6

After completing this chapter you should be able to integrate:

- a number of standard mathematical functions
- using the reverse of the chain rule
- using trigonometrical identities
- using partial fractions
- by substitution
- by parts
- to find areas and volumes
- to solve differential equations.

In addition some functions are too difficult to integrate and hence you should be able to

- use the trapezium rule to approximate their value.

Integration

The ability to integrate and solve differential equations has many fascinating uses in the world. Did you know that Forensic Investigators can use Newton's Law of Cooling (see page 44) to help solve cases?

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6.1 You need to be able to integrate standard functions.

You met the first result in the list below in your C1 book. The others are the reverse of ones you have already met in Chapter 8 of C3.

■ **You should be familiar with the following integrals:**

1. $\int x^n = \dfrac{x^{n+1}}{n+1} + C$
2. $\int e^x = e^x + C$
3. $\int \dfrac{1}{x} = \ln|x| + C$
4. $\int \cos x = \sin x + C$
5. $\int \sin x = -\cos x + C$
6. $\int \sec^2 x = \tan x + C$
7. $\int \operatorname{cosec} x \cot x = -\operatorname{cosec} x + C$
8. $\int \operatorname{cosec}^2 x = -\cot x + C$
9. $\int \sec x \tan x = \sec x + C$

Hint: When finding $\int \frac{1}{x}\,dx$ it is usual to write the answer as $\ln|x|$. The modulus sign removes difficulties that may arise when evaluating the integral. This will be explained in more detail in Section 6.5 but this form will be used throughout this chapter.

Example 1

Find the following integrals:

a $\int\left(2\cos x + \dfrac{3}{x} - \sqrt{x}\right)dx$ **b** $\int\left(\dfrac{\cos x}{\sin^2 x} - 2e^x\right)dx$

a $\int 2\cos x\,dx = 2\sin x$

Integrate each term separately.

Use 4.

$\int \dfrac{3}{x}\,dx = 3\ln|x|$

Use 3.

$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}}$

Use 1.

So $\int\left(2\cos x + \dfrac{3}{x} - \sqrt{x}\right)dx$

$= 2\sin x + 3\ln|x| - \frac{2}{3}x^{\frac{3}{2}} + C$

This is an indefinite integral so don't forget the $+C$.

b $\dfrac{\cos x}{\sin^2 x} = \dfrac{\cos x}{\sin x}\,\dfrac{1}{\sin x} = \cot x \operatorname{cosec} x$

Look at the list of integrals of standard functions and express the integrand in terms of these standard functions.

$$\int (\cot x \operatorname{cosec} x)\,\mathrm{d}x = -\operatorname{cosec} x$$

$$\int 2e^x\,\mathrm{d}x = 2e^x$$

Remember the minus sign.

$$\text{So } \int \left(\frac{\cos x}{\sin^2 x} - 2e^x \right) \mathrm{d}x$$
$$= -\operatorname{cosec} x - 2e^x + C$$

Exercise 6A

1 Integrate the following with respect to x:

a $3\sec^2 x + \dfrac{5}{x} + \dfrac{2}{x^2}$

b $5e^x - 4\sin x + 2x^3$

c $2(\sin x - \cos x + x)$

d $3\sec x \tan x - \dfrac{2}{x}$

e $5e^x + 4\cos x - \dfrac{2}{x^2}$

f $\dfrac{1}{2x} + 2\operatorname{cosec}^2 x$

g $\dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3}$

h $e^x + \sin x + \cos x$

i $2\operatorname{cosec} x \cot x - \sec^2 x$

j $e^x + \dfrac{1}{x} - \operatorname{cosec}^2 x$

2 Find the following integrals:

a $\displaystyle\int \left(\frac{1}{\cos^2 x} + \frac{1}{x^2} \right) \mathrm{d}x$

b $\displaystyle\int \left(\frac{\sin x}{\cos^2 x} + 2e^x \right) \mathrm{d}x$

c $\displaystyle\int \left(\frac{1 + \cos x}{\sin^2 x} + \frac{1 + x}{x^2} \right) \mathrm{d}x$

d $\displaystyle\int \left(\frac{1}{\sin^2 x} + \frac{1}{x} \right) \mathrm{d}x$

e $\displaystyle\int \sin x(1 + \sec^2 x)\,\mathrm{d}x$

f $\displaystyle\int \cos x(1 + \operatorname{cosec}^2 x)\,\mathrm{d}x$

g $\displaystyle\int \operatorname{cosec}^2 x(1 + \tan^2 x)\,\mathrm{d}x$

h $\displaystyle\int \sec^2 x(1 - \cot^2 x)\,\mathrm{d}x$

i $\displaystyle\int \sec^2 x(1 + e^x \cos^2 x)\,\mathrm{d}x$

j $\displaystyle\int \left(\frac{1 + \sin x}{\cos^2 x} + \cos^2 x \sec x \right) \mathrm{d}x$

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6.2 You can integrate some functions using the reverse of the chain rule.

This technique only works for linear transformations of functions such as $f(ax + b)$.

Example 2

Find the following integrals:

a $\int \cos(2x + 3)\,dx$ **b** $\int e^{4x+1}\,dx$ **c** $\int \sec^2 3x\,dx$

a Consider $y = \sin(2x + 3)$

So $\dfrac{dy}{dx} = \cos(2x + 3) \times 2$

So $\int \cos(2x + 3)\,dx = \frac{1}{2}\sin(2x + 3) + C$

Recall 4. So integrating a 'cos' function gives a 'sin' function.

Let $y = \sin(2x + 3)$ and differentiate using the chain rule.

Remember the 2 from the chain rule.

This is $2\times$ the required expression so you divide the $\sin(2x + 3)$ by 2.

b Consider $y = e^{4x+1}$

So $\dfrac{dy}{dx} = e^{4x+1} \times 4$

So $\int e^{4x+1}\,dx = \frac{1}{4}e^{4x+1} + C$

Recall 2. So integrating an 'exp' function gives an 'exp' function.

Let $y = e^{4x+1}$ and differentiate using the chain rule.

Remember the 4 comes from the chain rule.

This answer is 4 times the required expression so you divide by 4.

c Consider $y = \tan 3x$

So $\dfrac{dy}{dx} = \sec^2 3x \times 3$

So $\int \sec^2 3x\,dx = \frac{1}{3}\tan 3x + C$

Recall 6. Let $y = \tan 3x$ and differentiate using the chain rule.

This is 3 times the required expression so you divide by 3.

Example 2 illustrates the following general rule:

■ $\int f'(ax + b)\,dx = \dfrac{1}{a}f(ax + b) + C$

You can generalise the list in Section 6.1 to give:

10 $\int (ax+b)^n \, dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + C$

11 $\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$

12 $\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln |ax+b| + C$

13 $\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$

14 $\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$

15 $\int \sec^2(ax+b) \, dx = \frac{1}{a} \tan(ax+b) + C$

16 $\int \operatorname{cosec}(ax+b)\cot(ax+b) \, dx = -\frac{1}{a} \operatorname{cosec}(ax+b) + C$

17 $\int \operatorname{cosec}^2(ax+b) \, dx = -\frac{1}{a} \cot(ax+b) + C$

18 $\int \sec(ax+b)\tan(ax+b) \, dx = \frac{1}{a} \sec(ax+b) + C$

In C4 it is probably best to learn *how* to work out these results using the chain rule, rather than trying to remember lots of formulae.

