:

a $\frac{6x^2 + 7x - 3}{x^3 - x}$ **b** $\frac{5x^2 + 15x + 8}{x^3 + 3x^2 + 2x}$ **c** $\frac{5x^2 - 15x - 8}{x^3 - 4x^2 + x + 6}$

1.4 You can express a fraction that has repeated linear factors in its denominator as a partial fraction.

- **An expression with repeated linear terms such as $\frac{6x^2 - 29x - 29}{(x+1)(x-3)^2}$ can be split into the form $\frac{A}{(x+1)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2}$.**

Example 4

Split $\frac{11x^2 + 14x + 5}{(x+1)^2(2x+1)}$ into partial fraction form.

Let $\frac{11x^2 + 14x + 5}{(x+1)^2(2x+1)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(2x+1)}$

You need denominators of $(x + 1)$, $(x + 1)^2$ and $(2x + 1)$.

$\equiv \frac{A(x+1)(2x+1) + B(2x+1) + C(x+1)^2}{(x+1)^2(2x+1)}$

Add the three fractions.

Hence $11x^2 + 14x + 5$

$\equiv A(x+1)(2x+1) + B(2x+1) + C(x+1)^2$

Set the numerators equal.

Let $x = -1$

$11 - 14 + 5 = A \times 0 + B \times -1 + C \times 0$

To find B substitute $x = -1$.

$2 = -1B$

$B = -2$

Let $x = -\frac{1}{2}$

$\frac{11}{4} - 7 + 5 = A \times 0 + B \times 0 + C \times \frac{1}{4}$

To find C substitute $x = -\frac{1}{2}$.

$\frac{3}{4} = \frac{1}{4}C$

$C = 3$

$11 = 2A + C$

Equate terms in x^2. Term in x^2 is $A \times 2x^2 + C \times x^2$.

$11 = 2A + 3$

Substitute $C = 3$.

$2A = 8$

$A = 4$

Hence $\frac{11x^2 + 14x + 5}{(x+1)^2(2x+1)}$

$\equiv \frac{4}{(x+1)} - \frac{2}{(x+1)^2} + \frac{3}{(2x+1)}$

Exercise 1D

Put the following into partial fraction form:

1 $\frac{3x^2 + x + 2}{x^2(x+1)}$

2 $\frac{-x^2 - 10x - 5}{(x+1)^2(x-1)}$

3 $\frac{2x^2 + 2x - 18}{x(x-3)^2}$

4 $\frac{7x^2 - 42x + 64}{x(x-4)^2}$

5 $\frac{5x^2 - 2x - 1}{x^3 - x^2}$

6 $\frac{2x^2 + 2x - 18}{x^3 - 6x^2 + 9x}$

7 $\frac{2x}{(x+2)^2}$

8 $\frac{x^2 + 5x + 7}{(x+2)^3}$

1.5 You can split improper fractions into partial fractions by dividing the numerator by the denominator.

- **An algebraic fraction is improper when the degree of the numerator is equal to, or larger than, the degree of the denominator. An improper fraction must be divided first to obtain a number and a proper fraction before it can be expressed in partial fractions.**

$\frac{x^2}{x(x-3)}$, $\frac{x^3+4x^2+2}{(x+1)(x-3)}$ and $\frac{x^4}{(x-1)^2(x+2)}$ are all examples of improper fractions.

Example 5

Express $\frac{3x^2-3x-2}{(x-1)(x-2)}$ in partial fractions.

$$\frac{3x^2-3x-2}{(x-1)(x-2)} \equiv \frac{3x^2-3x-2}{x^2-3x+2}$$

Multiply out the denominator.

$$\begin{array}{r} 3 \\ \equiv x^2-3x+2\overline{)3x^2-3x-2} \\ \underline{3x^2-9x+6} \\ 6x-8 \end{array}$$

Divide the denominator into the numerator.

It goes in 3 times, with a remainder of $6x-8$.

