5

After completing this chapter you should be able to:

- know the difference between a scalar and a vector quantity
- draw vector diagrams
- perform simple vector arithmetic and know the definition of a unit vector
- use position vectors to describe points in two or three dimensions
- use Cartesian coordinates in three dimensions, including finding the distance between two points
- find a scalar product and know how it can be used to find the angle between two vectors
- write down the equation of a line in vector form
- determine whether or not two given straight lines intersect (if they do intersect you should be able to find the point of intersection)
- find the angle between two intersecting lines.

Vectors

Vectors are important when studying science and engineering to higher levels. They can be used to represent any quantity that has both a magnitude and a direction.

The flight path of an aeroplane can be described by a vector.

5.1 You need to know the difference between a scalar and a vector, and how to write down vectors and draw vector diagrams.

A scalar quantity can be described by using a single number (the *magnitude* or *size*).

■ **A vector quantity has both magnitude and direction.**

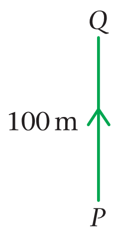
For example:

Scalar: The distance from P to Q is 100 metres.

Distance is a scalar.

Vector: From P to Q you go 100 metres north.

This is called the displacement from P to Q. Displacement is a vector.

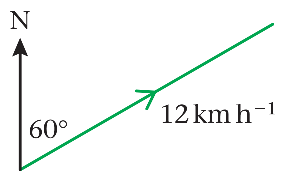


Scalar: A ship is sailing at $12\ \text{km h}^{-1}$.

Speed is a scalar.

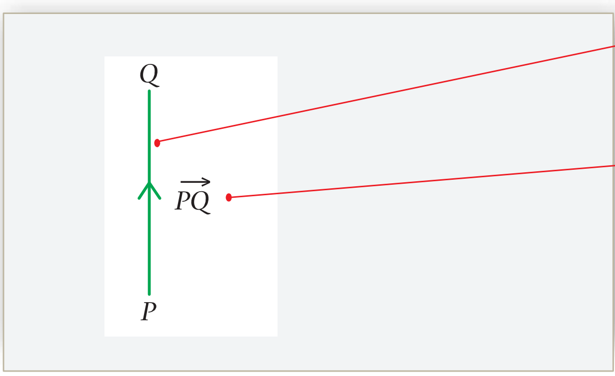
Vector: A ship is sailing at $12\ \text{km h}^{-1}$, on a bearing of 060°.

This is called the velocity of the ship. Velocity is a vector.



Example 1

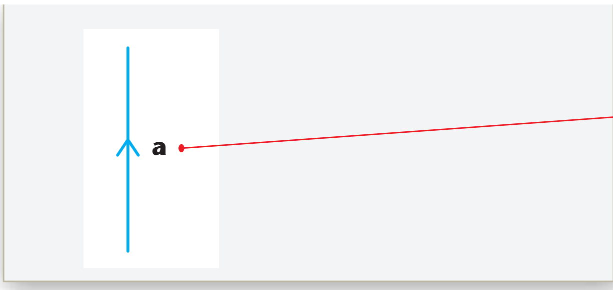
Show on a diagram the displacement vector from P to Q, where Q is 500 m due north of P.



This is called a 'directed line segment'. The direction of the arrow shows the direction of the vector.

The vector is written as $\overrightarrow{PQ}$.

The length of the line segment PQ represents distance 500 m. In accurate diagrams a scale could be used (e.g. 1 cm represents 100 m).

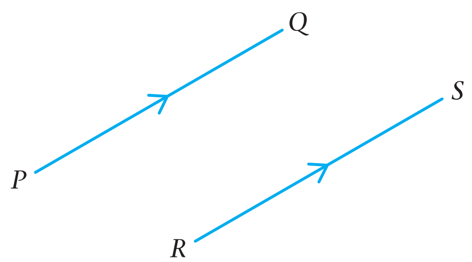


Sometimes, instead of using the endpoints *P* and *Q*, a small (lower case) letter is used.

In print, the small letter will be in **bold type**. In writing, you should underline the small letter to show it is a vector:

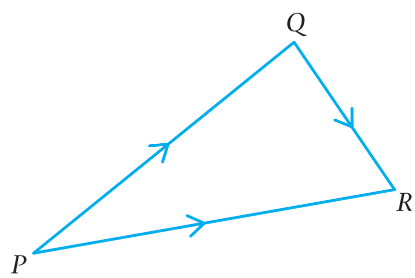
$\underline{a}$ or $\underset{\sim}{a}$

- **Vectors that are equal have both the same magnitude and the same direction.**



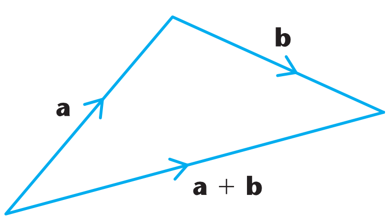
Here $\overrightarrow{PQ} = \overrightarrow{RS}$.

- **Two vectors are added using the 'triangle law'.**



Hint: Think of displacement vectors.
If you travel from *P* to *Q*, then from *Q* to *R*, the resultant journey is *P* to *R*:

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

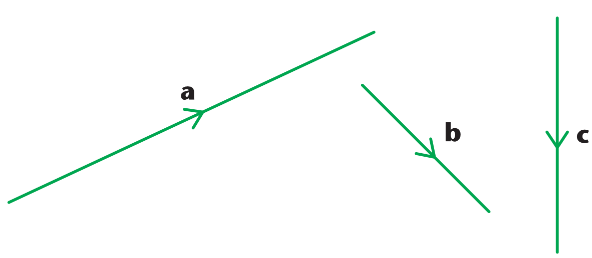


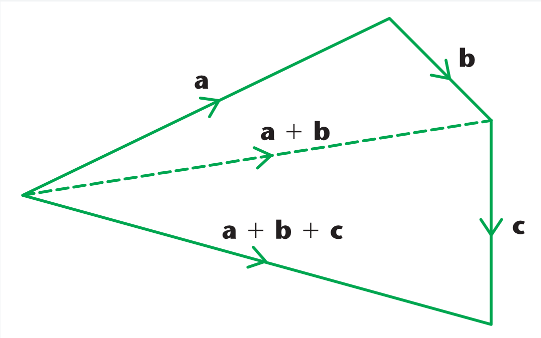
When you add the vectors **a** and **b**, the resultant vector **a** + **b** goes from 'the start of **a** to the finish of **b**'.

This is sometimes called the triangle law for vector addition.

Example 2

The diagram shows the vectors **a**, **b** and **c**. Draw another diagram to illustrate the vector addition **a** + **b** + **c**.

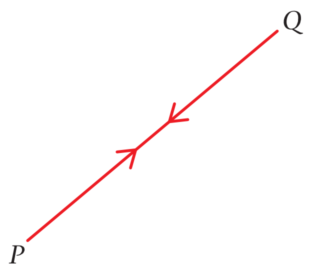




First use the triangle law for **a** + **b**, then use it again for (**a** + **b**) + **c**.

The resultant goes from the start of **a** to the finish of **c**.

- **Adding the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ gives the zero vector 0. $\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}$**



Hint: If you travel from P to Q, then back from Q to P, you are back where you started, so your displacement is zero.

The zero displacement vector is **0**.
It is printed in bold type, or underlined in written work.

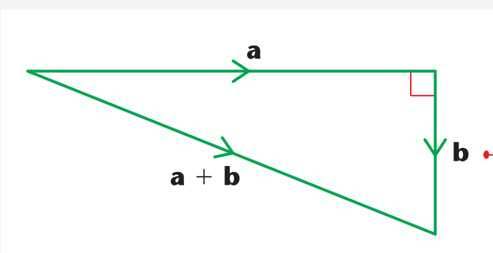
You can also write $\overrightarrow{QP}$ as $-\overrightarrow{PQ}$.

So $\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}$ or $\overrightarrow{PQ} - \overrightarrow{PQ} = \mathbf{0}$.

- **The modulus of a vector is another name for its magnitude.**
  - **The modulus of the vector a is written as $|\mathbf{a}|$.**
  - **The modulus of the vector $\overrightarrow{PQ}$ is written as $|\overrightarrow{PQ}|$.**

Example 3

The vector **a** is directed due east and $|\mathbf{a}| = 12$. The vector **b** is directed due south and $|\mathbf{b}| = 5$. Find $|\mathbf{a} + \mathbf{b}|$.



Use the triangle law for adding the vectors **a** and **b**.

$|\mathbf{a} + \mathbf{b}|^2 = 12^2 + 5^2 = 169$

$|\mathbf{a} + \mathbf{b}| = 13$

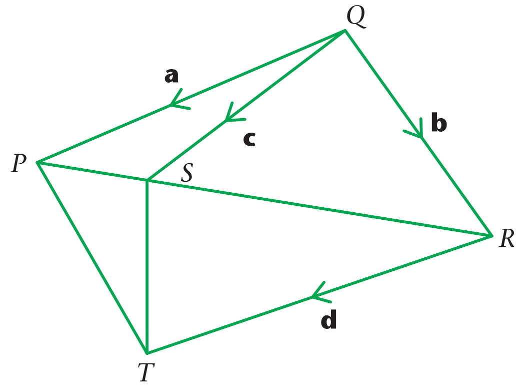
Use Pythagoras' Theorem.

Example 4

In the diagram, $\overrightarrow{QP} = \mathbf{a}$, $\overrightarrow{QR} = \mathbf{b}$, $\overrightarrow{QS} = \mathbf{c}$ and $\overrightarrow{RT} = \mathbf{d}$.

Find in terms of **a**, **b**, **c** and **d**:

a $\overrightarrow{PS}$ **b** $\overrightarrow{RP}$ **c** $\overrightarrow{PT}$ **d** $\overrightarrow{TS}$



a $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS} = -\mathbf{a} + \mathbf{c}$
$= \mathbf{c} - \mathbf{a}$

Add vectors using △*PQS*.

b $\overrightarrow{RP} = \overrightarrow{RQ} + \overrightarrow{QP} = -\mathbf{b} + \mathbf{a}$
$= \mathbf{a} - \mathbf{b}$

Add vectors using △*RQP*.

c $\overrightarrow{PT} = \overrightarrow{PR} + \overrightarrow{RT} = (\mathbf{b} - \mathbf{a}) + \mathbf{d}$
$= \mathbf{b} + \mathbf{d} - \mathbf{a}$

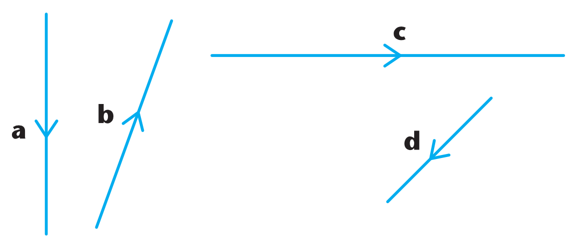
Add vectors using △*PRT*.
Use $\overrightarrow{PR} = -\overrightarrow{RP} = -(\mathbf{a} - \mathbf{b}) = \mathbf{b} - \mathbf{a}$.

d $\overrightarrow{TS} = \overrightarrow{TR} + \overrightarrow{RS} = -\mathbf{d} + (\overrightarrow{RQ} + \overrightarrow{QS})$
$= -\mathbf{d} + (-\mathbf{b} + \mathbf{c})$
$= \mathbf{c} - \mathbf{b} - \mathbf{d}$

Add vectors using △*TRS* and also △*RQS*.