Example 3

Find the following integrals:

a $\int \frac{1}{3x+2} \, dx$ **b** $\int (2x+3)^4 \, dx$

a $\int \frac{1}{3x+2} \, dx = \frac{1}{3} \ln |3x+2| + C$

Use 12.

b $\int (2x+3)^4 \, dx = \frac{1}{10} (2x+3)^5 + C$

Use 10.

You can only use the results in the list above for linear transformations of functions. Integrals of the form $\int \cos(2x^2+3) \, dx$ do not give an answer like $\frac{1}{4x} \sin(2x^2+3)$ since differentiating this expression would require the quotient rule and would not give $\cos(2x^2+3)$. Expressions similar to this will be investigated in Section 6.5.

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Exercise 6B

1 Integrate the following:

a $\sin(2x+1)$ **b** $3e^{2x}$ **c** $4e^{x+5}$

d $\cos(1-2x)$ **e** $\operatorname{cosec}^2 3x$ **f** $\sec 4x \tan 4x$

g $3\sin(\frac{1}{2}x+1)$ **h** $\sec^2(2-x)$ **i** $\operatorname{cosec} 2x \cot 2x$

j $\cos 3x - \sin 3x$

2 Find the following integrals:

a $\int (e^{2x} - \frac{1}{2}\sin(2x-1))\,dx$ **b** $\int (e^x+1)^2\,dx$

c $\int \sec^2 2x(1+\sin 2x)\,dx$ **d** $\int \left(\dfrac{3-2\cos(\frac{1}{2}x)}{\sin^2(\frac{1}{2}x)}\right)dx$

e $\int [e^{3-x} + \sin(3-x) + \cos(3-x)]\,dx$

3 Integrate the following:

a $\dfrac{1}{2x+1}$ **b** $\dfrac{1}{(2x+1)^2}$ **c** $(2x+1)^2$ **d** $\dfrac{3}{4x-1}$

e $\dfrac{3}{1-4x}$ **f** $\dfrac{3}{(1-4x)^2}$ **g** $(3x+2)^5$ **h** $\dfrac{3}{(1-2x)^3}$

i $\dfrac{6}{(3-2x)^4}$ **j** $\dfrac{5}{3-2x}$

4 Find the following integrals

a $\int \left(3\sin(2x+1) + \dfrac{4}{2x+1}\right)dx$ **b** $\int [e^{5x} + (1-x)^5]\,dx$

c $\int \left(\dfrac{1}{\sin^2 2x} + \dfrac{1}{1+2x} + \dfrac{1}{(1+2x)^2}\right)dx$ **d** $\int \left[(3x+2)^2 + \dfrac{1}{(3x+2)^2}\right]dx$

6.3 You can use trigonometric identities in integration.

Before you can integrate some trigonometric expressions you may need to replace the original expression with a function you can integrate from the lists on pages 90 and 91. You can do this using trigonometric identities.

Example 4

Find $\int \tan^2 x \, dx$.

Since $\sec^2 x \equiv 1 + \tan^2 x$

Then $\tan^2 x \equiv \sec^2 x - 1$

So $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$

$= \int \sec^2 x \, dx - \int 1 \, dx$

$= \tan x - x + C$

You cannot integrate $\tan^2 x$ but you can integrate $\sec^2 x$ using 6 in the list on page 00.

Use 6.

In Book C3 you met identities for $\cos 2x$ in terms of $\sin^2 x$ and $\cos^2 x$. To integrate $\sin^2 x$ or $\cos^2 x$ you need to use one of these identities.

Example 5

Find $\int \sin^2 x \, dx$.

Recall $\cos 2x \equiv 1 - 2\sin^2 x$

So $\sin^2 x \equiv \frac{1}{2}(1 - \cos 2x)$

So $\int \sin^2 x \, dx = \int (\frac{1}{2} - \frac{1}{2}\cos 2x) \, dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

You cannot integrate $\sin^2 x$ but you can write this in terms of $\cos 2x$.

Remember when integrating $\cos 2x$ you get an extra $\frac{1}{2}$. Use 13.

Example 6

Find:

a $\int \sin 3x \cos 3x \, dx$ **b** $\int (\sec x + \tan x)^2 \, dx$ **c** $\int \sin 3x \cos 2x \, dx$

a

$\int \sin 3x \cos 3x \, dx = \int \frac{1}{2} \sin 6x \, dx$

$= -\frac{1}{2} \times \frac{1}{6} \cos 6x + C$

$= -\frac{1}{12} \cos 6x + C$

Remember $\sin 2A = 2 \sin A \cos A$, so $\sin 6x = 2 \sin 3x \cos 3x$.

Use 14.

Simplify $\frac{1}{2} \times \frac{1}{6}$ to $\frac{1}{12}$.

b

$(\sec x + \tan x)^2$

$= \sec^2 x + 2\sec x \tan x + \tan^2 x$

Multiply out the bracket.

$= \sec^2 x + 2\sec x \tan x + (\sec^2 x - 1)$

$= 2\sec^2 x + 2\sec x \tan x - 1$

Write $\tan^2 x$ as $\sec^2 x - 1$. Then all the terms are standard integrals.

So $\int (\sec x + \tan x)^2\, dx$

$= \int (2\sec^2 x + 2\sec x \tan x - 1)\, dx$

$= 2\tan x + 2\sec x - x + C$

Integrate each term using 6 and 9.

c

$\sin(3x + 2x) = \sin 3x \cos 2x + \cos 3x \sin 2x$

$\sin(3x - 2x) = \sin 3x \cos 2x - \cos 3x \sin 2x$

Remember $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$. So you need to use $A = 3x$ and $B = 2x$ to get a $\sin 3x \cos 2x$ term.

Adding gives

$\sin 5x + \sin x = 2\sin 3x \cos 2x$

So $\int \sin 3x \cos 2x\, dx$

$= \int \frac{1}{2}(\sin 5x + \sin x)\, dx$

$= \frac{1}{2}(-\frac{1}{5}\cos 5x - \cos x) + C$

$= -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C$

Exercise 6C

1 Integrate the following:

a $\cot^2 x$ **b** $\cos^2 x$ **c** $\sin 2x \cos 2x$

d $(1 + \sin x)^2$ **e** $\tan^2 3x$ **f** $(\cot x - \operatorname{cosec} x)^2$

g $(\sin x + \cos x)^2$ **h** $\sin^2 x \cos^2 x$ **i** $\dfrac{1}{\sin^2 x \cos^2 x}$

j $(\cos 2x - 1)^2$

2 Find the following integrals:

a $\displaystyle\int\left(\frac{1-\sin x}{\cos^2 x}\right)\mathrm{d}x$ **b** $\displaystyle\int\left(\frac{1+\cos x}{\sin^2 x}\right)\mathrm{d}x$ **c** $\displaystyle\int\frac{\cos 2x}{\cos^2 x}\,\mathrm{d}x$

d $\displaystyle\int\frac{\cos^2 x}{\sin^2 x}\,\mathrm{d}x$ **e** $\displaystyle\int\frac{(1+\cos x)^2}{\sin^2 x}\,\mathrm{d}x$ **f** $\displaystyle\int\frac{(1+\sin x)^2}{\cos^2 x}\,\mathrm{d}x$

