C4 VECTORS — Answers - Worksheet B

1 **a** $6\mathbf{i} + \mathbf{j}$ **b** $-4\mathbf{i} + 2\mathbf{j}$ **c** $-6\mathbf{i}$ **d** $10\mathbf{i} - 2\mathbf{j}$

2 **a** $= 4(\mathbf{i} - 3\mathbf{j})$
$= 4\mathbf{i} - 12\mathbf{j}$

b $= (4\mathbf{i} + 2\mathbf{j}) - (\mathbf{i} - 3\mathbf{j})$
$= 3\mathbf{i} + 5\mathbf{j}$

c $= 2(\mathbf{i} - 3\mathbf{j}) + 3(4\mathbf{i} + 2\mathbf{j})$
$= 14\mathbf{i}$

d $= 4(\mathbf{i} - 3\mathbf{j}) - 2(4\mathbf{i} + 2\mathbf{j})$
$= -4\mathbf{i} - 16\mathbf{j}$

3 **a** $= \sqrt{9+16} = 5$

b $= 2\sqrt{1+4} = 2\sqrt{5}$

c $\mathbf{p} + 2\mathbf{q} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$
$|\,\mathbf{p} + 2\mathbf{q}\,| = 5$

d $3\mathbf{q} - 2\mathbf{p} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -3 \\ 14 \end{pmatrix}$
$|\,3\mathbf{q} - 2\mathbf{p}\,| = \sqrt{9+196} = \sqrt{205} = 14.3$ (3sf)

4 **a** $= \tan^{-1} \frac{1}{2} = 26.6°$

b $= \tan^{-1} 3 = 71.6°$

c $5\mathbf{p} + \mathbf{q} = 5(2\mathbf{i} + \mathbf{j}) + (\mathbf{i} - 3\mathbf{j}) = 11\mathbf{i} + 2\mathbf{j}$
angle $= \tan^{-1} \frac{2}{11} = 10.3°$

d $\mathbf{p} - 3\mathbf{q} = (2\mathbf{i} + \mathbf{j}) - 3(\mathbf{i} - 3\mathbf{j}) = -\mathbf{i} + 10\mathbf{j}$
angle $= 180° - \tan^{-1} 10 = 95.7°$

5 **a** $\left|\begin{pmatrix} 4 \\ 3 \end{pmatrix}\right| = \sqrt{16+9} = 5$
$\therefore\ \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

b $\left|\begin{pmatrix} 7 \\ -24 \end{pmatrix}\right| = \sqrt{49+576} = 25$
$\therefore\ \frac{1}{25}\begin{pmatrix} 7 \\ -24 \end{pmatrix}$

c $\left|\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right| = \sqrt{1+1} = \sqrt{2}$
$\therefore\ \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{2}\sqrt{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

d $\left|\begin{pmatrix} 2 \\ 4 \end{pmatrix}\right| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$
$\therefore\ \frac{1}{2\sqrt{5}}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \frac{1}{5}\sqrt{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

6 **a** $|\,5\mathbf{i} + 12\mathbf{j}\,| = \sqrt{25+144} = 13$
$\therefore\ \frac{26}{13}(5\mathbf{i} + 12\mathbf{j}) = 10\mathbf{i} + 24\mathbf{j}$

b $|\,-6\mathbf{i} - 8\mathbf{j}\,| = \sqrt{36+64} = 10$
$\therefore\ \frac{15}{10}(-6\mathbf{i} - 8\mathbf{j}) = -9\mathbf{i} - 12\mathbf{j}$

c $|\,2\mathbf{i} - 4\mathbf{j}\,| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$
$\therefore\ \frac{5}{2\sqrt{5}}(2\mathbf{i} - 4\mathbf{j}) = \sqrt{5}\,(\mathbf{i} - 2\mathbf{j})$

7 **a** $(2\mathbf{i} + \lambda\mathbf{j}) + (\mu\mathbf{i} - 5\mathbf{j}) = 3\mathbf{i} - \mathbf{j}$
$2 + \mu = 3$ and $\lambda - 5 = -1$
$\therefore\ \lambda = 4,\ \mu = 1$

b $2(2\mathbf{i} + \lambda\mathbf{j}) - (\mu\mathbf{i} - 5\mathbf{j}) = -3\mathbf{i} + 8\mathbf{j}$
$4 - \mu = -3$ and $2\lambda + 5 = 8$
$\therefore\ \lambda = \frac{3}{2},\ \mu = 7$