Exercise 5A

1 The diagram shows the vectors **a**, **b**, **c** and **d**. Draw a diagram to illustrate the vector addition **a** + **b** + **c** + **d**.

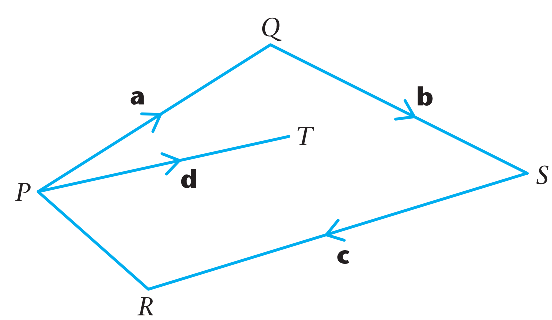


2 The vector **a** is directed due north and $|\mathbf{a}| = 24$. The vector **b** is directed due west and $|\mathbf{b}| = 7$. Find $|\mathbf{a} + \mathbf{b}|$.

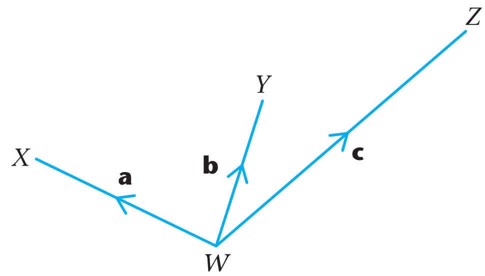
3 The vector **a** is directed north-east and $|\mathbf{a}| = 20$. The vector **b** is directed south-east and $|\mathbf{b}| = 13$. Find $|\mathbf{a} + \mathbf{b}|$.

4 In the diagram, $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{QS} = \mathbf{b}$, $\overrightarrow{SR} = \mathbf{c}$ and $\overrightarrow{PT} = \mathbf{d}$. Find in terms of **a**, **b**, **c** and **d**:

a $\overrightarrow{QT}$
b $\overrightarrow{PR}$
c $\overrightarrow{TS}$
d $\overrightarrow{TR}$



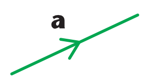
5 In the diagram, $\overrightarrow{WX} = \mathbf{a}$, $\overrightarrow{WY} = \mathbf{b}$ and $\overrightarrow{WZ} = \mathbf{c}$. It is given that $\overrightarrow{XY} = \overrightarrow{YZ}$.
Prove that $\mathbf{a} + \mathbf{c} = 2\mathbf{b}$.
($2\mathbf{b}$ is equivalent to $\mathbf{b} + \mathbf{b}$).

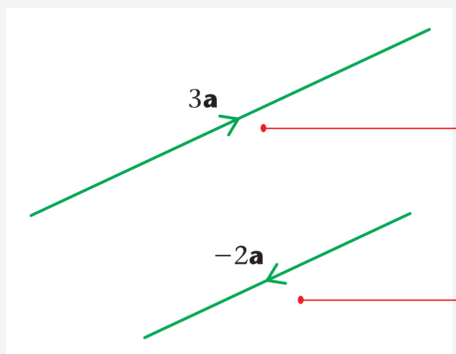


5.2 You need to be able to perform simple vector arithmetic, and to know the definition of a unit vector.

Example 5

The diagram shows the vector **a**. Draw diagrams to illustrate the vectors $3\mathbf{a}$ and $-2\mathbf{a}$.





Vector $3\mathbf{a}$ is $\mathbf{a} + \mathbf{a} + \mathbf{a}$, so is in the same direction as **a** with 3 times its magnitude. The vector **a** has been multiplied by the scalar 3 (a scalar multiple).

Vector $-2\mathbf{a}$ is $-\mathbf{a} - \mathbf{a}$, so is in the opposite direction to **a** with 2 times its magnitude.

■ **Any vector parallel to the vector a may be written as $\lambda\mathbf{a}$, where λ is a non-zero scalar.**

Example 6

Show that the vectors $6\mathbf{a} + 8\mathbf{b}$ and $9\mathbf{a} + 12\mathbf{b}$ are parallel.

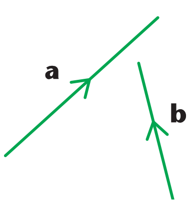
$9\mathbf{a} + 12\mathbf{b}$

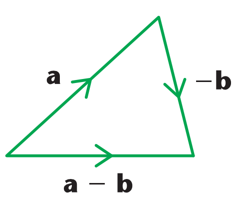
$= \frac{3}{2}(6\mathbf{a} + 8\mathbf{b})$

$\therefore$ the vectors are parallel.

Here $\lambda = \frac{3}{2}$.

■ **Subtracting a vector is equivalent to 'adding a negative vector', so $\mathbf{a} - \mathbf{b}$ is defined to be $\mathbf{a} + (-\mathbf{b})$.**





Hint: To subtract **b**, you reverse the direction of **b** then add.

■ **A unit vector is a vector which has magnitude (or modulus) 1 unit.**

Example 7

The vector **a** has magnitude 20 units. Write down a unit vector that is parallel to **a**.

The unit vector is $\frac{\mathbf{a}}{20}$ or $\frac{1}{20}\mathbf{a}$.

Divide **a** by the magnitude. In general, the unit vector is $\frac{\mathbf{a}}{|\mathbf{a}|}$.

■ **If $\lambda\mathbf{a} + \mu\mathbf{b} = \alpha\mathbf{a} + \beta\mathbf{b}$, and the non-zero vectors a and b are not parallel, then $\lambda = \alpha$ and $\mu = \beta$.**

The above result can be shown as follows:
$\lambda\mathbf{a} + \mu\mathbf{b} = \alpha\mathbf{a} + \beta\mathbf{b}$ can be written as $(\lambda - \alpha)\mathbf{a} = (\beta - \mu)\mathbf{b}$, but two vectors cannot be equal unless they are parallel or zero.
Since **a** and **b** are not parallel or zero, $(\lambda - \alpha) = 0$ and $(\beta - \mu) = 0$, so $\lambda = \alpha$ and $\beta = \mu$.

Example 8

Given that $5\mathbf{a} - 4\mathbf{b} = (2s + t)\mathbf{a} + (s - t)\mathbf{b}$, where **a** and **b** are non-zero, non-parallel vectors, find the values of the scalars s and t.

$2s + t = 5$
$s - t = -4$

Equate the **a** and **b** coefficients.

$3s = 1$

Solve simultaneously (add).

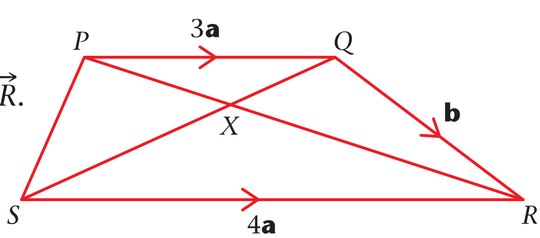
$s = \frac{1}{3}$
$t = 5 - 2s = 4\frac{1}{3}$
So $s = \frac{1}{3}$ and $t = 4\frac{1}{3}$.

Example 9

In the diagram, $\overrightarrow{PQ} = 3\mathbf{a}$, $\overrightarrow{QR} = \mathbf{b}$, $\overrightarrow{SR} = 4\mathbf{a}$ and $\overrightarrow{PX} = k\overrightarrow{PR}$.
Find, in terms of **a**, **b** and k:

a $\overrightarrow{PS}$ **b** $\overrightarrow{PX}$ **c** $\overrightarrow{SQ}$ **d** $\overrightarrow{SX}$

Use the fact that X lies on SQ to find the value of k.



a $\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS} = \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS}$
$= 3\mathbf{a} + \mathbf{b} - 4\mathbf{a} = \mathbf{b} - \mathbf{a}$

Add using the triangle law.

b $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = 3\mathbf{a} + \mathbf{b}$
$\overrightarrow{PX} = k\overrightarrow{PR} = k(3\mathbf{a} + \mathbf{b})$

Find $\overrightarrow{PR}$ and use $\overrightarrow{PX} = k\overrightarrow{PR}$.

c $\overrightarrow{SQ} = 4\mathbf{a} - \mathbf{b}$ — Use the triangle law on $\triangle SRQ$.

d $\overrightarrow{SX} = \overrightarrow{SP} + \overrightarrow{PX}$

$= -(\mathbf{b} - \mathbf{a}) + k(3\mathbf{a} + \mathbf{b})$ — Use $\overrightarrow{SP} = -\overrightarrow{PS}$, and the answers to parts (a) and (b).

$= -\mathbf{b} + \mathbf{a} + k(3\mathbf{a} + \mathbf{b})$

$= (3k + 1)\mathbf{a} + (k - 1)\mathbf{b}$

X lies on SQ, so $\overrightarrow{SQ}$ and $\overrightarrow{SX}$ are parallel.

$(3k + 1)\mathbf{a} + (k - 1)\mathbf{b} = \lambda(4\mathbf{a} - \mathbf{b})$ — Use the fact that, for parallel vectors, one is a scalar multiple of the other.

$(3k + 1)\mathbf{a} + (k - 1)\mathbf{b} = 4\lambda\mathbf{a} - \lambda\mathbf{b}$

So $(3k + 1) = 4\lambda$ and $(k - 1) = -\lambda$ — **a** and **b** are non-parallel and non-zero, so equate coefficients.

$(3k + 1) = 4(1 - k)$

$k = \frac{3}{7}$ — Eliminate λ and solve for k.

Exercise 5B

1 In the triangle PQR, $\overrightarrow{PQ} = 2\mathbf{a}$ and $\overrightarrow{QR} = 2\mathbf{b}$. The mid-point of PR is M. Find, in terms of **a** and **b**:

a $\overrightarrow{PR}$ **b** $\overrightarrow{PM}$ **c** $\overrightarrow{QM}$.

2 $ABCD$ is a trapezium with AB parallel to DC and $DC = 3AB$. M is the mid-point of DC, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$. Find, in terms of **a** and **b**:

a $\overrightarrow{AM}$ **b** $\overrightarrow{BD}$ **c** $\overrightarrow{MB}$ **d** $\overrightarrow{DA}$.