g $\displaystyle\int(\cot x-\tan x)^2\,\mathrm{d}x$ **h** $\displaystyle\int(\cos x-\sin x)^2\,\mathrm{d}x$ **i** $\displaystyle\int(\cos x-\sec x)^2\,\mathrm{d}x$

j $\displaystyle\int\frac{\cos 2x}{1-\cos^2 2x}\,\mathrm{d}x$

3 Find the following integrals:

a $\displaystyle\int\cos 2x\cos x\,\mathrm{d}x$ **b** $\displaystyle\int 2\sin 5x\cos 3x\,\mathrm{d}x$ **c** $\displaystyle\int 2\sin 3x\cos 5x\,\mathrm{d}x$

d $\displaystyle\int 2\sin 2x\sin 5x\,\mathrm{d}x$ **e** $\displaystyle 4\int\cos 3x\cos 7x\,\mathrm{d}x$ **f** $\displaystyle\int 2\cos 4x\cos 4x\,\mathrm{d}x$

g $\displaystyle\int 2\cos 4x\sin 4x\,\mathrm{d}x$ **h** $\displaystyle\int 2\sin 4x\sin 4x\,\mathrm{d}x$

6.4 You can use partial fractions to integrate expressions.

In Chapter 1 you saw how to split certain rational expressions into partial fractions. This process is helpful in integration.

Example 7

Use partial fractions to find the following integrals:

a $\displaystyle\int\frac{x-5}{(x+1)(x-2)}\,\mathrm{d}x$ **b** $\displaystyle\int\frac{8x^2-19x+1}{(2x+1)(x-2)^2}\,\mathrm{d}x$ **c** $\displaystyle\int\frac{2}{(1-x^2)}\,\mathrm{d}x$

a $\displaystyle\frac{x-5}{(x+1)(x-2)}\equiv\frac{A}{x+1}+\frac{B}{x-2}$

Split the expression to be integrated into partial fractions.

So $x-5\equiv A(x-2)+B(x+1)$

Put over the same denominator and compare numerators.

Let $x=-1$: $-6=A(-3)$ so $A=2$

Let $x=2$: $-3=B(3)$ so $B=-1$

Let $x=-1$ and 2.

So $\displaystyle\int\frac{x-5}{(x+1)(x-2)}\,\mathrm{d}x$

$\displaystyle=\int\left(\frac{2}{x+1}-\frac{1}{x-2}\right)\mathrm{d}x$

Rewrite the integral and integrate each term as in Section 6.2.

$=2\ln|x+1|-\ln|x-2|+C$

Remember to use the modulus when using ln in integration.

$\displaystyle=\ln\left|\frac{(x+1)^2}{x-2}\right|+C$

The answer could be left in this form, but sometimes you may be asked to combine the ln terms using the rules of logarithms met in Book C2.

b Let $I = \int \frac{8x^2 - 19x + 1}{(2x+1)(x-2)^2}\,dx$

It is sometimes useful to label the integral as I.

$$\frac{8x^2 - 19x + 1}{(2x+1)(x-2)^2} \equiv \frac{A}{2x+1} + \frac{B}{(x-2)^2} + \frac{C}{x-2}$$

Remember the partial fraction form for a repeated factor on the denominator.

$$8x^2 - 19x + 1 \equiv A(x-2)^2 + B(2x+1) + C(2x+1)(x-2)$$

Put over same denominator and compare numerators, then find the values of A, B and C.

Let $x = 2$

Then $-5 = 0 + 5B + 0$ so $B = -1$

Let $x = -\frac{1}{2}$

Then $12\frac{1}{2} = \frac{25}{4}A + 0 + 0$ so $A = 2$

Let $x = 0$ Then $1 = 4A - 2C + B$

So $1 = 8 - 2C - 1$ so $C = 3$

$$I = \int \left(\frac{2}{2x+1} - \frac{1}{(x-2)^2} + \frac{3}{x-2} \right) dx$$

Rewrite the integral using the partial fractions. Note that using I saves copying the question again.

$$I = \tfrac{2}{2} \ln|2x+1| + \frac{1}{x-2} + 3\ln|x-2| + C$$

Don't forget to divide by 2 when integrating $\frac{1}{2x+1}$ and remember that the integral of $\frac{1}{(x-2)^2}$ does not involve ln.

$$I = \ln|2x+1| + \frac{1}{x-2} + \ln|x-2|^3 + C$$

$$I = \ln|(2x+1)(x-2)^3| + \frac{1}{x-2} + C$$

Simplify using the laws of logarithms.

c Let $I = \int \frac{2}{(1-x^2)}\,dx$

$$\frac{2}{(1-x^2)} = \frac{2}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$$

Remember that $(1 - x^2)$ can be factorised using the difference of two squares.

$$2 = A(1+x) + B(1-x)$$

$x = -1$ gives $2 = 2B$ so $B = 1$

$x = 1$ gives $2 = 2A$ so $A = 1$

So $I = \int \left(\frac{1}{1+x} + \frac{1}{1-x} \right) dx$ ✻

Rewrite the integral using the partial fractions.

$$I = \ln|1+x| - \ln|1-x| + C$$

Notice the minus sign that comes from integrating $\frac{1}{1-x}$.

or $I = \ln\left|\frac{1+x}{1-x}\right| + C$

You should notice that because the modulus sign is used in writing this integral, the answer could be

$$I = \ln\left|\frac{1+x}{x-1}\right| + C$$

since $|1-x| = |x-1|$. This can be found from line ✻ in the above example.

$$I = \int\left(\frac{1}{1+x} + \frac{1}{1-x}\right)\mathrm{d}x$$

$$I = \int\left(\frac{1}{1+x} - \frac{1}{x-1}\right)\mathrm{d}x$$

Since $\frac{1}{1-x} = -\frac{1}{x-1}$.

So $\quad I = \ln|1+x| - \ln|x-1| + C$

So $\quad I = \ln\left|\frac{1+x}{x-1}\right| + C$

Notice that no extra minus sign is needed when integrating $\frac{1}{x-1}$.

This use of the modulus sign, mentioned in Section 6.1, means that both cases can be incorporated in the one expression and this is one reason why the convention is used.

To integrate an improper fraction, you need to divide the numerator by the denominator.

Example 8

Find $\int \frac{9x^2 - 3x + 2}{9x^2 - 4}\,\mathrm{d}x$.

Let $I = \int \frac{9x^2 - 3x + 2}{9x^2 - 4}\,\mathrm{d}x$

First divide the numerator by $9x^2 - 4$.

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\;\; 1 \\ (9x^2-4) & \overline{\,9x^2 - 3x + 2} \\ & \underline{9x^2 \quad\quad\;\; -4} \\ & \quad\quad -3x + 6 \end{array}$$

$9x^2 \div 9x^2$ gives 1, so put this on top and subtract $9x^2 - 4$. This leaves a remainder of $-3x + 6$.

so $\quad I = \int\left(1 + \frac{6-3x}{9x^2-4}\right)\mathrm{d}x$

$$\frac{6-3x}{9x^2-4} = \frac{A}{3x-2} + \frac{B}{3x+2}$$

Factorise $9x^2 - 4$ and then split into partial fractions.

