C4 VECTORS — Answers - Worksheet E

1 **a** $1 + 2\lambda = 7 \quad \therefore \lambda = 3$

$p - 3\lambda = -1 \quad \therefore p = 8$

$-4 + q\lambda = -1 \quad \therefore q = 1$

b $|2\mathbf{i} + \mathbf{j} - 3\mathbf{k}| = \sqrt{4+1+9} = \sqrt{14}$

$|-4\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}| = \sqrt{16+25+4} = \sqrt{45}$

$(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}).(-4\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$

$= -8 + 5 + 6 = 3$

$\theta = \cos^{-1}\left|\frac{3}{\sqrt{14}\sqrt{45}}\right| = 83.1°$ (1dp)

2 **a** $\overrightarrow{AB} = \begin{pmatrix}5\\0\\-6\end{pmatrix} - \begin{pmatrix}1\\6\\4\end{pmatrix} = \begin{pmatrix}4\\-6\\-10\end{pmatrix}$

$\therefore \mathbf{r} = \begin{pmatrix}1\\6\\4\end{pmatrix} + s\begin{pmatrix}4\\-6\\-10\end{pmatrix}$

b $1 + 4s = 5 + t \quad (1)$

$6 - 6s = -5 - 4t \quad (2)$

$4\times(1) + (2) \Rightarrow 10 + 10s = 15$

$s = \frac{1}{2}$

$\therefore$ pos. vector of $C = \begin{pmatrix}3\\3\\-1\end{pmatrix}$

c pos. vector of mid-point of AB

$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$= \begin{pmatrix}1\\6\\4\end{pmatrix} + \frac{1}{2}\begin{pmatrix}4\\-6\\-10\end{pmatrix} = \begin{pmatrix}3\\3\\-1\end{pmatrix}$

$\therefore C$ is mid-point of AB

3 **a** $\overrightarrow{PQ} = (3\mathbf{i} + \mathbf{j}) - (5\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$

$= -2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$\therefore \mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \lambda(-2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$

b $5 - 2\lambda = 4 + 5\mu \quad (1)$

$-2 + 3\lambda = 6 - \mu \quad (2)$

$2 - 2\lambda = -1 + 3\mu \quad (3)$

$(1) - (3) \Rightarrow 3 = 5 + 2\mu$

$\mu = -1, \ \lambda = 3$

check (2) $\quad -2 + 3(3) = 6 - (-1)$

true $\therefore$ intersect

pos. vector of int. $= -\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$

c $|-2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}| = \sqrt{4+9+4} = \sqrt{17}$

$|5\mathbf{i} - \mathbf{j} + 3\mathbf{k}| = \sqrt{25+1+9} = \sqrt{35}$

$(-2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}).(5\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

$= -10 - 3 - 6 = -19$

$\theta = \cos^{-1}\left|\frac{-19}{\sqrt{17}\sqrt{35}}\right| = 38.8°$

4 **a** $5 + 2\lambda = 7 - \mu \quad (1)$

$-\lambda = -3 + \mu \quad (2)$

$1 + 2\lambda = 7 - 2\mu \quad (3)$

$(1) + (2) \Rightarrow 5 + \lambda = 4$

$\lambda = -1, \ \mu = 4$

check (3) $\quad 1 + 2(-1) = 7 - 2(4)$

true $\therefore$ intersect

pos. vector of int. $= 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

b diagonals bisect each other

let M be point of intersection

$\therefore \overrightarrow{AM} = (3\mathbf{i} + \mathbf{j} - \mathbf{k}) - (9\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})$

$= -6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$

$\overrightarrow{OC} = \overrightarrow{OA} + 2\overrightarrow{AM}$

$= (9\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}) + 2(-6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$

$= -3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$

c area of triangle $ABC = \frac{1}{2} \times 54 = 27$

$\overrightarrow{AC} = 2(-6\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) = 6(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

$|\overrightarrow{AC}| = 6\sqrt{4+1+4} = 18$

let distance of B from $l_1 = d$

$\therefore \frac{1}{2} \times 18 \times d = 27$

$d = 3$

5 **a** $\overrightarrow{AB} = (2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$

$= -2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

$\therefore\ \mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + \lambda(-2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$

b $\mathbf{r} = 4\mathbf{i} - 7\mathbf{j} - \mathbf{k} + \mu(6\mathbf{j} - 2\mathbf{k})$

c $-7 + 6\mu = 2 \Rightarrow \mu = \frac{3}{2}$

sub. $\mu = \frac{3}{2}$ in l_2

$\mathbf{r} = 4\mathbf{i} - 7\mathbf{j} - \mathbf{k} + \frac{3}{2}(6\mathbf{j} - 2\mathbf{k})$