8 **a** $6\mathbf{i} + c\mathbf{j} = 3(2\mathbf{i} + \mathbf{j})$
$\therefore\ c = 3$

b $6\mathbf{i} + c\mathbf{j} = -\frac{2}{3}(-9\mathbf{i} - 6\mathbf{j})$
$\therefore\ c = 4$

c $36 + c^2 = 10^2 = 100$
$\therefore\ c^2 = 64$
$c > 0\ \therefore\ c = 8$

d $36 + c^2 = (3\sqrt{5})^2 = 45$
$\therefore\ c^2 = 9$
$c > 0\ \therefore\ c = 3$

9 **a** $a(\mathbf{i} + 3\mathbf{j}) + b(4\mathbf{i} - 2\mathbf{j}) = -5\mathbf{i} + 13\mathbf{j}$

$\therefore \quad a + 4b = -5 \quad (1)$

and $\quad 3a - 2b = 13 \quad (2)$

$(1) + 2\times(2) \quad \Rightarrow \quad 7a = 21$

$\therefore \; a = 3, \; b = -2$

b $c(\mathbf{i} + 3\mathbf{j}) + (4\mathbf{i} - 2\mathbf{j}) = k\mathbf{j}$

$\therefore \; c + 4 = 0$

$\therefore \; c = -4$

c $(\mathbf{i} + 3\mathbf{j}) + d(4\mathbf{i} - 2\mathbf{j}) = k(3\mathbf{i} - \mathbf{j})$

$\therefore \quad 1 + 4d = 3k$

and $\quad 3 - 2d = -k$

$(1) + 2\times(2) \quad \Rightarrow \quad 7 = k$

$\therefore \quad d = 5$

10 **a** $\overrightarrow{AB} = \begin{pmatrix}-5\\2\end{pmatrix} - \begin{pmatrix}3\\6\end{pmatrix} = \begin{pmatrix}-8\\-4\end{pmatrix}$

b $|\overrightarrow{AB}| = \sqrt{64+16} = \sqrt{80} = 4\sqrt{5}$

c $= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$= \begin{pmatrix}3\\6\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-8\\-4\end{pmatrix} = \begin{pmatrix}-1\\4\end{pmatrix}$

d $\overrightarrow{OC} = \overrightarrow{AB}$

$\therefore$ pos. vector $= \begin{pmatrix}-8\\-4\end{pmatrix}$

11 **a** $= \sqrt{(2-4)^2 + (-3-0)^2 + (3-9)^2}$

$= \sqrt{4+9+36}$

$= 7$

b $= \sqrt{(7-11)^2 + (-1+3)^2 + (3-5)^2}$

$= \sqrt{16+4+4}$

$= \sqrt{24} = 2\sqrt{6} = 4.90$ (3sf)

12 **a** $= \sqrt{16+4+16}$

$= 6$

b $= \sqrt{1+1+1}$

$= \sqrt{3} = 1.73$ (3sf)

c $= \sqrt{64+1+16}$

$= 9$

d $= \sqrt{9+25+1}$

$= \sqrt{35} = 5.92$ (3sf)

13 **a** $|5\mathbf{i} - 2\mathbf{j} + 14\mathbf{k}| = \sqrt{25+4+196} = 15$

$\therefore \; \frac{1}{15}(5\mathbf{i} - 2\mathbf{j} + 14\mathbf{k})$

b $|2\mathbf{i} + 11\mathbf{j} - 10\mathbf{k}| = \sqrt{4+121+100} = 15$

$\therefore \; \frac{10}{15}(2\mathbf{i} + 11\mathbf{j} - 10\mathbf{k}) = \frac{2}{3}(2\mathbf{i} + 11\mathbf{j} - 10\mathbf{k})$

c $|-5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}| = \sqrt{25+16+4} = \sqrt{45} = 3\sqrt{5}$

$\therefore \; \frac{20}{3\sqrt{5}}(-5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = \frac{4}{3}\sqrt{5}\,(-5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$

14 $\lambda^2 + 144 + 16 = 14^2 = 196$

$\lambda^2 = 36$

$\lambda = \pm\, 6$

15 **a** $= \begin{pmatrix}1\\3\\-1\end{pmatrix} + 2\begin{pmatrix}4\\-2\\1\end{pmatrix} = \begin{pmatrix}9\\-1\\1\end{pmatrix}$

b $= \begin{pmatrix}1\\3\\-1\end{pmatrix} - \begin{pmatrix}-2\\5\\-3\end{pmatrix} = \begin{pmatrix}3\\-2\\2\end{pmatrix}$

c $= \begin{pmatrix}1\\3\\-1\end{pmatrix} + \begin{pmatrix}4\\-2\\1\end{pmatrix} + \begin{pmatrix}-2\\5\\-3\end{pmatrix} = \begin{pmatrix}3\\6\\-3\end{pmatrix}$

d $= 2\begin{pmatrix}1\\3\\-1\end{pmatrix} - 3\begin{pmatrix}4\\-2\\1\end{pmatrix} + \begin{pmatrix}-2\\5\\-3\end{pmatrix} = \begin{pmatrix}-12\\17\\-8\end{pmatrix}$