$x = -\frac{2}{3} \Rightarrow 8 = -4B$ so $B = -2$

$x = \frac{2}{3} \Rightarrow 4 = 4A$ so $A = 1$

So $\quad I = \int\left(1 + \frac{1}{3x-2} - \frac{2}{3x+2}\right)\mathrm{d}x$

Rewrite the integral using the partial fractions.

So $\quad I = x + \frac{1}{3}\ln|3x-2| - \frac{2}{3}\ln|3x+2| + C$

Integrate and don't forget the $\frac{1}{3}$.

Or $\quad I = x + \frac{1}{3}\ln\left|\frac{3x-2}{(3x+2)^2}\right| + C$

Simplify using the laws of logarithms.

Exercise 6D

1 Use partial fractions to integrate the following:

a $\dfrac{3x+5}{(x+1)(x+2)}$ **b** $\dfrac{3x-1}{(2x+1)(x-2)}$ **c** $\dfrac{2x-6}{(x+3)(x-1)}$ **d** $\dfrac{3}{(2+x)(1-x)}$

e $\dfrac{4}{(2x+1)(1-2x)}$ **f** $\dfrac{3(x+1)}{9x^2-1}$ **g** $\dfrac{3-5x}{(1-x)(2-3x)}$ **h** $\dfrac{x^2-3}{(2+x)(1+x)^2}$

i $\dfrac{5+3x}{(x+2)(x+1)^2}$ **j** $\dfrac{17-5x}{(3+2x)(2-x)^2}$

2 Find the following integrals:

a $\displaystyle\int \frac{2(x^2+3x-1)}{(x+1)(2x-1)}\,dx$ **b** $\displaystyle\int \frac{x^3+2x^2+2}{x(x+1)}\,dx$

c $\displaystyle\int \frac{x^2}{x^2-4}\,dx$ **d** $\displaystyle\int \frac{x^2+x+2}{3-2x-x^2}\,dx$

e $\displaystyle\int \frac{6+3x-x^2}{x^3+2x^2}\,dx$

6.5 You can use standard patterns to integrate some expressions.

In Section 6.2 you saw how to integrate $\dfrac{1}{2x+3}$ but you could not apply the technique to integrals of the form $\dfrac{1}{x^2+1}$. However there are families of expressions similar to this that can be integrated easily.

Example 9

Find:

a $\displaystyle\int \frac{2x}{x^2+1}\,dx$ **b** $\displaystyle\int \frac{\cos x}{3+2\sin x}\,dx$ **c** $\displaystyle\int 3\cos x\sin^2 x\,dx$ **d** $\displaystyle\int x(x^2+5)^3\,dx$

a Let $I = \displaystyle\int \frac{2x}{x^2+1}\,dx$

Consider $y = \ln|x^2+1|$

Then $\dfrac{dy}{dx} = \dfrac{1}{x^2+1} \times 2x$

Remember the $2x$ comes from differentiating x^2+1 using the chain rule.

So $I = \ln|x^2+1| + C$

Since integration is the reverse of differentiation.

b Let $I = \int \frac{\cos x}{3 + 2\sin x}\,dx$

Try differentiating $y = \ln|3 + 2\sin x|$.

Let $y = \ln|3 + 2\sin x|$

$$\frac{dy}{dx} = \frac{1}{3 + 2\sin x} \times 2\cos x$$

The $2\cos x$ comes from differentiating $3 + 2\sin x$ with the chain rule.

So $I = \frac{1}{2}\ln|3 + 2\sin x| + C$

This is 2 times the required answer so, since integration is the reverse of differentiation you need to divide by 2.

c Let $I = \int 3\cos x \sin^2 x\,dx$

Try differentiating $\sin^3 x$.

Let $y = \sin^3 x$

$$\frac{dy}{dx} = 3\sin^2 x \cos x$$

The $\cos x$ comes from differentiating $\sin x$ in the chain rule.

So $I = \sin^3 x + C$

d Let $I = \int x(x^2 + 5)^3\,dx$

Try differentiating $(x^2 + 5)^4$.

Let $y = (x^2 + 5)^4$

$$\frac{dy}{dx} = 4(x^2 + 5)^3 \times 2x$$

The $2x$ comes from differentiating $x^2 + 5$.

$$= 8x(x^2 + 5)^3$$

This is 8 times the required expression so you divide by 8.

So $I = \frac{1}{8}(x^2 + 5)^4 + C$

You should notice that these examples fall into two types. In **a** and **b** you had $k\frac{f'(x)}{f(x)}$, for some function $f(x)$ and constant k. In **c** and **d** you had $kf'(x)[f(x)]^n$ for some function $f(x)$, constant k and power n.

■ **You should remember the following general patterns:**

- **To integrate expressions of the form $\int k\frac{f'(x)}{f(x)}\,dx$, try $\ln|f(x)|$ and differentiate to check and adjust any constant.**
- **To integrate an expression of the form $\int kf'(x)[f(x)]^n\,dx$, try $[f(x)]^{n+1}$ and differentiate to check and adjust any constant.**

Example 10

Find the following integrals:

a $\int \frac{\operatorname{cosec}^2 x}{(2 + \cot x)^3}\,dx$ **b** $\int 5\tan x \sec^4 x\,dx$

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a Let $I = \int \frac{\operatorname{cosec}^2 x}{(2 + \cot x)^3}\,dx$

Let $y = (2 + \cot x)^{-2}$

$\frac{dy}{dx} = -2(2 + \cot x)^{-3} \times (-\operatorname{cosec}^2 x)$

$= 2(2 + \cot x)^{-3} \operatorname{cosec}^2 x$

So $I = \frac{1}{2}(2 + \cot x)^{-2} + C$

Notice that $2 + \cot x$ is $f(x)$ and $f'(x) = -\operatorname{cosec}^2 x$, $n = -3$.

Use the chain rule.

This is 2 times the required answer so you need to divide by 2.

b Let $I = \int 5 \tan x \sec^4 x\,dx$

Let $y = \sec^4 x$

$\frac{dy}{dx} = 4 \sec^3 x \times \sec x \tan x$

$= 4 \sec^4 x \tan x$

So $I = \frac{5}{4} \sec^4 x + C$

If $f(x) = \sec x$, then $f'(x)$ is $\sec x \tan x$, so $n = 3$ and $k = 5$.

Use the chain rule.

This is $\frac{4}{5}$ times the required answer so you need to divide by $\frac{4}{5}$.

Exercise 6E

1 Integrate the following functions:

a $\frac{x}{x^2 + 4}$ **b** $\frac{e^{2x}}{e^{2x} + 1}$ **c** $\frac{x}{(x^2 + 4)^3}$ **d** $\frac{e^{2x}}{(e^{2x} + 1)^3}$

e $\frac{\cos 2x}{3 + \sin 2x}$ **f** $\frac{\sin 2x}{(3 + \cos 2x)^3}$ **g** xe^{x^2} **h** $\cos 2x(1 + \sin 2x)^4$

i $\sec^2 x \tan^2 x$ **j** $\sec^2 x(1 + \tan^2 x)$

2 Find the following integrals:

a $\int (x + 1)(x^2 + 2x + 3)^4\,dx$ **b** $\int \operatorname{cosec}^2 2x \cot 2x\,dx$

c $\int \sin^5 3x \cos 3x\,dx$ **d** $\int \cos x e^{\sin x}\,dx$

e $\int \frac{e^{2x}}{e^{2x} + 3}\,dx$ **f** $\int x(x^2 + 1)^{\frac{3}{2}}\,dx$

g $\int (2x + 1)\sqrt{x^2 + x + 5}\,dx$ **h** $\int \frac{2x + 1}{\sqrt{x^2 + x + 5}}\,dx$

i $\int \frac{\sin x \cos x}{\sqrt{\cos 2x + 3}}\,dx$ **j** $\int \frac{\sin x \cos x}{\cos 2x + 3}\,dx$

6.6 **Sometimes you can simplify an integral by changing the variable. This process is similar to using the chain rule in differentiation and is called integration by substitution.**

Example 11

Use the substitution $u = 2x + 5$ to find $\int x\sqrt{(2x+5)}\,dx$.