$= 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}\ \therefore A$ lies on l_2

d $|-2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}| = \sqrt{4+9+36} = 7$

$|6\mathbf{j} - 2\mathbf{k}| = \sqrt{36+4} = \sqrt{40}$

$(-2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}).(6\mathbf{j} - 2\mathbf{k})$

$= 0 - 18 - 12 = -30$

$\theta = \cos^{-1}\left|\frac{-30}{7\sqrt{40}}\right| = 47.3°$ (1dp)

6 **a** $\overrightarrow{AB} = \begin{pmatrix}4\\1\\-8\end{pmatrix} - \begin{pmatrix}5\\-1\\-10\end{pmatrix} = \begin{pmatrix}-1\\2\\2\end{pmatrix}$

$\therefore\ \mathbf{r} = \begin{pmatrix}5\\-1\\-10\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\2\end{pmatrix}$

b $5 - \lambda = 0 \Rightarrow \lambda = 5$

sub. $\lambda = 5$ in l

$\mathbf{r} = \begin{pmatrix}0\\9\\0\end{pmatrix}\ \therefore C\ (0, 9, 0)$

c $\overrightarrow{OD} = \begin{pmatrix}5-\lambda\\-1+2\lambda\\-10+2\lambda\end{pmatrix}$

$\overrightarrow{OD}\,.\begin{pmatrix}-1\\2\\2\end{pmatrix} = 0$

$-(5 - \lambda) + 2(-1 + 2\lambda) + 2(-10 + 2\lambda) = 0$

$9\lambda - 27 = 0$

$\lambda = 3,\ \overrightarrow{OD} = \begin{pmatrix}2\\5\\-4\end{pmatrix}$

$\therefore\ D\ (2, 5, -4)$

d $OD = \sqrt{4+25+16} = \sqrt{45} = 3\sqrt{5}$

$CD = \sqrt{4+16+16} = 6$

area $= \frac{1}{2} \times 6 \times 3\sqrt{5} = 9\sqrt{5}$

7 **a** $-6 + 4s = 6 \Rightarrow s = 3$

sub. $s = 3$ in l_1

$$\mathbf{r} = \begin{pmatrix} 1 \\ -6 \\ -2 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ -5 \end{pmatrix}$$

$\therefore P\ (1, 6, -5)$ lies on l_1

b $1 = 4 + 3t \quad \Rightarrow t = -1$

sub. $t = -1$ in l_2

$$\mathbf{r} = \begin{pmatrix} 4 \\ -4 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$$

$$\overrightarrow{OQ} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

c $PQ = \sqrt{0+64+4} = \sqrt{68} = 2\sqrt{17}$

$$\left|\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}\right| = \sqrt{9+4+4} = \sqrt{17}$$

$$\therefore\ \overrightarrow{OR} = \overrightarrow{OQ} \pm 2\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 2 \\ -7 \end{pmatrix} \text{ or } \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix}$$

8 **a** $\overrightarrow{AB} = (4\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}) - (4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$

$= \mathbf{j} - 4\mathbf{k}$

$\therefore\ \mathbf{r} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k} + \lambda(\mathbf{j} - 4\mathbf{k})$

b $4 = 1 + \mu$ (1)

$5 + \lambda = 5 + \mu$ (2)

$6 - 4\lambda = -3 - \mu$ (3)

(1) $\Rightarrow$ $\mu = 3$

sub. (2) $\Rightarrow$ $\lambda = 3$

check (3) $6 - 4(3) = -3 - (3)$

true $\therefore$ intersect

pos. vector of int. $= 4\mathbf{i} + 8\mathbf{j} - 6\mathbf{k}$

c $|(\mathbf{j} - 4\mathbf{k})| = \sqrt{1+16} = \sqrt{17}$

$|(\mathbf{i} + \mathbf{j} - \mathbf{k})| = \sqrt{1+1+1} = \sqrt{3}$

$(\mathbf{j} - 4\mathbf{k}).(\mathbf{i} + \mathbf{j} - \mathbf{k}) = 0 + 1 + 4 = 5$

$\theta = \cos^{-1}\left|\frac{5}{\sqrt{3}\sqrt{17}}\right| = 45.6°$ (1dp)

d let closest point be C

$\overrightarrow{OC} = (1 + \mu)\mathbf{i} + (5 + \mu)\mathbf{j} + (-3 - \mu)\mathbf{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (-3 + \mu)\mathbf{i} + \mu\mathbf{j} + (-9 - \mu)\mathbf{k}$

AC must be perpendicular to l_2

$\therefore\ \overrightarrow{AC}.(\mathbf{i} + \mathbf{j} - \mathbf{k}) = 0$

$(-3 + \mu) + \mu - (-9 - \mu) = 0$

$\mu = -2$

$\therefore\ \overrightarrow{OC} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$