3 In each part, find whether the given vector is parallel to $\mathbf{a} - 3\mathbf{b}$:

a $2\mathbf{a} - 6\mathbf{b}$ **b** $4\mathbf{a} - 12\mathbf{b}$ **c** $\mathbf{a} + 3\mathbf{b}$

d $3\mathbf{b} - \mathbf{a}$ **e** $9\mathbf{b} - 3\mathbf{a}$ **f** $\frac{1}{2}\mathbf{a} - \frac{2}{3}\mathbf{b}$

4 The non-zero vectors **a** and **b** are not parallel. In each part, find the value of λ and the value of μ:

a $\mathbf{a} + 3\mathbf{b} = 2\lambda\mathbf{a} - \mu\mathbf{b}$

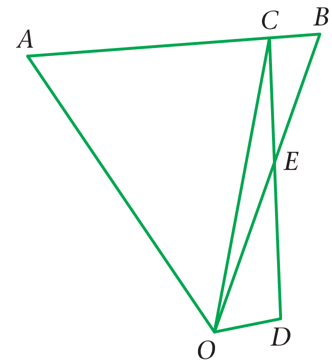
b $(\lambda + 2)\mathbf{a} + (\mu - 1)\mathbf{b} = 0$

c $4\lambda\mathbf{a} - 5\mathbf{b} - \mathbf{a} + \mu\mathbf{b} = 0$

d $(1 + \lambda)\mathbf{a} + 2\lambda\mathbf{b} = \mu\mathbf{a} + 4\mu\mathbf{b}$

e $(3\lambda + 5)\mathbf{a} + \mathbf{b} = 2\mu\mathbf{a} + (\lambda - 3)\mathbf{b}$

5 In the diagram, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and C divides AB in the ratio 5:1.



a Write down, in terms of **a** and **b**, expressions for $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{OC}$.

Given that $\overrightarrow{OE} = \lambda\mathbf{b}$, where λ is a scalar:

b Write down, in terms of **a**, **b** and λ, an expression for $\overrightarrow{CE}$.

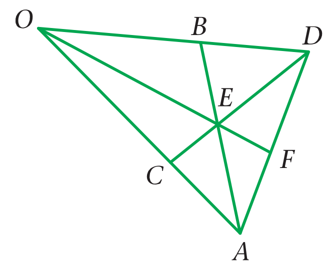
Given that $\overrightarrow{OD} = \mu(\mathbf{b} - \mathbf{a})$, where μ is a scalar:

c Write down, in terms of **a**, **b**, λ and μ, an expression for $\overrightarrow{ED}$.

Given also that E is the mid-point of CD:

d Deduce the values of λ and μ. *E*

6 In the diagram $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $3\overrightarrow{OC} = 2\overrightarrow{OA}$ and $4\overrightarrow{OD} = 7\overrightarrow{OB}$.



The line DC meets the line AB at E.

a Write down, in terms of **a** and **b**, expressions for
i $\overrightarrow{AB}$ **ii** $\overrightarrow{DC}$

Given that $\overrightarrow{DE} = \lambda\overrightarrow{DC}$ and $\overrightarrow{EB} = \mu\overrightarrow{AB}$ where λ and μ are constants:

b Use $\triangle EBD$ to form an equation relating to **a**, **b**, λ and μ.

Hence:

c Show that $\lambda = \frac{9}{13}$. **d** Find the exact value of μ.

e Express $\overrightarrow{OE}$ in terms of **a** and **b**.

The line $\overrightarrow{OE}$ produced meets the line AD at F.
Given that $\overrightarrow{OF} = k\overrightarrow{OE}$ where k is a constant and that $\overrightarrow{AF} = \frac{1}{10}(7\mathbf{b} - 4\mathbf{a})$:

f Find the value of k. *E*

7 In $\triangle OAB$, P is the mid-point of AB and Q is the point on OP such that $Q = \frac{3}{4}P$. Given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, find, in terms of **a** and **b**:

a $\overrightarrow{AB}$ **b** $\overrightarrow{OP}$ **c** $\overrightarrow{OQ}$ **d** $\overrightarrow{AQ}$

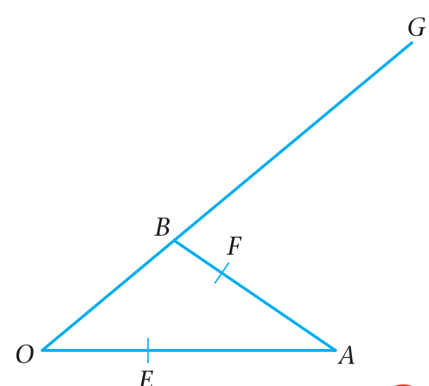
The point R on OB is such that $OR = kOB$, where $0 < k < 1$.

e Find, in terms of **a**, **b** and k, the vector $\overrightarrow{AR}$.

Given that AQR is a straight line:

f Find the ratio in which Q divides AR and the value of k. *E*

8 In the figure $OE : EA = 1 : 2$, $AF : FB = 3 : 1$ and $OG : OB = 3 : 1$. The vector $\overrightarrow{OA} = \mathbf{a}$ and the vector $\overrightarrow{OB} = \mathbf{b}$.



Find, in terms of **a**, **b** or **a** and **b**, expressions for:

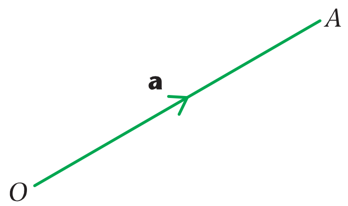
a $\overrightarrow{OE}$ **b** $\overrightarrow{OF}$ **c** $\overrightarrow{EF}$
d $\overrightarrow{BG}$ **e** $\overrightarrow{FB}$ **f** $\overrightarrow{FG}$

g Use your results in **c** and **f** to show that the points E, F and G are collinear and find the ratio $EF{:}FG$.

h Find $\overrightarrow{EB}$ and $\overrightarrow{AG}$ and hence prove that EB is parallel to AG. *E*

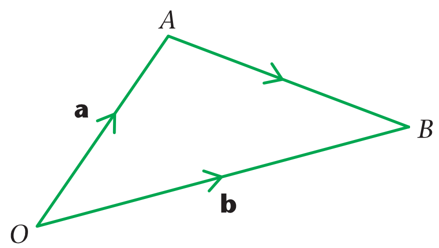
5.3 You need to be able to use vectors to describe the position of a point in two or three dimensions.

■ **The position vector of a point A is the vector $\overrightarrow{OA}$, where O is the origin. $\overrightarrow{OA}$ is usually written as vector $\mathbf{a}$.**



$\overrightarrow{OA} = \mathbf{a}$

■ **$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of A and B respectively.**

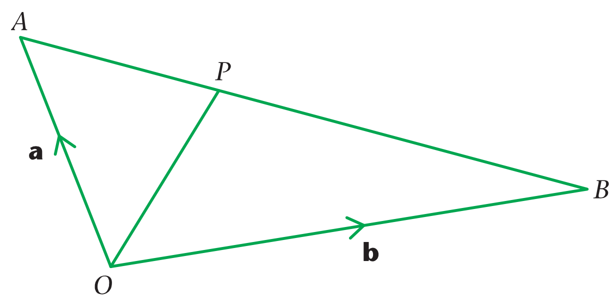


Hint: Use the triangle law to give
$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b}$
So $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

Example 10

In the diagram the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively (referred to the origin O). The point P divides AB in the ratio $1:2$.

Find the position vector of P.



$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ — $\overrightarrow{OP}$ is the position vector of P.

$\overrightarrow{AP} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$ — Use the 1:2 ratio (AP is one third of AB).

$\overrightarrow{OP} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a})$

$\overrightarrow{OP} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$ — You could write $\mathbf{p} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$.

Exercise 5C

1 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively (referred to the origin O).
The point P divides AB in the ratio 1:5.
Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of P.

2 The points A, B and C have position vectors **a**, **b** and **c** respectively (referred to the origin O). The point P is the mid-point of AB.
Find, in terms of **a**, **b** and **c**, the vector $\overrightarrow{PC}$.

3 $OABCDE$ is a regular hexagon. The points A and B have position vectors **a** and **b** respectively, referred to the origin O.
Find, in terms of **a** and **b**, the position vectors of C, D and E.

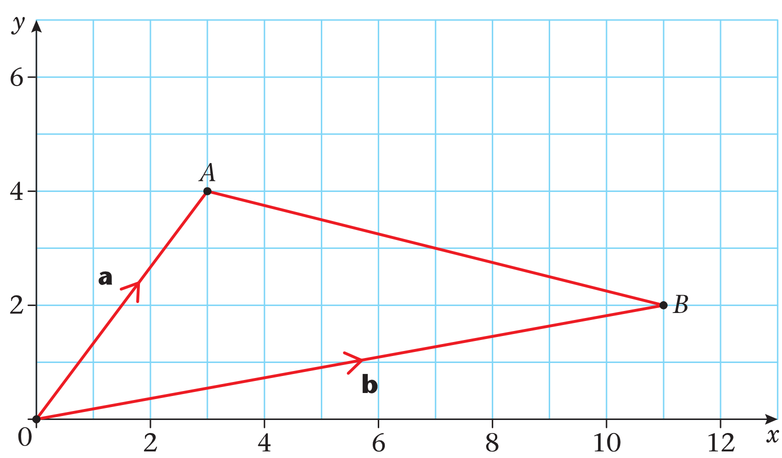
5.4 You need to know how to write down and use the Cartesian components of a vector in two dimensions.

- **The vectors i and j are unit vectors parallel to the x-axis and the y-axis, and in the direction of x increasing and y increasing, respectively.**

Example 11

The points A and B in the diagram have coordinates (3, 4) and (11, 2) respectively.
Find, in terms of **i** and **j**:

a the position vector of A **b** the position vector of B **c** the vector $\overrightarrow{AB}$



a $\mathbf{a} = \overrightarrow{OA} = 3\mathbf{i} + 4\mathbf{j}$

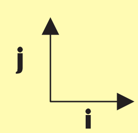
b $\mathbf{b} = \overrightarrow{OB} = 11\mathbf{i} + 2\mathbf{j}$

c $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$= (11\mathbf{i} + 2\mathbf{j}) - (3\mathbf{i} + 4\mathbf{j})$

$= 8\mathbf{i} - 2\mathbf{j}$

i goes 1 unit 'across', **j** goes 1 unit 'up':



You can see from the diagram that the vector $\overrightarrow{AB}$ goes 8 units 'across' and 2 units 'down'.

- **You can write a vector with Cartesian components as a column matrix:**

$$x\mathbf{i} + y\mathbf{j} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Hint: This standard notation is easy to read and also avoids the need to write out lengthy expressions with **i** and **j** terms.

Example 12

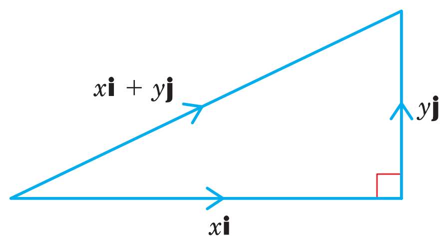
Given that $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$, $\mathbf{b} = 12\mathbf{i} - 10\mathbf{j}$ and $\mathbf{c} = -3\mathbf{i} + 9\mathbf{j}$, find $\mathbf{a} + \mathbf{b} + \mathbf{c}$, using column matrix notation in your working.