Let $I = \int x\sqrt{(2x+5)}\,dx$

Let $u = 2x + 5$

So $\dfrac{du}{dx} = 2$

So 'dx' can be replaced by '$\frac{1}{2}du$'.

$\sqrt{(2x+5)} = \sqrt{u} = u^{\frac{1}{2}}$

$x = \dfrac{u-5}{2}$

So $I = \int \left(\dfrac{u-5}{2}\right)u^{\frac{1}{2}} \times \frac{1}{2}\,du$

$= \int \frac{1}{4}(u-5)u^{\frac{1}{2}}\,du$

$= \int \frac{1}{4}(u^{\frac{3}{2}} - 5u^{\frac{1}{2}})\,du$

$= \frac{1}{4}\dfrac{u^{\frac{5}{2}}}{\frac{5}{2}} - \dfrac{5u^{\frac{3}{2}}}{4 \times \frac{3}{2}} + C$

$= \dfrac{u^{\frac{5}{2}}}{10} - \dfrac{5u^{\frac{3}{2}}}{6} + C$

So $I = \dfrac{(2x+5)^{\frac{5}{2}}}{10} - \dfrac{5(2x+5)^{\frac{3}{2}}}{6} + C$

You need to replace each 'x' term with a corresponding 'u' term. First replace dx with a term in du.

So $dx = \frac{1}{2}du$.

Next rewrite the function in terms of $u = 2x + 5$.

Rearrange $u = 2x + 5$ to get $2x = u - 5$ and hence $x = \dfrac{u-5}{2}$.

Rewrite I in terms of u and simplify.

Multiply out the bracket and integrate using rules from your C1 book.

Simplify.

Finally rewrite the answer in terms of x.

Example 12

Use the substitution $u = \sin x + 1$ to find $\int \cos x \sin x\,(1 + \sin x)^3\,dx$.

Let $\quad I = \int \cos x \sin x (1 + \sin x)^3\,dx$

Let $\quad u = \sin x + 1$

$\frac{du}{dx} = \cos x$

So substitute 'cos x dx' with 'du'.

$(\sin x + 1)^3 = u^3$

$\sin x = u - 1$

So $\quad I = \int (u - 1)u^3\,du$

$= \int (u^4 - u^3)\,du$

$= \frac{u^5}{5} - \frac{u^4}{4} + C$

So $\quad I = \frac{(\sin x + 1)^5}{5} - \frac{(\sin x + 1)^4}{4} + C$

First replace the 'dx'.

Notice that this could be split as du = cos x dx. The cos x term can be combined with the dx when substituting.

Use $u = \sin x + 1$ to substitute for the remaining terms, rearranging where required to get $\sin x = u - 1$.

Rewrite I in terms of u.

Multiply out the bracket and integrate in the usual way.

Notice that this example might have been disguised by asking you to use the substitution $u = \sin x + 1$ to find $\int \sin 2x(1 + \sin x)^3\,dx$. You then need to write $\sin 2x$ as $2 \sin x \cos x$ and then proceed as in Example 12. Similar examples might appear in the C4 examination.

In the previous examples the substitution was given. In very simple cases you may be left to choose a substitution of your own.

Example 13

Use integration by substitution to find $\int 6xe^{x^2}\,dx$.

Let $\quad I = \int 6xe^{x^2}\,dx.$

Let $\quad u = x^2$

So $\quad \frac{du}{dx} = 2x$

So replace $2x$ dx with du and replace $3e^{x^2}$ with $3e^u$.

So $\quad I = \int 3e^u\,du$

$= 3e^u + C$

So $\quad I = 3e^{x^2} + C$

Since you know how to integrate $\int e^u\,du$, try substituting $u = x^2$.

First aim to replace the dx. You can also use the x and a 2 from the expression. This leaves $3e^{x^2}$.

Write I in terms of u.

Rewrite the answer in terms of x.

Notice that this integral is one that could have been answered by the methods of Section 6.5. All the integrals in that section can be answered by using substitution but if you can learn how to identify those forms then it is quicker to use the method outlined in that section.

Sometimes implicit differentiation may be needed to help you in the first step of an integration by substitution.

Example 14

Use the substitution $u^2 = 2x + 5$ to find $\int x\sqrt{2x+5}\,dx$. (This solution should be compared with Example 11.)

Let $\quad I = \int x\sqrt{2x+5}\,dx$

First aim to replace the dx.

$$u^2 = 2x + 5$$

$$2u\frac{du}{dx} = 2$$

Using implicit differentiation, cancel 2 and rearrange to get d$x = u$du.

So replace dx with udu.

$$\sqrt{2x+5} = u$$

and $\quad x = \dfrac{u^2 - 5}{2}$

Substitute the remaining expressions. You will need to make x the subject of $u^2 = 2x + 5$.

So $\quad I = \int\left(\dfrac{u^2-5}{2}\right)u \times udu$

$$= \int\left(\frac{u^4}{2} - \frac{5u^2}{2}\right)du$$

Multiply out the brackets and integrate.

$$= \frac{u^5}{10} - \frac{5u^3}{6} + C$$

So $\quad I = \dfrac{(2x+5)^{\frac{5}{2}}}{10} - \dfrac{5(2x+5)^{\frac{3}{2}}}{6} + C$

Rewrite answer in terms of x.

If you compare this solution with Example 11 the integration step is a little easier (since you are dealing with integer powers not fractions) but the first and last steps you might consider a little more difficult. Unless the substitution is specified in the question you can choose which sort of substitution you wish to use.

Integration by substitution can also be used to evaluate definite integrals by changing the limits of the integral as well as the expression being integrated.

Example 15

Use integration by substitution to evaluate:

a $\int_0^2 x(x+1)^3\,dx$ **b** $\int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\sin x}\,dx$

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a Let $I = \int_0^2 x(x+1)^3\,dx$

Let $u = x + 1$

$\frac{du}{dx} = 1$

so replace dx with du and replace $(x+1)^3$ with u^3, and x with $u - 1$.

Replace each term in x with a term in u in the usual way.

x	u
2	3
0	1

Change the limits. When $x = 2$, $u = 2 + 1 = 3$ and when $x = 0$, $u = 1$.

So $I = \int_1^3 (u-1)u^3\,du$

Note that the new u limits replace their corresponding x limits.

$= \int_1^3 (u^4 - u^3)\,du$

$= \left[\frac{u^5}{5} - \frac{u^4}{4}\right]_1^3$

Multiply out and integrate. Remember there is no need for a $+C$.

$= (\frac{243}{5} - \frac{81}{4}) - (\frac{1}{5} - \frac{1}{4})$

The integral can now be evaluated using the limits for u without having to change back into x.