16 **a** $-2\mathbf{i} + \lambda\mathbf{j} + \mu\mathbf{k} = -\tfrac{1}{2}(4\mathbf{i} + 2\mathbf{j} - 8\mathbf{k})$

$\therefore\ \lambda = -1,\ \mu = 4$

b $-2\mathbf{i} + \lambda\mathbf{j} + \mu\mathbf{k} = \tfrac{2}{5}(-5\mathbf{i} + 20\mathbf{j} - 10\mathbf{k})$

$\therefore\ \lambda = 8,\ \mu = -4$

17 **a** $2\mathbf{p} - \mathbf{q} = 2(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) - (-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$

$= 3\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$

$\therefore\ |\,2\mathbf{p} - \mathbf{q}\,| = \sqrt{9+36+36} = 9$

b $(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) + k(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$

$= l(2\mathbf{i} - 4\mathbf{j} - 7\mathbf{k})$

$\therefore\quad 1 - k = 2l \quad (1)$

$-2 + 2k = -4l \quad (2)$

$4 + 2k = -7l \quad (3)$

[(1) and (2) are the same equation]

$(2) - (3) \Rightarrow -6 = 3l$

$\therefore\ l = -2$

$\therefore\ k = 5$

18 **a** $\overrightarrow{AB} = (-4\mathbf{i} + \mathbf{j} + 8\mathbf{k}) - (-2\mathbf{i} + 7\mathbf{j} + 4\mathbf{k})$

$= -2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$

pos. vec of mid-point $= \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AB}$

$= (-2\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}) + \tfrac{1}{2}(-2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k})$

$= -3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

b $\overrightarrow{AC} = (6\mathbf{i} - 5\mathbf{j}) - (-2\mathbf{i} + 7\mathbf{j} + 4\mathbf{k})$

$= 8\mathbf{i} - 12\mathbf{j} - 4\mathbf{k}$

$\overrightarrow{AD} = \overrightarrow{OA} + \tfrac{3}{4}\overrightarrow{AC}$

$= (-2\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}) + \tfrac{3}{4}(8\mathbf{i} - 12\mathbf{j} - 4\mathbf{k})$

$= 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

19 **a** $(\lambda\mathbf{i} - 2\lambda\mathbf{j} + \mu\mathbf{k}) = k(2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$

$\therefore\quad \lambda = 2k \quad (1)$

$-2\lambda = -4k \quad (2)$

$\mu = -3k \quad (3)$

[(1) and (2) are the same equation]

$3\times(1) + 2\times(3) \quad \Rightarrow \quad 3\lambda + 2\mu = 0$

b $\lambda^2 + (-2\lambda)^2 + \mu^2 = (\,2\sqrt{29}\,)^2$

$5\lambda^2 + \mu^2 = 116$

$\mu = -\tfrac{3}{2}\lambda \quad \Rightarrow \quad 5\lambda^2 + \tfrac{9}{4}\lambda^2 = 116$

$\lambda^2 = 16$

$\lambda = \pm 4$

$\mu = -\tfrac{3}{2}\lambda$ and $\mu > 0\ \therefore\ \lambda = -4,\ \mu = 6$

20 **a** $\overrightarrow{BC} = \begin{pmatrix}6\\1\\-8\end{pmatrix} - \begin{pmatrix}12\\-7\\-4\end{pmatrix} = \begin{pmatrix}-6\\8\\-4\end{pmatrix}$

$\overrightarrow{OM} = \overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{BC}$

$= \begin{pmatrix}12\\-7\\-4\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix}-6\\8\\-4\end{pmatrix} = \begin{pmatrix}9\\-3\\-6\end{pmatrix}$

b $\overrightarrow{OM} = \tfrac{3}{2}\overrightarrow{OA}$

$\therefore\ \overrightarrow{OM}$ and $\overrightarrow{OA}$ are parallel

common point O

$\therefore$ O, A and M are collinear

21 **a** $d^2 = (9 - t)^2 + (1 + 2t)^2 + (5 - t)^2$

$= 81 - 18t + t^2 + 1 + 4t + 4t^2 + 25 - 10t + t^2$

$= 6t^2 - 24t + 107$

b $d^2 = 6(t^2 - 4t) + 107 = 6[(t - 2)^2 - 4] + 107$

$= 6(t - 2)^2 + 83$

$\therefore$ closest when $t = 2$

min. $d = \sqrt{83} = 9.11$ m (3sf)