Therefore

$$\frac{3x^2-3x-2}{(x-1)(x-2)} \equiv 3+\frac{6x-8}{x^2-3x+2}$$

Write $\frac{3x^2-3x-2}{(x-1)(x-2)}$ as a mixed number fraction.

$$= 3+\frac{6x-8}{(x-1)(x-2)}$$

Factorise x^2-3x+2.

$$\text{Let } \frac{6x-8}{(x-1)(x-2)} = \frac{A}{(x-1)}+\frac{B}{(x-2)}$$

The denominators must be $(x-1)$ and $(x-2)$.

$$\equiv \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

Add the two fractions.

$$6x-8 \equiv A(x-2)+B(x-1)$$

Set the numerators equal.

$$\text{Let } x=2,\quad 12-8 = A\times 0+B\times 1$$
$$B=4$$

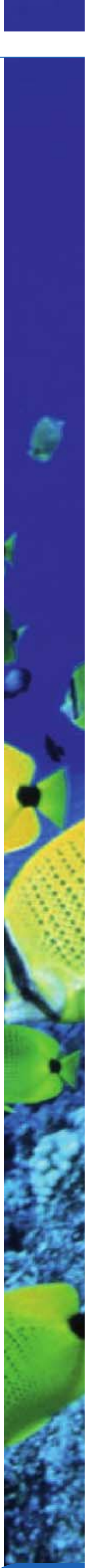
Substitute $x=2$ to find B.

$$\text{Let } x=1,\quad 6-8 = A\times -1+B\times 0$$
$$A=2$$

Substitute $x=1$ to find A.

$$\frac{3x^2-3x-2}{(x-1)(x-2)} = 3+\frac{6x-8}{(x-1)(x-2)}$$
$$\equiv 3+\frac{2}{(x-1)}+\frac{4}{(x-2)}$$

Write out full solution.



Exercise 1E

1 Express the following improper fractions as a partial fraction:

a $\dfrac{x^2 + 3x - 2}{(x + 1)(x - 3)}$ **b** $\dfrac{x^2 - 10}{(x - 2)(x + 1)}$

c $\dfrac{x^3 - x^2 - x - 3}{x(x - 1)}$ **d** $\dfrac{2x^2 - 1}{(x + 1)^2}$

2 By factorising the denominator, express the following as partial fraction:

a $\dfrac{4x^2 + 17x - 11}{x^2 + 3x - 4}$ **b** $\dfrac{x^4 - 4x^3 + 9x^2 - 17x + 12}{x^3 - 4x^2 + 4x}$

3 Show that $\dfrac{-3x^3 - 4x^2 + 19x + 8}{x^2 + 2x - 3}$ can be expressed in the form

$A + Bx + \dfrac{C}{(x - 1)} + \dfrac{D}{(x + 3)}$, where A, B, C and D are constants to be found.

Mixed exercise 1F

1 Express the following as a partial fraction:

a $\dfrac{x - 3}{x(x - 1)}$ **b** $\dfrac{7x^2 + 2x - 2}{x^2(x + 1)}$

c $\dfrac{-15x + 21}{(x - 2)(x + 1)(x - 5)}$ **d** $\dfrac{x^2 + 1}{x(x - 2)}$

2 Write the following algebraic fractions as a partial fraction:

a $\dfrac{3x + 1}{x^2 + 2x + 1}$ **b** $\dfrac{2x^2 + 2x - 8}{x^2 + 2x - 3}$

c $\dfrac{3x^2 + 12x + 8}{(x + 2)^3}$ **d** $\dfrac{x^4}{x^2 - 2x + 1}$

3 Given that $f(x) = 2x^3 + 9x^2 + 10x + 3$:

a Show that -3 is a root of $f(x)$.

b Express $\dfrac{10}{f(x)}$ as partial fractions.

E

(adapted)

Summary of key points

1 An algebraic fraction can be written as a sum of two or more simpler fractions. This technique is called splitting into partial fractions.

2 An expression with two linear terms in the denominator such as $\frac{11}{(x-3)(x+2)}$ can be split by converting into the form $\frac{A}{(x-3)} + \frac{B}{(x+2)}$.

3 An expression with three or more linear terms such as $\frac{4}{(x+1)(x-3)(x+4)}$ can be split by converting into the form $\frac{A}{(x+1)} + \frac{B}{(x-3)} + \frac{C}{(x+4)}$ and so on if there are more terms.

4 An expression with repeated terms in the denominator such as $\frac{6x^2-29x-29}{(x+1)(x-3)^2}$ can be split by converting into the form $\frac{A}{(x+1)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2}$.

5 An improper fraction is one where the index of the numerator is equal to or higher than the index of the denominator. An improper fraction must be divided first to obtain a number and a proper fraction before you can express it in partial fractions.

- For example, $\frac{x^2+3x+4}{x^2+3x+2} = 1 + \frac{2}{x^2+3x+2} = 1 + \frac{A}{(x+1)} + \frac{B}{(x+2)}$.