$= 48.4 - 20 = 28.4$

b $\int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\sin x}\,dx$

Use $u = 1 + \sin x$.

$u = 1 + \sin x \Rightarrow \frac{du}{dx} = \cos x$, so replace $\cos x\,dx$ with du and replace $\sqrt{(1+\sin x)}$ with $u^{\frac{1}{2}}$.

x	u
$\frac{\pi}{2}$	2
0	1

Remember that limits for integrals involving trigonometric functions will always be in radians.

$x = \frac{\pi}{2}$, means $u = 1 + 1 = 2$ and $x = 0$, means $u = 1 + 0 = 1$.

So $I = \int_1^2 u^{\frac{1}{2}}\,du$

Rewrite the integral in terms of u.

$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^2$

$= \left(\frac{2}{3}2^{\frac{3}{2}}\right) - \left(\frac{2}{3}\right)$

Remember that $2^{\frac{3}{2}} = \sqrt{8} = 2\sqrt{2}$.

So $I = \frac{2}{3}(2\sqrt{2} - 1)$

Exercise 6F

1 Use the given substitution to find the following integrals:

a $\int x\sqrt{1+x}\,\mathrm{d}x;\ u = 1 + x$

b $\int \frac{x}{\sqrt{1+x}}\,\mathrm{d}x;\ u = 1 + x$

c $\int \frac{1+\sin x}{\cos x}\,\mathrm{d}x;\ u = \sin x$

d $\int x(3+2x)^5\,\mathrm{d}x;\ u = 3 + 2x$

e $\int \sin^3 x\,\mathrm{d}x;\ u = \cos x$

2 Use the given substitution to find the following integrals:

a $\int x\sqrt{2+x}\,\mathrm{d}x;\ u^2 = 2 + x$

b $\int \frac{2}{\sqrt{x}(x-4)}\,\mathrm{d}x;\ u = \sqrt{x}$

c $\int \sec^2 x \tan x\sqrt{1+\tan x}\,\mathrm{d}x;\ u^2 = 1 + \tan x$

d $\int \frac{\sqrt{x^2+4}}{x}\,\mathrm{d}x;\ u^2 = x^2 + 4$

e $\int \sec^4 x\,\mathrm{d}x;\ u = \tan x$

3 Evaluate the following:

a $\int_0^5 x\sqrt{x+4}\,\mathrm{d}x$

b $\int_0^{\frac{\pi}{3}} \sec x \tan x\sqrt{\sec x + 2}\,\mathrm{d}x$

c $\int_2^5 \frac{1}{1+\sqrt{x-1}}\,\mathrm{d}x;$ let $u^2 = x - 1$

d $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\,\mathrm{d}\theta;$ let $u = 1 + \cos\theta$

e $\int_0^1 x(2+x)^3\,\mathrm{d}x$

f $\int_1^4 \frac{1}{\sqrt{x}(4x-1)}\,\mathrm{d}x;$ let $u = \sqrt{x}$

6.7 You can use integration by parts to integrate some expressions.

In the C3 book you met the product rule for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

Rearranging gives

$$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(uv) - v\frac{\mathrm{d}u}{\mathrm{d}x}$$

Now integrating each term with respect to x gives

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}(uv)\,\mathrm{d}x - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Now, since differentiating a function and then integrating it leaves the function unchanged, you can simplify $\int \frac{\mathrm{d}}{\mathrm{d}x}(uv)\,\mathrm{d}x$ to uv, and this gives the **integration by parts** formula:

■ $$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

This formula enables you to exchange an integral which is complicated $\left(\int u \frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x\right)$ for a simpler one $\left(\int v \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x\right)$. You will not be expected to produce this proof in the C4 examination.

Example 16

Find $\int x \cos x \,\mathrm{d}x$.

Let $\quad I = \int x \cos x \,\mathrm{d}x$

$$u = x \quad \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$v = \sin x \Leftarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$$

Using the integration by parts formula:

$$I = x \sin x - \int \sin x \times 1 \,\mathrm{d}x$$

$$= x \sin x + \cos x + C$$

Let $u = x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$.

Complete the table for u, v, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$.

Take care to differentiate u but integrate $\frac{\mathrm{d}v}{\mathrm{d}x}$.

Notice that $\int v \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$ is a simpler integral than $\int u \frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x$.

In general you will *usually* let u = any terms of the form x^n, but there is one exception to this and that is when there is a $\ln x$ term. In this case you should let u = the $\ln x$ term.

Example 17

Find $\int x^2 \ln x \,\mathrm{d}x$.

Let $\quad I = \int x^2 \ln x \,\mathrm{d}x$

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$

$$v = \frac{x^3}{3} \Leftarrow \frac{\mathrm{d}v}{\mathrm{d}x} = x^2$$

$$I = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \times \frac{1}{x}\,\mathrm{d}x$$

$$= \frac{x^3}{3} \ln x - \int \frac{x^2}{3}\,\mathrm{d}x$$

$$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C$$

Since there is a $\ln x$ term, let $u = \ln x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = x^2$.

Complete the table for u, v, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$.

Take care to differentiate u but integrate $\frac{\mathrm{d}v}{\mathrm{d}x}$.

Apply the integration by parts formula.

Simplify the $v \frac{\mathrm{d}u}{\mathrm{d}x}$ term.

Sometimes you may have to use integration by parts twice.

Example 18

Find $\int x^2 e^x \, dx$.

Let $\quad I = \int x^2 e^x \, dx$

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x$$

$$v = e^x \Leftarrow \frac{dv}{dx} = e^x$$

So $\quad I = x^2 e^x - \int 2x e^x \, dx$

$$u = 2x \Rightarrow \frac{du}{dx} = 2$$

$$v = e^x \Leftarrow \frac{dv}{dx} = e^x$$

So $\quad I = x^2 e^x - \left[2xe^x - \int 2e^x \, dx \right]$

$$= x^2 e^x - 2xe^x + \int 2e^x \, dx$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

There is no $\ln x$ term so let $u = x^2$ and $\frac{dv}{dx} = e^x$.

Complete the table for u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$.

Take care to differentiate u but integrate $\frac{dv}{dx}$.

Apply the integration by parts formula.

Notice that this integral is simpler than I but still not one you can write down. It has a similar structure to I and so you can use integration by parts again with $u = 2x$ and $\frac{dv}{dx} = e^x$.

Apply the integration by parts formula a second time.

Integration by parts involves integrating in two separate stages (first the uv term then the $\int v \frac{du}{dx} dx$). Any limits can be applied separately to each part.

Example 19

Evaluate $\int_1^2 \ln x \, dx$, leaving your answer in terms of natural logarithms.

Let $I = \int_1^2 \ln x \, dx = \int_1^2 \ln x \times 1 \, dx$

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$v = x \quad \Leftarrow \frac{dv}{dx} = 1$$

$$I = [x \ln x]_1^2 - \int_1^2 x \times \frac{1}{x} \, dx$$

$$= (2 \ln 2) - (1 \ln 1) - \int_1^2 1 \, dx$$

$$= 2 \ln 2 - [x]_1^2$$

$$= 2 \ln 2 - 2 - 1$$

$$= 2 \ln 2 - 1$$

Write the expression to be integrated as $\ln x \times 1$, then $u = \ln x$ and $\frac{dv}{dx} = 1$.