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 12 \\ -10 \end{pmatrix} + \begin{pmatrix} -3 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} 11 \\ 4 \end{pmatrix}$$

Add the numbers in the top line to get 11 (the x component), and the bottom line to get 4 (the y component). This is $11\mathbf{i} + 4\mathbf{j}$.

- **The modulus (or magnitude) of $x\mathbf{i} + y\mathbf{j}$ is $\sqrt{x^2 + y^2}$**



Hint: From Pythagoras' Theorem, the magnitude of $x\mathbf{i} + y\mathbf{j}$, represented by the hypotenuse, is $\sqrt{x^2 + y^2}$.

Example 13

The vector **a** is equal to $5\mathbf{i} - 12\mathbf{j}$.
Find $|\mathbf{a}|$, and find a unit vector in the same direction as **a**.

$$|\mathbf{a}| = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13$$

Unit vector is $\dfrac{\mathbf{a}}{|\mathbf{a}|}$

Look back to Section 5.2.

$$= \frac{5\mathbf{i} - 12\mathbf{j}}{13}$$

$$= \tfrac{1}{13}(5\mathbf{i} - 12\mathbf{j})$$

$$\text{or } \tfrac{5}{13}\mathbf{i} - \tfrac{12}{13}\mathbf{j}$$

$$\text{or } \tfrac{1}{13}\begin{pmatrix} 5 \\ -12 \end{pmatrix}$$

Example 14

Given that $\mathbf{a} = 5\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = -2\mathbf{i} - 4\mathbf{j}$, find the exact value of $|2\mathbf{a} + \mathbf{b}|$.

$$2\mathbf{a} + \mathbf{b} = 2\begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ -2 \end{pmatrix}$$

$$|2\mathbf{a} + \mathbf{b}| = \sqrt{8^2 + (-2)^2}$$

$$= \sqrt{68}$$

$$= \sqrt{4}\sqrt{17}$$

$$= 2\sqrt{17}$$

You must give the answer as a surd because the question asks for an exact answer.

Exercise 5D

1 Given that $\mathbf{a} = 9\mathbf{i} + 7\mathbf{j}$, $\mathbf{b} = 11\mathbf{i} - 3\mathbf{j}$ and $\mathbf{c} = -8\mathbf{i} - \mathbf{j}$, find:

a $\mathbf{a} + \mathbf{b} + \mathbf{c}$

b $2\mathbf{a} - \mathbf{b} + \mathbf{c}$

c $2\mathbf{b} + 2\mathbf{c} - 3\mathbf{a}$

(Use column matrix notation in your working.)

2 The points A, B and C have coordinates $(3, -1)$, $(4, 5)$ and $(-2, 6)$ respectively, and O is the origin.

Find, in terms of $\mathbf{i}$ and $\mathbf{j}$:

a the position vectors of A, B and C

b $\overrightarrow{AB}$

c $\overrightarrow{AC}$

Find, in surd form:

d $|\overrightarrow{OC}|$

e $|\overrightarrow{AB}|$

f $|\overrightarrow{AC}|$

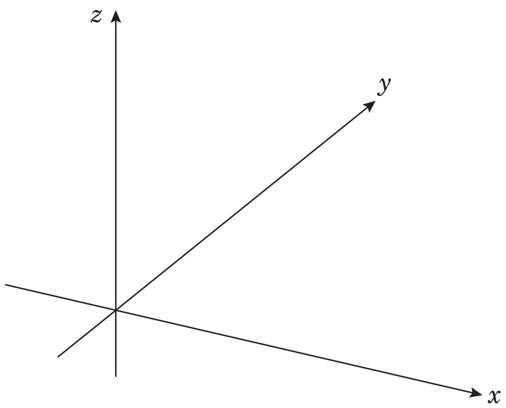
3 Given that $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} - 12\mathbf{j}$, $\mathbf{c} = -7\mathbf{i} + 24\mathbf{j}$ and $\mathbf{d} = \mathbf{i} - 3\mathbf{j}$, find a unit vector in the direction of **a**, **b**, **c** and **d**.

4 Given that $\mathbf{a} = 5\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \lambda\mathbf{i} + 3\mathbf{j}$, and that $|3\mathbf{a} + \mathbf{b}| = 10$, find the possible values of λ.

5.5 You need to know how to use Cartesian coordinates in three dimensions.

Cartesian coordinate axes in three dimensions are usually called x, y and z axes, each being at right angles to each of the others.

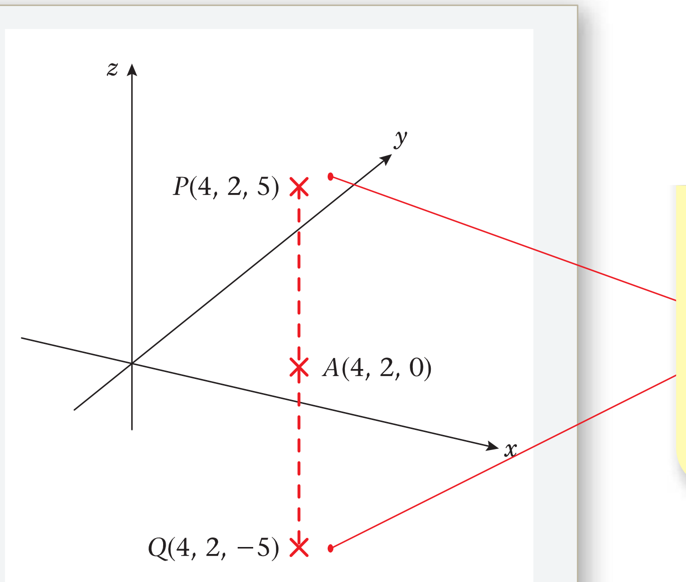
The coordinates of a point in three dimensions are written as (x, y, z).



Hint: To visualise this, think of the x and y axes being drawn on a flat surface and the z axis sticking up from the surface.

Example 15

Find the distance between the points $P(4, 2, 5)$ and $Q(4, 2, -5)$.



The point $A(4, 2, 0)$ is on the 'flat surface' (the xy plane).

(4, 2, 5) is 5 units 'above the surface'.

(4, 2, −5) is 5 units 'below the surface'.

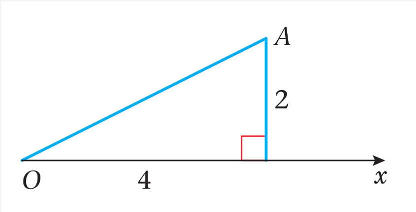
So the line joining these 2 points is parallel to the z-axis.

The distance between the points (4, 2, 5) and (4, 2, −5) is $5 + 5 = 10$ units.

Example 16

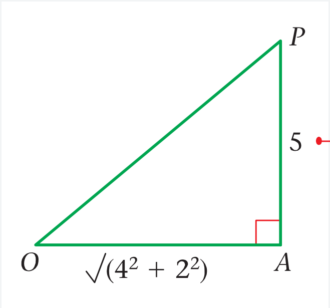
Find the distance from the origin to the point $P(4, 2, 5)$.

Let A be the point $(4, 2, 0)$.



$OA^2 = 4^2 + 2^2$ — Use Pythagoras' Theorem in the xy plane.

$OA = \sqrt{(4^2 + 2^2)}$



Next look at $\triangle OAP$, with OA on the xy plane and AP parallel to the z-axis.

$OP = \sqrt{(OA^2 + 5^2)}$ — Use Pythagoras' Theorem again.

$OP = \sqrt{(4^2 + 2^2 + 5^2)}$ — Notice this method just gives the three-dimensional version of Pythagoras' Theorem.

$OP = \sqrt{45} = \sqrt{9}\ \sqrt{5} = 3\sqrt{5}.$

■ **The distance from the origin to the point (x, x, z) is $\sqrt{x^2 + y^2 + z^2}$.**

Pythagoras' Theorem in three dimensions.

Example 17

Find the distance from the origin to the point $P(4, -7, -1)$.

$OA = \sqrt{4^2 + (-7)^2 + (-1)^2}$ — Straight from the formula.

$OA = \sqrt{16 + 49 + 1} = \sqrt{66}$

$= 8.12$ (2 d.p.)

■ **The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.**

This is the three-dimensional version of the formula $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example 18

Find the distance between the points $A(1, 3, 4)$ and $B(8, 6, -5)$, giving your answer to 1 decimal place.

$$AB = \sqrt{(1-8)^2 + (3-6)^2 + (4-(-5))^2}$$

Straight from the formula.

$$= \sqrt{(-7)^2 + (-3)^2 + (9)^2}$$

$$= \sqrt{139} = 11.8 \text{ (1 d.p.)}$$

Example 19

The coordinates of A and B are $(5, 0, 3)$ and $(4, 2, k)$ respectively.
Given that the distance from A to B is 3 units, find the possible values of k.

$$AB = \sqrt{(5-4)^2 + (0-2)^2 + (3-k)^2} = 3$$

$$\sqrt{1 + 4 + (9 - 6k + k^2)} = 3$$

$$1 + 4 + 9 - 6k + k^2 = 9$$

Square both sides of the equation.

$$k^2 - 6k + 5 = 0$$

$$(k-5)(k-1) = 0$$

Solve to find the two possible values of k.

$$k = 1 \text{ or } k = 5$$

Exercise 5E

1 Find the distance from the origin to the point $P(2, 8, -4)$.

2 Find the distance from the origin to the point $P(7, 7, 7)$.

3 Find the distance between A and B when they have the following coordinates:

a $A(3, 0, 5)$ and $B(1, -1, 8)$

b $A(8, 11, 8)$ and $B(-3, 1, 6)$

c $A(3, 5, -2)$ and $B(3, 10, 3)$

d $A(-1, -2, 5)$ and $B(4, -1, 3)$

4 The coordinates of A and B are $(7, -1, 2)$ and $(k, 0, 4)$ respectively.
Given that the distance from A to B is 3 units, find the possible values of k.

5 The coordinates of A and B are $(5, 3, -8)$ and $(1, k, -3)$ respectively.
Given that the distance from A to B is $3\sqrt{10}$ units, find the possible values of k.

5.6 You can extend the two-dimensional vector results to three dimensions, using k as the unit vector parallel to the *z*-axis, in the direction of *z* increasing.

Extending the results gives the following key points:

- **The vectors i, j and k are unit vectors parallel to the *x*-axis, the *y*-axis and the *z*-axis and in the direction of *x* increasing, *y* increasing and *z* increasing, respectively.**
- **The vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, may be written as a column matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.**
- **The modulus (or magnitude) of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is $\sqrt{x^2 + y^2 + z^2}$.**

Example 20

The points *A* and *B* have position vectors $4\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} - 1\mathbf{k}$ respectively, and *O* is the origin. Find $|\overrightarrow{AB}|$ and show that $\triangle OAB$ is isosceles.