Complete the usual table.

Apply the limits to the uv term and to $\int v \frac{du}{dx} dx$.

Evaluate the limits on uv and remember $\ln 1 = 0$.

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■ **The following integrals should be given in your formulae booklet and they can easily be verified by differentiation. Some of them are used in the next exercise.**

- $\int \tan x \, dx = \ln|\sec x| + C$
- $\int \sec x \, dx = \ln|\sec x + \tan x| + C$
- $\int \cot x \, dx = \ln|\sin x| + C$
- $\int \operatorname{cosec} x \, dx = -\ln|\operatorname{cosec} x + \cot x| + C$

Exercise 6G

1 Find the following integrals:

a $\int x \sin x \, dx$ **b** $\int xe^x \, dx$ **c** $\int x \sec^2 x \, dx$

d $\int x \sec x \tan x \, dx$ **e** $\int \frac{x}{\sin^2 x} \, dx$

2 Find the following integrals:

a $\int x^2 \ln x \, dx$ **b** $\int 3 \ln x \, dx$ **c** $\int \frac{\ln x}{x^3} \, dx$

d $\int (\ln x)^2 \, dx$ **e** $\int (x^2 + 1) \ln x \, dx$

3 Find the following integrals:

a $\int x^2 e^{-x} \, dx$ **b** $\int x^2 \cos x \, dx$ **c** $\int 12x^2(3 + 2x)^5 \, dx$

d $\int 2x^2 \sin 2x \, dx$ **e** $\int x^2 2 \sec^2 x \tan x \, dx$

4 Evaluate the following:

a $\int_0^{\ln 2} xe^{2x} \, dx$ **b** $\int_0^{\frac{\pi}{2}} x \sin x \, dx$ **c** $\int_0^{\frac{\pi}{2}} x \cos x \, dx$ **d** $\int_1^2 \frac{\ln x}{x^2} \, dx$

e $\int_0^1 4x(1 + x)^3 \, dx$ **f** $\int_0^{\pi} x \cos\left(\frac{1}{4}x\right) dx$ **g** $\int_0^{\frac{\pi}{3}} \sin x \ln(\sec x) \, dx$

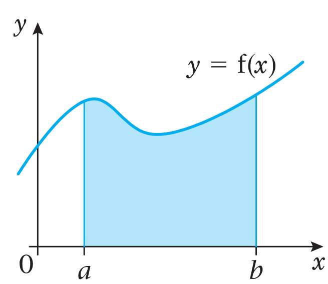
6.8 You can use numerical integration.

In the C2 book you met the **trapezium rule** for finding an approximate value for a definite integral. In C4 you may be asked to use the trapezium rule for integrals involving some of the new functions met in C3 and C4.

■ **Remember: the trapezium rule is**

$$\int_a^b y \, dx \approx \tfrac{1}{2} h[y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n]$$

where $h = \frac{b-a}{n}$ **and** $y_i = f(a + ih)$



Example 20

For the integral $I = \int_0^{\frac{\pi}{3}} \sec x \, dx$:

a Find the exact value of I.

b Use the trapezium rule with two strips to estimate I.

c Use the trapezium rule with four strips to find a second estimate for I.

d Find the percentage error in using these two estimates for I.

a $I = \int_0^{\frac{\pi}{3}} \sec x \, dx$:

$= \left[\ln|\sec x + \tan x|\right]_0^{\frac{\pi}{3}}$

$= (\ln|2 + \sqrt{3}|) - (\ln|1 + 0|)$

$= \ln(2 + \sqrt{3})$

Use the formulae booklet or see Section 6.7.

$\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{0.5} = 2$ and $\tan\frac{\pi}{3} = \sqrt{3}$.

b

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$
y	1	1.155	2

Complete the table to find the values of y.

$I \approx \frac{1}{2}\frac{\pi}{6}[1 + 2 \times 1.155 + 2]$

$= \frac{\pi}{12} \times 5.31 = 1.390\ldots$

$= 1.39$ (3 s.f.)

c

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	1	1.035	1.155	1.414	2

Complete the table to find the values of y.

$I \approx \frac{1}{2}\frac{\pi}{12}[1 + 2(1.035 + 1.155 + 1.414) + 2]$

$= \frac{\pi}{24}[10.208]$

$= 1.336\,224\,075\ldots = 1.34$ (3 s.f.)

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d

Percentage error in using **b** is

$$\frac{(1.390\ldots - \ln|2+\sqrt{3}|)}{\ln|2+\sqrt{3}|} \times 100 = 5.6\%$$

Percentage error in using **c** is

$$\frac{(1.336\ldots - \ln|2+\sqrt{3}|)}{\ln|2+\sqrt{3}|} \times 100 = 1.5\%$$

So **c** is more accurate.

Exercise 6H

1 Use the trapezium rule with n strips to estimate the following:

a $\int_0^3 \ln(1+x^2)\,dx$; $n = 6$

b $\int_0^{\frac{\pi}{3}} \sqrt{(1+\tan x)}\,dx$; $n = 4$

c $\int_0^2 \frac{1}{\sqrt{(e^x+1)}}\,dx$; $n = 4$

d $\int_{-1}^1 \operatorname{cosec}^2(x^2+1)\,dx$; $n = 4$

e $\int_{0.1}^{1.1} \sqrt{\cot x}\,dx$; $n = 5$

2 **a** Find the exact value of $I = \int_1^4 x \ln x\,dx$.

b Find approximate values for I using the trapezium rule with

i 3 strips **ii** 6 strips

c Compare the percentage error for these two approximations.

3 **a** Find an approximate value for $I = \int_0^1 e^x \tan x\,dx$ using

i 2 strips **ii** 4 strips **iii** 8 strips.

b Suggest a possible value for I.

4 **a** Find the exact value of $I = \int_0^2 x\sqrt{(2-x)}\,dx$.

b Find an approximate value for I using the trapezium rule with

i 4 strips **ii** 6 strips.

c Compare the percentage error for these two approximations.

6.9 You can use integration to find areas and volumes.

In the C2 book you saw how to find the area of a region R between a curve and the x-axis:

- **Area of region between $y = f(x)$, the x-axis and $x = a$ and $x = b$ is given by:**

  $$\text{Area} = \int_a^b y \, dx$$

This area can be thought of as a limit of a sum of approximate rectangular strips of width δx and length y.

Hint: Since the strip is roughly rectangular the area is approximately $y\delta x$.

Thus the area is the limit of $\sum y\delta x$ as $\delta x \to 0$. The integration symbol $\int$ is an elongated 'S' to represent this idea of a sum.

If each strip is now revolved through 2π radians (or 360 degrees) about the x-axis, it will form a shape that is approximately cylindrical. The volume of each cylinder will be $\pi y^2 \delta x$ since the radius is y and the height is δx.

The limit of the sum $\sum \pi y^2 \delta x$, as $\delta x \to 0$, is given by $\pi \int y^2 \, dx$ and this formula can be used to find the volume of the solid formed when the region R is rotated through 2π radians about the x-axis.