$|\overrightarrow{OA}| = \mathbf{a} = \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}, |\overrightarrow{OB}| = \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ — Write down the position vectors of *A* and *B*.

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -8 \end{pmatrix}$ — Use $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

$|\overrightarrow{AB}| = \sqrt{(-1)^2 + 2^2 + (-8)^2} = \sqrt{69}$ — Use the vector magnitude formula.

$|\overrightarrow{OA}| = \sqrt{4^2 + 2^2 + 7^2} = \sqrt{69}$

$|\overrightarrow{OB}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26}$ — Find the lengths of the other sides *OA* and *OB* of $\triangle OAB$.

So $\triangle OAB$ is isosceles, with $AB = OA$.

Example 21

The points *A* and *B* have coordinates $(t, 5, t - 1)$ and $(2t, t, 3)$ respectively.

a Find $|\overrightarrow{AB}|$.

b By differentiating $|\overrightarrow{AB}|^2$, find the value of *t* for which $|\overrightarrow{AB}|$ is a minimum.

c Find the minimum value of $|\overrightarrow{AB}|$.

a $\mathbf{a} = \begin{pmatrix} t \\ 5 \\ t-1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2t \\ t \\ 3 \end{pmatrix}$ — Write down the position vectors of *A* and *B*.

$\overrightarrow{AB} = \begin{pmatrix} 2t \\ t \\ 3 \end{pmatrix} - \begin{pmatrix} t \\ 5 \\ t-1 \end{pmatrix} = \begin{pmatrix} t \\ t-5 \\ 4-t \end{pmatrix}$ — Use $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

$|\overrightarrow{AB}| = \sqrt{t^2 + (t-5)^2 + (4-t)^2}$ — Use the vector magnitude formula.

$= \sqrt{t^2 + t^2 - 10t + 25 + 16 - 8t + t^2}$

$= \sqrt{3t^2 - 18t + 41}$

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b $|\overrightarrow{AB}|^2 = 3t^2 - 18t + 41$ — Call this p, and differentiate.

$\frac{dp}{dt} = 6t - 18$

For a minimum, $\frac{dp}{dt} = 0$

so $6t - 18 = 0$ — Use the fact that $\frac{dp}{dt} = 0$ at a minimum.

$t = 3$

$\frac{d^2p}{dt^2} = 6$, positive, ∴ minimum. — Use the fact that if the second derivative is positive, the value is a minimum.

c $|\overrightarrow{AB}| = \sqrt{3t^2 - 18t + 41}$ — Substitute $t = 3$ back into $|\overrightarrow{AB}|$.

$= \sqrt{27 - 54 + 41}$

$= \sqrt{14}$

Exercise 5F

1 Find the modulus of:

a $3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ **b** $4\mathbf{i} - 2\mathbf{k}$ **c** $\mathbf{i} + \mathbf{j} - \mathbf{k}$

d $5\mathbf{i} - 9\mathbf{j} - 8\mathbf{k}$ **e** $\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$

2 Given that $\mathbf{a} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix}$, find in column matrix form:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{b} - \mathbf{c}$ **c** $\mathbf{a} + \mathbf{b} + \mathbf{c}$

d $3\mathbf{a} - \mathbf{c}$ **e** $\mathbf{a} - 2\mathbf{b} + \mathbf{c}$ **f** $|\mathbf{a} - 2\mathbf{b} + \mathbf{c}|$

3 The position vector of the point A is $2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{AB} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.
Find the position of the point B.

4 Given that $\mathbf{a} = t\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, and that $|\mathbf{a}| = 7$, find the possible values of t.

5 Given that $\mathbf{a} = 5t\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$, and that $|\mathbf{a}| = 3\sqrt{10}$, find the possible values of t.

6 The points A and B have position vectors $\begin{pmatrix} 2 \\ 9 \\ t \end{pmatrix}$ and $\begin{pmatrix} 2t \\ 5 \\ 3t \end{pmatrix}$ respectively.

a Find $\overrightarrow{AB}$.

b Find, in terms of t, $|\overrightarrow{AB}|$.

c Find the value of t that makes $|\overrightarrow{AB}|$ a minimum.

d Find the minimum value of $|\overrightarrow{AB}|$.

7 The points A and B have position vectors $\begin{pmatrix} 2t+1 \\ t+1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} t+1 \\ 5 \\ 2 \end{pmatrix}$ respectively.

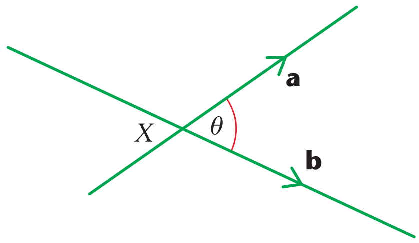
a Find $\overrightarrow{AB}$.

b Find, in terms of t, $|\overrightarrow{AB}|$.

c Find the value of t that makes $|\overrightarrow{AB}|$ a minimum.

d Find the minimum value of $|\overrightarrow{AB}|$.

5.7 You need to know the definition of the scalar product of two vectors (in either two or three dimensions), and how it can be used to find the angle between two vectors.

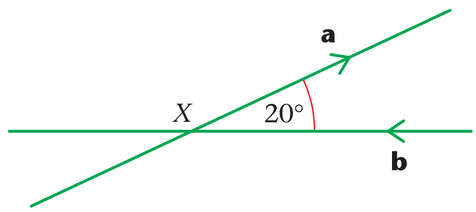


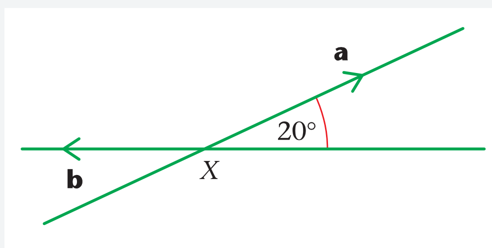
On the diagram, the angle between the vectors **a** and **b** is θ.

Notice that **a** and **b** are both directed **away from** the point X.

Example 22

Find the angle between the vectors **a** and **b** on the diagram:





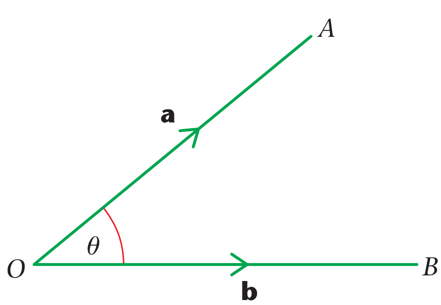
The angle between **a** and **b** is $180° - 20° = 160°$.

For the correct angle, **a** and **b** must both be pointing away from X, so re-draw to show this.

■ **The scalar product of two vectors a and b is written as a.b (say 'a dot b'), and defined by**

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$$

where θ is the angle between a and b.



You can see from this diagram that if **a** and **b** are the position vectors of A and B, then the angle between **a** and **b** is $\angle AOB$.

■ **If a and b are the position vectors of the points *A* and *B*, then**

$$\cos AOB = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

If two vectors **a** and **b** are perpendicular, the angle between them is 90°.
Because $\cos 90° = 0$, then $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos 90° = 0$.

■ **The non-zero vectors a and b are perpendicular if and only if a.b = 0.**

Also, because $\cos 0° = 1$,

■ **If a and b are parallel, a.b = |a| |b|.**

- **In particular, a.a = |a|^2.**

|**a**| |**b**| cos 0°

|**a**| |**a**| cos 0°

Example 23

Find the value of

a **i**.**j** **b** **k**.**k** **c** (4**j**).**k** + (3**i**).(3**i**)

a **i**.**j** = 1 × 1 × cos 90° = 0

i and **j** are unit vectors (magnitude 1), and are perpendicular.

b **k**.**k** = 1 × 1 × cos 0° = 1

k is a unit vector (magnitude 1), the angle between **k** and itself is 0°.

c (4**j**).**k** + (3**i**).(3**i**)

= (4 × 1 × cos 90°) + (3 × 3 × cos 0°)

= 0 + 9 = 9

It can be shown that:

i **a**.**b** = **b**.**a** and **ii** **a**.(**b** + **c**) = **a**.**b** + **a**.**c**.

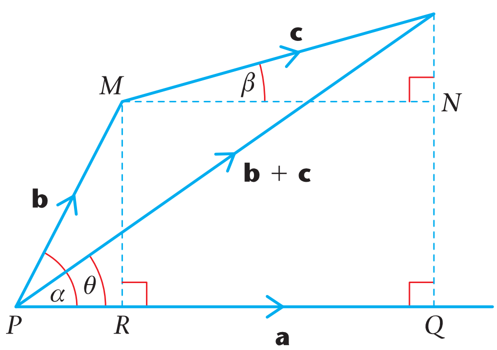
Because of these results, many processes which you are familiar with in ordinary algebra can be applied to the algebra of scalar products.

The proofs of results **i** and **ii** are shown below:

i **a**.**b** = |**a**| |**b**| cos θ, where θ is the angle between **a** and **b**.

b.**a** = |**b**| |**a**| cos θ = |**a**| |**b**| cos θ

So **a**.**b** = **b**.**a**.



ii From the diagram,

a.(**b** + **c**) = |**a**| |**b** + **c**| cos θ

but $\cos\theta = \dfrac{PQ}{|\mathbf{b}+\mathbf{c}|}$, so **a**.(**b** + **c**) = |**a**| × PQ

a.**b** = |**a**| |**b**| cos α, but $\cos\alpha = \dfrac{PR}{|\mathbf{b}|}$, so **a**.**b** = |**a**| × PR

a.**c** = |**a**| |**c**| cos β,

but $\cos\beta = \dfrac{MN}{|\mathbf{c}|} = \dfrac{RQ}{|\mathbf{c}|}$,

Since MN is parallel to RQ, the angle between **c** and **a** is the same as that between **c** and MN, i.e. β.

so **a**.**c** = |**a**| × RQ

so **a**.(**b** + **c**) = |**a**| × PQ = |**a**| × (PR + RQ) = (|**a**| × PR) + (|**a**| × RQ) = **a**.**b** + **a**.**c**

∴ **a**.(**b** + **c**) = **a**.**b** + **a**.**c**

Example 24

Given that $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ find **a**.**b**

$$\mathbf{a}.\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$
$$= a_1\mathbf{i}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$
$$+ a_2\mathbf{j}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$
$$+ a_3\mathbf{k}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$
$$= (a_1\mathbf{i}).(b_1\mathbf{i}) + (a_1\mathbf{i}).(b_2\mathbf{j}) + (a_1\mathbf{i}).(b_3\mathbf{k})$$
$$+ (a_2\mathbf{j}).(b_1\mathbf{i}) + (a_2\mathbf{j}).(b_2\mathbf{j}) + (a_2\mathbf{j}).(b_3\mathbf{k})$$
$$+ (a_3\mathbf{k}).(b_1\mathbf{i}) + (a_3\mathbf{k}).(b_2\mathbf{j})$$
$$+ (a_3\mathbf{k}).(b_3\mathbf{k})$$
$$= (a_1b_1)\mathbf{i}.\mathbf{i} + (a_1b_2)\mathbf{i}.\mathbf{j} + (a_1b_3)\mathbf{i}.\mathbf{k}$$
$$+ (a_2b_1)\mathbf{j}.\mathbf{i} + (a_2b_2)\mathbf{j}.\mathbf{j} + (a_2b_3)\mathbf{j}.\mathbf{k}$$
$$+ (a_3b_1)\mathbf{k}.\mathbf{i} + (a_3b_2)\mathbf{k}.\mathbf{j} + (a_3b_3)\mathbf{k}.\mathbf{k}$$
$$= a_1b_1 + a_2b_2 + a_3b_3$$

Use the results for parallel and perpendicular unit vectors:

$\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$

$\mathbf{i}.\mathbf{j} = \mathbf{i}.\mathbf{k} = \mathbf{j}.\mathbf{i} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = \mathbf{k}.\mathbf{j} = 0$

The above example leads to a very simple formula for finding the scalar product of 2 vectors that are given in Cartesian component form:

- **If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,**

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

Example 25

Given that $\mathbf{a} = 8\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$:

a Find **a**.**b**

b Find the angle between **a** and **b**, giving your answer in degrees to 1 decimal place.

a $\mathbf{a}.\mathbf{b} = \begin{pmatrix} 8 \\ -5 \\ -4 \end{pmatrix} . \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}$

Write in column matrix form.

$= (8 \times 5) + (-5 \times 4) + (-4 \times -1)$

Use $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

$= 40 - 20 + 4$

$= 24$

b $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$ — Use the scalar product definition.

$|\mathbf{a}| = \sqrt{8^2 + (-5)^2 + (-4)^2} = \sqrt{105}$

$|\mathbf{b}| = \sqrt{5^2 + 4^2 + (-1)^2} = \sqrt{42}$ — Find the modulus of **a** and of **b**.

$\sqrt{105}\ \sqrt{42}\cos\theta = 24$ — Use $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$.

$\cos\theta = \frac{24}{\sqrt{105}\ \sqrt{42}}$

$\theta = 68.8°$ (1 d.p.)

Example 26

Given that $\mathbf{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, find the angle between **a** and **b**, giving your answer in degrees to 1 decimal place.

$\mathbf{a}.\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} . \begin{pmatrix} 7 \\ -2 \\ 2 \end{pmatrix} = -7 - 2 + 6 = -3$

$|\mathbf{a}| = \sqrt{(-1)^2 + 1^2 + 3^2} = \sqrt{11}$

$|\mathbf{b}| = \sqrt{(-7)^2 + (-2)^2 + 2^2} = \sqrt{57}$

For the scalar product formula, you need to find **a.b**, |**a**| and |**b**|.

$\sqrt{11}\ \sqrt{57}\cos\theta = -3$ — Use $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$.

$\cos\theta = \frac{-3}{\sqrt{11}\ \sqrt{57}}$ — The cosine is negative, so the angle is obtuse.

$\theta = 96.9°$ (1 d.p.)

Example 27

Given that the vectors $\mathbf{a} = 2\mathbf{i} - 6\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$ are perpendicular, find the value of λ.

$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 2 \\ -6 \\ 1 \end{pmatrix} . \begin{pmatrix} 5 \\ 2 \\ \lambda \end{pmatrix}$

$= 10 - 12 + \lambda$

$= -2 + \lambda$

Find the scalar product.

$-2 + \lambda = 0$ — For perpendicular vectors, the scalar product is zero.

$\lambda = 2$

Example 28

Given that $\mathbf{a} = -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 8\mathbf{j} + 5\mathbf{k}$, find a vector which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Let the required vector be $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\mathbf{a}.\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \text{ and } \mathbf{b}.\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Both scalar products are zero.

$$\begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix}.\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \text{ and } \begin{pmatrix} 4 \\ -8 \\ 5 \end{pmatrix}.\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$-2x + 5y - 4z = 0$$
$$4x - 8y + 5z = 0$$

Let $z = 1$

Choose a (non-zero) value for z (or for x, or for y).

$$-2x + 5y = 4 \quad (\times 2)$$
$$4x - 8y = -5$$
$$-4x + 10y = 8$$
$$4x - 8y = -5$$

Adding, $2y = 3$

$$y = \tfrac{3}{2}$$
$$-2x + \tfrac{15}{2} = 4,\ 2x = \tfrac{7}{2}$$
$$x = \tfrac{7}{4}$$

So $x = \frac{7}{4}$, $y = \frac{3}{2}$ and $z = 1$

Solve simultaneously, by multiplying the first equation by 2, and eliminating x.

A possible vector is $\frac{7}{4}\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k}$

Another possible vector is $4(\frac{7}{4}\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k})$
$= 7\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$

You can multiply by a scalar constant to find another vector which is also perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Exercise 5G

1 The vectors **a** and **b** each have magnitude 3 units, and the angle between **a** and **b** is 60°. Find **a**.**b**.

2 In each part, find **a**.**b**:

a $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

b $\mathbf{a} = 10\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - 12\mathbf{k}$

c $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = -\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

d $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{b} = 6\mathbf{i} - 5\mathbf{j} - 8\mathbf{k}$

e $\mathbf{a} = 3\mathbf{j} + 9\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 12\mathbf{j} - 4\mathbf{k}$

3 In each part, find the angle between **a** and **b**, giving your answer in degrees to 1 decimal place:

a $\mathbf{a} = 3\mathbf{i} + 7\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} + \mathbf{j}$

b $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$, $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j}$

c $\mathbf{a} = \mathbf{i} - 7\mathbf{j} + 8\mathbf{k}$, $\mathbf{b} = 12\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

d $\mathbf{a} = -\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 11\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

e $\mathbf{a} = 6\mathbf{i} - 7\mathbf{j} + 12\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

f $\mathbf{a} = 4\mathbf{i} + 5\mathbf{k}$, $\mathbf{b} = 6\mathbf{i} - 2\mathbf{j}$

g $\mathbf{a} = -5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} + 11\mathbf{k}$

h $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

4 Find the value, or values, of λ for which the given vectors are perpendicular:

a $3\mathbf{i} + 5\mathbf{j}$ and $\lambda\mathbf{i} + 6\mathbf{j}$

b $2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ and $\lambda\mathbf{i} - 4\mathbf{j} - 14\mathbf{k}$

c $3\mathbf{i} + \lambda\mathbf{j} - 8\mathbf{k}$ and $7\mathbf{i} - 5\mathbf{j} + \mathbf{k}$

d $9\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\lambda\mathbf{i} + \lambda\mathbf{j} + 3\mathbf{k}$

e $\lambda\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\lambda\mathbf{i} + \lambda\mathbf{j} + 5\mathbf{k}$

5 Find, to the nearest tenth of a degree, the angle that the vector $9\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$ makes with:

a the positive x-axis **b** the positive y-axis

6 Find, to the nearest tenth of a degree, the angle that the vector $\mathbf{i} + 11\mathbf{j} - 4\mathbf{k}$ makes with:

a the positive y-axis **b** the positive z-axis

7 The angle between the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ is θ.
Calculate the exact value of $\cos\theta$.

8 The angle between the vectors $\mathbf{i} + 3\mathbf{j}$ and $\mathbf{j} + \lambda\mathbf{k}$ is 60°.

Show that $\lambda = \pm\sqrt{\frac{13}{5}}$.

9 Simplify as far as possible:

a $\mathbf{a}.(\mathbf{b} + \mathbf{c}) + \mathbf{b}.(\mathbf{a} - \mathbf{c})$, given that $\mathbf{b}$ is perpendicular to $\mathbf{c}$.

b $(\mathbf{a} + \mathbf{b}).(\mathbf{a} + \mathbf{b})$, given that $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 3$.

c $(\mathbf{a} + \mathbf{b}).(2\mathbf{a} - \mathbf{b})$, given that $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

10 Find a vector which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, where:

a $\mathbf{a} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

b $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$

c $\mathbf{a} = 4\mathbf{i} - 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = -2\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$

11 The points A and B have position vectors $2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $6\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ respectively, and O is the origin.
Calculate each of the angles in $\triangle OAB$, giving your answers in degrees to 1 decimal place.

12 The points A, B and C have position vectors $\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $2\mathbf{i} + 7\mathbf{j} - 3\mathbf{k}$ and $4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ respectively.

a Find, as surds, the lengths of AB and BC.

b Calculate, in degrees to 1 decimal place, the size of $\angle ABC$.

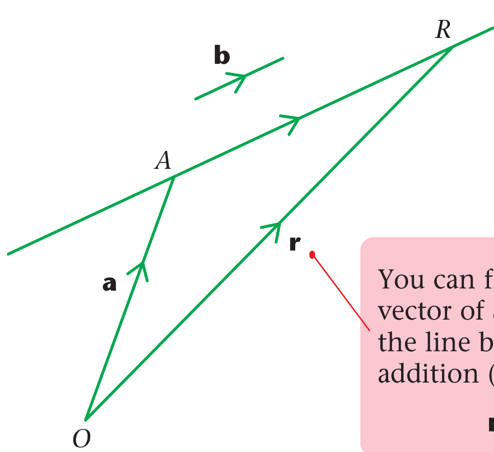
13 Given that the points A and B have coordinates $(7, 4, 4)$ and $(2, -2, -1)$ respectively, use a vector method to find the value of $\cos AOB$, where O is the origin.
Prove that the area of $\triangle AOB$ is $\dfrac{5\sqrt{29}}{2}$.

14 AB is a diameter of a circle centred at the origin O, and P is any point on the circumference of the circle.
Using the position vectors of A, B and P, prove (using a scalar product) that AP is perpendicular to BP (i.e. the angle in the semicircle is a right angle).

15 Use a vector method to prove that the diagonals of the square $OABC$ cross at right angles.

5.8 You need to know how to write the equation of a straight line in vector form.

Suppose a straight line passes through a given point A, with position vector $\mathbf{a}$, and is parallel to the given vector $\mathbf{b}$. Only one such line is possible.



Since $\overrightarrow{AR}$ is parallel to $\mathbf{b}$, $\overrightarrow{AR} = t\mathbf{b}$, where t is a scalar.
The vector $\mathbf{b}$ is called the direction vector of the line.
So the position vector $\mathbf{r}$ can be written as $\mathbf{a} + t\mathbf{b}$.

You can find the position vector of any point R on the line by using vector addition ($\triangle OAR$):

$\mathbf{r} = \mathbf{a} + \overrightarrow{AR}$

■ **A vector equation of a straight line passing through the point A with position vector $\mathbf{a}$, and parallel to the vector $\mathbf{b}$, is**

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}$$

where t is a scalar parameter.

By taking different values of the parameter t, you can find the position vectors of different points that lie on the straight line.

Example 29

Find a vector equation of the straight line which passes through the point A, with position vector $3\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$, and is parallel to the vector $7\mathbf{i} - 3\mathbf{k}$.

Here $\mathbf{a} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix}$

$\mathbf{b}$ is the direction vector.

An equation of the line is

$$\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} + t\begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix}$$

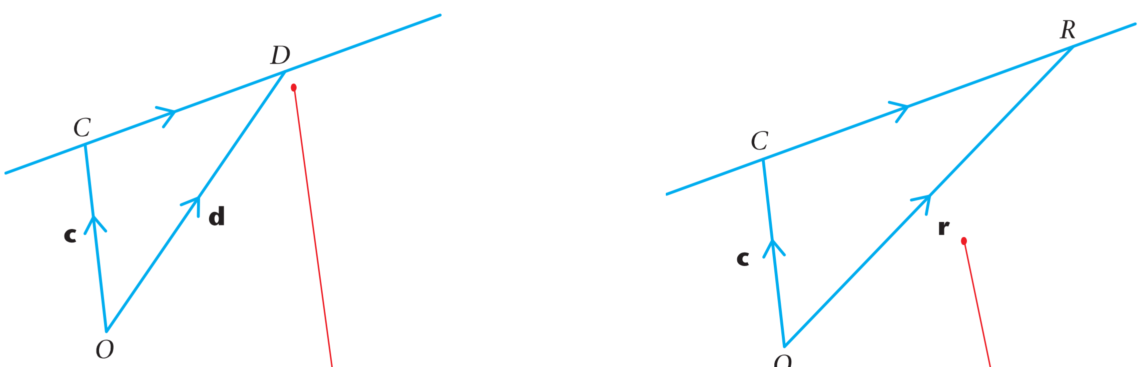
or $\mathbf{r} = (3\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}) + t(7\mathbf{i} - 3\mathbf{k})$

or $\mathbf{r} = (3 + 7t)\mathbf{i} + (-5)\mathbf{j} + (4 - 3t)\mathbf{k}$

or $\mathbf{r} = \begin{pmatrix} 3 + 7t \\ -5 \\ 4 - 3t \end{pmatrix}$

You sometimes need to show the separate x, y, z components in terms of t.

Now suppose a straight line passes through two given points C and D, with position vectors $\mathbf{c}$ and $\mathbf{d}$ respectively. Again, only one such line is possible.



You can use CD as a direction vector for the line:

$\overrightarrow{CD} = \mathbf{d} - \mathbf{c}$ (see Section 5.3).

You can now use one of the two given points and the direction vector to form an equation for the straight line.

- **A vector equation of a straight line passing through the points *C* and *D*, with position vectors c and d respectively, is**

  $$\mathbf{r} = \mathbf{c} + t(\mathbf{d} - \mathbf{c})$$

  where *t* is a scalar parameter.

You could also have used the point *D*, giving $\mathbf{r} = \mathbf{d} + t(\mathbf{d} - \mathbf{c})$.

Example 30

Find a vector equation of the straight line which passes through the points *A* and *B*, with coordinates (4, 5, −1) and (6, 3, 2) respectively.

$$\mathbf{a} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$$

Write down the position vectors of *A* and *B*.

$$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

Find a direction vector for the line.

$$\mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

Use one of the given points to form the equation.

The equation could be written in other ways:

$$\mathbf{r} = (4\mathbf{i} + 5\mathbf{j} - \mathbf{k}) + t(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$$

$$\mathbf{r} = (4 + 2t)\mathbf{i} + (5 - 2t)\mathbf{j} + (-1 + 3t)\mathbf{k}$$

$$\mathbf{r} = \begin{pmatrix} 4 + 2t \\ 5 - 2t \\ -1 + 3t \end{pmatrix}$$

Example 31

The straight line *l* has vector equation $\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) + t(\mathbf{i} - 6\mathbf{j} - 2\mathbf{k})$.
Given that the point $(a, b, 0)$ lies on *l*, find the value of *a* and the value of *b*.

$$\mathbf{r} = \begin{pmatrix} 3 + t \\ 2 - 6t \\ -5 - 2t \end{pmatrix}$$

You can write the equation in this form.

$$-5 - 2t = 0$$

Use the *z*-coordinate (zero) to find the value of *t*.

$$t = -2\tfrac{1}{2}$$

$$a = 3 + t = \tfrac{1}{2}$$

$$b = 2 - 6t = 17$$

$$a = \tfrac{1}{2} \text{ and } b = 17$$

Find *a* and *b* using the value of *t*.

Example 32

The straight line l has vector equation $\mathbf{r} = (2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}) + t(6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$.
Show that another vector equation of l is $\mathbf{r} = (8\mathbf{i} + 3\mathbf{j} + \mathbf{k}) + t(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$.

Since $\begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, these two vectors are parallel.

So $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ can also be used as the direction vector.

So another form of the equation of l is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$

If $t = 2$, $\mathbf{r} = \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix}$

You need to show that (8, 3, 1) lies on l. Look for a t-value that gives the position vector of this point.

So the point (8, 3, 1) also lies on l.

So another form of the equation of l is

$$\mathbf{r} = \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$

Exercise 5H

1 Find a vector equation of the straight line which passes through the point A, with position vector $\mathbf{a}$, and is parallel to the vector $\mathbf{b}$:

a $\mathbf{a} = 6\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$

b $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

c $\mathbf{a} = -7\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

d $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

e $\mathbf{a} = \begin{pmatrix} 6 \\ -11 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$

2 Calculate, to 1 decimal place, the distance between the point P, where $t = 1$, and the point Q, where $t = 5$, on the line with equation:

a $\mathbf{r} = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) + t(3\mathbf{i} - 8\mathbf{j} - \mathbf{k})$

b $\mathbf{r} = (\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + t(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

c $\mathbf{r} = (2\mathbf{i} + 5\mathbf{k}) + t(-3\mathbf{i} + 4\mathbf{j} - \mathbf{k})$

3 Find a vector equation for the line which is parallel to the z-axis and passes through the point $(4, -3, 8)$.

4 Find a vector equation for the line which passes through the points:

a $(2, 1, 9)$ and $(4, -1, 8)$

b $(-3, 5, 0)$ and $(7, 2, 2)$

c $(1, 11, -4)$ and $(5, 9, 2)$

d $(-2, -3, -7)$ and $(12, 4, -3)$

5 The point $(1, p, q)$ lies on the line l. Find the values of p and q, given that the equation is l is:

a $\mathbf{r} = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) + t(\mathbf{i} - 4\mathbf{j} - 9\mathbf{k})$

b $\mathbf{r} = (-4\mathbf{i} + 6\mathbf{j} - \mathbf{k}) + t(2\mathbf{i} - 5\mathbf{j} - 8\mathbf{k})$

c $\mathbf{r} = (16\mathbf{i} - 9\mathbf{j} - 10\mathbf{k}) + t(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

5.9 You need to be able to determine whether two given straight lines intersect.

When you need to deal with more than one straight line in the same question, use a different parameter for each line.

The letters t and s are often used as parameters.
Greek letters λ and μ are also commonly used as parameters.

In three dimensions, two straight lines will not generally intersect. The next example, however, deals with two straight lines that do intersect, and shows you how to prove this.

Example 33

Show that the lines with vector equations

$$\mathbf{r} = (3\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$$
$$\text{and } \mathbf{r} = (7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) + s(2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$$

intersect, and find the position vector of their point of intersection.

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$$\mathbf{r} = \begin{pmatrix} 3+2t \\ 8-t \\ -2+3t \end{pmatrix} \qquad \mathbf{r} = \begin{pmatrix} 7+2s \\ 4+s \\ 3+4s \end{pmatrix}$$

At an intersection point,

$$\begin{pmatrix} 3+2t \\ 8-t \\ -2+3t \end{pmatrix} = \begin{pmatrix} 7+2s \\ 4+s \\ 3+4s \end{pmatrix}$$

$$3 + 2t = 7 + 2s$$
$$8 - t = 4 + s$$

Equate the x components.
Equate the y components.

$$3 + 2t = 7 + 2s$$
$$16 - 2t = 8 + 2s$$
$$19 \quad = 15 + 4s$$
$$s = 1$$
$$3 + 2t = 7 + 2$$
$$t = 3$$

Solve simultaneously.

If the lines intersect,

$-2 + 3t = 3 + 4s$ must be true.

z components must also be equal

$$-2 + 3t = -2 + 9 = 7$$
$$3 + 4s = 3 + 4 \quad = 7$$

Check that $s = 1$, $t = 3$ gives equal z components.

The z components are also equal, so the lines do intersect.

The intersection point has position vector:

$$\begin{pmatrix} 3+2t \\ 8-t \\ -2+3t \end{pmatrix}$$

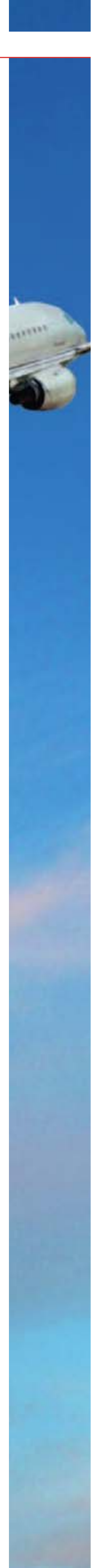
With $t = 3$: $\mathbf{r} = \begin{pmatrix} 9 \\ 5 \\ 7 \end{pmatrix}$ or $\mathbf{r} = 9\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$

Exercise 5I

In each question, determine whether the lines with the given equations intersect. If they do intersect, find the coordinates of their point of intersection.

1 $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -7 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 14 \\ 16 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

2 $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 9 \\ -2 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$



3 $\mathbf{r} = \begin{pmatrix} 12 \\ 4 \\ -6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 8 \\ -2 \\ 6 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$

4 $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -2 \\ -9 \\ 12 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

5 $\mathbf{r} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + s\begin{pmatrix} 6 \\ -4 \\ 1 \end{pmatrix}$

5.10 You need to be able to calculate the angle between two straight lines.

- **The acute angle θ between two straight lines is given by**

$$\cos\theta = \left|\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\right|$$

where a and b are direction vectors of the lines.

Example 34

Find, to 1 decimal place, the acute angle between the lines with vector equations

$$\mathbf{r} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(3\mathbf{i} - 8\mathbf{j} - \mathbf{k})$$
and $$\mathbf{r} = (7\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + s(2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

$$\mathbf{a} = \begin{pmatrix} 3 \\ -8 \\ -1 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

Use the direction vectors.

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

Find the angle between the 2 vectors.

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 3 \\ -8 \\ -1 \end{pmatrix} . \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$$= 6 - 16 - 3 = -13$$

$$|\mathbf{a}| = \sqrt{3^2 + (-8)^2 + (-1)^2} = \sqrt{74}$$

$$|\mathbf{b}| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{17}$$

$$\cos\theta = -\frac{13}{\sqrt{74}\ \sqrt{17}}$$

Use the formula for cos θ.

$$\theta = 68.5^\circ \text{ (1 d.p.)}$$

This is the angle between 2 *vectors*.

So the acute angle between the lines is
$180^\circ - 111.5^\circ = 68.5^\circ$ (1 d.p.)

Exercise 5J

In Questions 1 to 5, find, to 1 decimal place, the acute angle between the lines with the given vector equations:

1 $\mathbf{r} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(3\mathbf{i} - 5\mathbf{j} - \mathbf{k})$
and $\mathbf{r} = (7\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + s(2\mathbf{i} + \mathbf{j} - 9\mathbf{k})$

2 $\mathbf{r} = (\mathbf{i} - \mathbf{j} + 7\mathbf{k}) + t(-2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$
and $\mathbf{r} = (8\mathbf{i} + 5\mathbf{j} - \mathbf{k}) + s(-4\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

3 $\mathbf{r} = (3\mathbf{i} + 5\mathbf{j} - \mathbf{k}) + t(\mathbf{i} + \mathbf{j} + \mathbf{k})$
and $\mathbf{r} = (-\mathbf{i} + 11\mathbf{j} + 5\mathbf{k}) + s(2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k})$

4 $\mathbf{r} = (\mathbf{i} + 6\mathbf{j} - \mathbf{k}) + t(8\mathbf{i} - \mathbf{j} - 2\mathbf{k})$
and $\mathbf{r} = (6\mathbf{i} + 9\mathbf{j}) + s(\mathbf{i} + 3\mathbf{j} - 7\mathbf{k})$

5 $\mathbf{r} = (2\mathbf{i} + \mathbf{k}) + t(11\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$
and $\mathbf{r} = (\mathbf{i} + \mathbf{j}) + s(-3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$

6 The straight lines l_1 and l_2 have vector equations $\mathbf{r} = (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + t(8\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + s(3\mathbf{i} + \mathbf{j})$ respectively, and P is the point with coordinates (1, 4, 2).

a Show that the point $Q(9, 9, 3)$ lies on l_1.

b Find the cosine of the acute angle between l_1 and l_2.

c Find the possible coordinates of the point R, such that R lies on l_2 and $PQ = PR$.

Mixed exercise 5K

1 With respect to an origin O, the position vectors of the points L, M and N are $(4\mathbf{i} + 7\mathbf{j} + 7\mathbf{k})$, $(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ and $(2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k})$ respectively.

a Find the vectors $\overrightarrow{ML}$ and $\overrightarrow{MN}$.

b Prove that $\cos \angle LMN = \frac{9}{10}$. E

2 The position vectors of the points A and B relative to an origin O are $5\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, $-\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ respectively. Find the position vector of the point P which lies on AB produced such that $AP = 2BP$. E

3 Points A, B, C, D in a plane have position vectors $\mathbf{a} = 6\mathbf{i} + 8\mathbf{j}$, $\mathbf{b} = \frac{3}{2}\mathbf{a}$, $\mathbf{c} = 6\mathbf{i} + 3\mathbf{j}$, $\mathbf{d} = \frac{5}{3}\mathbf{c}$ respectively. Write down vector equations of the lines AD and BC and find the position vector of their point of intersection. E

4 Find the point of intersection of the line through the points (2, 0, 1) and (−1, 3, 4) and the line through the points (−1, 3, 0) and (4, −2, 5). Calculate the acute angle between the two lines. E

5 Show that the lines

$\mathbf{r} = (-2\mathbf{i} + 5\mathbf{j} - 11\mathbf{k}) + \lambda(3\mathbf{i} + \mathbf{j} + 3\mathbf{k})$
$\mathbf{r} = 8\mathbf{i} + 9\mathbf{j} + \mu(4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$

intersect. Find the position vector of their common point. *E*

6 Find a vector that is perpendicular to both $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. *E*

7 State a vector equation of the line passing through the points A and B whose position vectors are $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ respectively. Determine the position vector of the point C which divides the line segment AB internally such that $AC = 2CB$. *E*

8 Vectors $\mathbf{r}$ and $\mathbf{s}$ are given by

$\mathbf{r} = \lambda\mathbf{i} + (2\lambda - 1)\mathbf{j} - \mathbf{k}$
$\mathbf{s} = (1 - \lambda)\mathbf{i} + 3\lambda\mathbf{j} + (4\lambda - 1)\mathbf{k}$

where λ is a scalar.

a Find the values of λ for which $\mathbf{r}$ and $\mathbf{s}$ are perpendicular.

When $\lambda = 2$, $\mathbf{r}$ and $\mathbf{s}$ are the position vectors of the points A and B respectively, referred to an origin O.

b Find $\overrightarrow{AB}$.

c Use a scalar product to find the size of $\angle BAO$, giving your answer to the nearest degree. *E*

9 With respect to an origin O, the position vectors of the points L and M are $2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ and $5\mathbf{i} + \mathbf{j} + c\mathbf{k}$ respectively, where c is a constant. The point N is such that $OLMN$ is a rectangle.

a Find the value of c.

b Write down the position vector of N.

c Find, in the form $\mathbf{r} = \mathbf{p} + t\mathbf{q}$, an equation of the line MN. *E*

10 The point A has coordinates $(7, -1, 3)$ and the point B has coordinates $(10, -2, 2)$. The line l has vector equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} + \mathbf{k})$, where λ is a real parameter.

a Show that the point A lies on the line l.

b Find the length of AB.

c Find the size of the acute angle between the line l and the line segment AB, giving your answer to the nearest degree.

d Hence, or otherwise, calculate the perpendicular distance from B to the line l, giving your answer to two significant figures. *E*

11 Referred to a fixed origin O, the points A and B have position vectors $(5\mathbf{i} - \mathbf{j} - \mathbf{k})$ and $(\mathbf{i} - 5\mathbf{j} + 7\mathbf{k})$ respectively.

a Find an equation of the line AB.

b Show that the point C with position vector $4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ lies on AB.

c Show that OC is perpendicular to AB.

d Find the position vector of the point D, where $D \not\equiv A$, on AB such that $|\overrightarrow{OD}| = |\overrightarrow{OA}|$.

E

12 Referred to a fixed origin O, the points A, B and C have position vectors $(9\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, $(6\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$ and $(3\mathbf{i} + p\mathbf{j} + q\mathbf{k})$ respectively, where p and q are constants.

a Find, in vector form, an equation of the line l which passes through A and B.

Given that C lies on l:

b Find the value of p and the value of q.

c Calculate, in degrees, the acute angle between OC and AB.

The point D lies on AB and is such that OD is perpendicular to AB.

d Find the position vector of D.

E

13 Referred to a fixed origin O, the points A and B have position vectors $(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ and $(5\mathbf{i} - 3\mathbf{j})$ respectively.

a Find, in vector form, an equation of the line l_1 which passes through A and B.

The line l_2 has equation $\mathbf{r} = (4\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$, where μ is a scalar parameter.

b Show that A lies on l_2.

c Find, in degrees, the acute angle between the lines l_1 and l_2.

The point C with position vector $(2\mathbf{i} - \mathbf{k})$ lies on l_2.

d Find the shortest distance from C to the line l_1.

E

14 Two submarines are travelling in straight lines through the ocean. Relative to a fixed origin, the vector equations of the two lines, l_1 and l_2, along which they travel are

$$\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$
$$\text{and } \mathbf{r} = 9\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \mu(4\mathbf{i} + \mathbf{j} - \mathbf{k})$$

where λ and μ are scalars.

a Show that the submarines are moving in perpendicular directions.

b Given that l_1 and l_2 intersect at the point A, find the position vector of A.

The point B has position vector $10\mathbf{j} - 11\mathbf{k}$.

c Show that only one of the submarines passes through the point B.

d Given that 1 unit on each coordinate axis represents 100 m, find, in km, the distance AB.

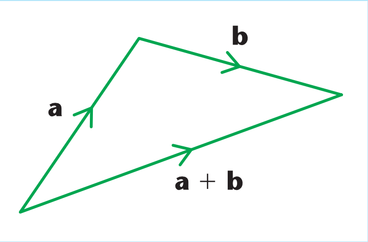
E

Summary of key points

1 A vector is a quantity that has both magnitude and direction.

2 Vectors that are equal have both the same magnitude and the same direction.

3 Two vectors are added using the ‘triangle law’.



4 Adding the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ gives the zero vector $\mathbf{0}$.
($\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}$)

5 The modulus of a vector is another name for its magnitude.
- The modulus of the vector $\mathbf{a}$ is written as $|\mathbf{a}|$.
- The modulus of the vector $\overrightarrow{PQ}$ is written as $|\overrightarrow{PQ}|$.

6 The vector $-\mathbf{a}$ has the same magnitude as the vector $\mathbf{a}$ but is in the opposite direction.

7 Any vector parallel to the vector $\mathbf{a}$ may be written as $\lambda\mathbf{a}$, where λ is a non-zero scalar.

8 $\mathbf{a} - \mathbf{b}$ is defined to be $\mathbf{a} + (-\mathbf{b})$.

9 A unit vector is a vector which has magnitude (or modulus) 1 unit.

10 If $\lambda\mathbf{a} + \mu\mathbf{b} = \alpha\mathbf{a} + \beta\mathbf{b}$, and the non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are not parallel, then $\lambda = \alpha$ and $\mu = \beta$.

11 The position vector of a point A is the vector $\overrightarrow{OA}$, where O is the origin. $\overrightarrow{OA}$ is usually written as vector $\mathbf{a}$.

12 $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of A and B respectively.

13 The vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors parallel to the x-axis, the y-axis and the z-axis and in the direction of x increasing, y increasing and z increasing, respectively.

14 The modulus (or magnitude) of $x\mathbf{i} + y\mathbf{j}$ is $\sqrt{x^2 + y^2}$.

15 The vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ may be written as a column matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

16 The distance from the origin to the point (x, y, z) is $\sqrt{x^2 + y^2 + z^2}$.

17 The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

18 The modulus (or magnitude) of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is $\sqrt{x^2 + y^2 + z^2}$.

19 The scalar product of two vectors **a** and **b** is written as **a**.**b**, and defined by

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta,$$

where θ is the angle between **a** and **b**.

20 If **a** and **b** are the position vectors of the points A and B, then

$$\cos AOB = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

21 The non-zero vectors **a** and **b** are perpendicular if and only if $\mathbf{a}.\mathbf{b} = 0$.

22 If **a** and **b** are parallel, $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$.

- In particular, $\mathbf{a}.\mathbf{a} = |\mathbf{a}|^2$.

23 If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

24 A vector equation of a straight line passing through the point A with position vector **a**, and parallel to the vector **b**, is

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}$$

where t is a scalar parameter.

25 A vector equation of a straight line passing through the points C and D, with position vectors **c** and **d** respectively, is

$$\mathbf{r} = \mathbf{c} + t(\mathbf{d} - \mathbf{c})$$

where t is a scalar parameter.

26 The acute angle θ between two straight lines is given by

$$\cos\theta = \left| \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \right|$$

where **a** and **b** are direction vectors of the lines.