- **Volume of revolution formed when $y = f(x)$ is rotated about the x-axis between $x = a$ and $x = b$ is given by:**

  $$\text{Volume} = \pi \int_a^b y^2 \, dx$$

Example 21

The region R is bounded by the curve with equation $y = \sin 2x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$.

a Find the area of R.

b Find the volume of the solid formed when the region R is rotated through 2π radians about the x-axis.

a $\text{Area} = \int_0^{\frac{\pi}{2}} \sin 2x \, dx$

$$= \left[-\tfrac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left(-\tfrac{1}{2}(-1)\right) - \left(-\tfrac{1}{2}\right)$$

$$= 1$$

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b Volume $= \pi\int_0^{\frac{\pi}{2}} \sin^2 2x \, dx$

$= \pi\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 4x)\, dx$

$= \pi\left[\frac{1}{2}x - \frac{1}{8}\sin 4x\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{\pi^2}{4} - 0\right) - (0)$

$= \frac{\pi^2}{4}$

Use $\cos 2A = 1 - 2\sin^2 A$.
Rearrange to give $\sin^2 A = \ldots$

Note that $2 \times 2x$ gives $4x$ in the cos term.

Multiply out and integrate.

Sometimes the equation of the curve may be given in terms of parameters. You can integrate in terms of the parameter by changing the variable in a manner similar to that described in Section 6.6.

Example 22

The curve C has parametric equations

$x = t(1 + t)$

$y = \frac{1}{1 + t}$

where t is the parameter and $t \geqslant 0$.
The region R is bounded by C, the x-axis and the lines $x = 0$ and $x = 2$.

a Find the exact area of R.

b Find the exact volume of the solid formed when R is rotated through 2π radians about the x-axis.

a Area $= \int_0^2 y\, dx$

By the chain rule $\int y\, dx = \int y \frac{dx}{dt}\, dt$

$x = t(1 + t) \Rightarrow \frac{dx}{dt} = 1 + 2t$

$x = 0$ so $t(1 + t) = 0$ so $t = 0$ or -1,
but since $t \geqslant 0$, $t = 0$

$x = 2$ so $t^2 + t - 2 = 0$

so $(t + 2)(t - 1) = 0$, so $t = 1$ or -2,
but since $t \geqslant 0$, $t = 1$

You need to change the integral into one in terms of t.

Write $x = t + t^2$ and then differentiate.

Change the limits.

$$\text{So Area} = \int y \frac{dx}{dt}\,dt$$

$$= \int_0^1 \frac{1}{(1+t)}(1+2t)\,dt$$

Divide $(1 + t)$ into the numerator.

$$= \int_0^1 \left(2 - \frac{1}{1+t}\right)dt$$

Simplify using algebraic division.

$$= \Big[2t - \ln|1+t|\Big]_0^1$$

$$= (2 - \ln 2) - (0 - \ln 1)$$

$$= 2 - \ln 2$$

b

$$\text{Volume} = \pi\int y^2\,dx = \pi\int y^2 \frac{dx}{dt}\,dt$$

By the chain rule as above.

$$\text{Volume} = \pi\int_0^1 \frac{1}{(1+t)^2}(1+2t)\,dt$$

Limits will be the same as for the area.

$$\frac{1+2t}{(1+t)^2} \equiv \frac{A}{(1+t)^2} + \frac{B}{(1+t)}$$

Use partial fractions.

$$1 + 2t = A + B(1+t)$$

$$B = 2$$

$$A = -1$$

Substitute values of t or compare coefficients.

$$\text{So Volume} = \pi\int_0^1 \left(\frac{2}{1+t} - \frac{1}{(1+t)^2}\right)dt$$

$$= \pi\left[2\ln|1+t| + \frac{1}{(1+t)}\right]_0^1$$

$$= \pi[(2\ln 2 + \tfrac{1}{2}) - (0 + 1)]$$

$$= \pi(2\ln 2 - \tfrac{1}{2})$$

Exercise 6I

1 The region R is bounded by the curve with equation $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. In each of the following cases find the exact value of:

i the area of R,

ii the volume of the solid of revolution formed by rotating R through 2π radians about the x-axis.

a $f(x) = \dfrac{2}{1+x}$; $a = 0$, $b = 1$

b $f(x) = \sec x$; $a = 0$, $b = \dfrac{\pi}{3}$

c $f(x) = \ln x$; $a = 1$, $b = 2$

d $f(x) = \sec x \tan x$; $a = 0$, $b = \dfrac{\pi}{4}$

e $f(x) = x\sqrt{4 - x^2}$; $a = 0$, $b = 2$

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2 Find the exact area between the curve $y = \mathrm{f}(x)$, the x-axis and the lines $x = a$ and $x = b$ where:

a $\mathrm{f}(x) = \dfrac{4x+3}{(x+2)(2x-1)}$; $a = 1$, $b = 2$

b $\mathrm{f}(x) = \dfrac{x}{(x+1)^2}$; $a = 0$, $b = 2$

c $\mathrm{f}(x) = x\sin x$; $a = 0$, $b = \dfrac{\pi}{2}$

d $\mathrm{f}(x) = \cos x\sqrt{2\sin x + 1}$; $a = 0$, $b = \dfrac{\pi}{6}$

e $\mathrm{f}(x) = x\mathrm{e}^{-x}$; $a = 0$, $b = \ln 2$

3 The region R is bounded by the curve C, the x-axis and the lines $x = -8$ and $x = +8$. The parametric equations for C are $x = t^3$ and $y = t^2$. Find:

a the area of R,

b the volume of the solid of revolution formed when R is rotated through 2π radians about the x-axis.

4 The curve C has parametric equations $x = \sin t$, $y = \sin 2t$, $0 \leqslant t \leqslant \dfrac{\pi}{2}$.

a Find the area of the region bounded by C and the x-axis.

If this region is revolved through 2π radians about the x-axis,

b find the volume of the solid formed.

6.10 You can use integration to solve differential equations.

In Chapter 4 you met differential equations. In this section you will learn how to solve simple first order differential equations by the process called **separation of variables**.

■ **When $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\mathrm{g}(y)$ you can write**

$$\int \frac{1}{\mathrm{g}(y)}\,\mathrm{d}y = \int \mathrm{f}(x)\,\mathrm{d}x$$

This is called separating the variables.

Example 23

Find a general solution to the differential equation $(1 + x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\tan y$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{1+x^2}\tan y$$

Write the equation in the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\mathrm{g}(y)$

$$\int \frac{1}{\tan y}\,\mathrm{d}y = \int \frac{x}{1+x^2}\,\mathrm{d}x$$

Now **separate the variables** so that $\dfrac{1}{\mathrm{g}(y)}\,\mathrm{d}y = \mathrm{f}(x)\,\mathrm{d}x$.

$$\int \cot y\,\mathrm{d}y = \int \frac{x}{1+x^2}\,\mathrm{d}x$$

Use $\cot y = \dfrac{1}{\tan y}$.

$$\ln|\sin y| = \tfrac{1}{2}\ln|1+x^2| + C$$

Integrate, remembering that the integral of $\cot y$ is in the formulae book (or see page 108).

or $\quad \ln|\sin y| = \frac{1}{2}\ln|1 + x^2| + \ln k$

Don't forget the $+C$ which could be written as $\ln k$.

$$\ln|\sin y| = \ln|k\sqrt{1 + x^2}|$$

Combining logs.

so $\quad \sin y = k\sqrt{1 + x^2}$

Finally remove the ln. Sometimes you might be asked to give your answer in the form $y = \mathrm{f}(x)$. This question did not specify that so it is acceptable to give the answer in this form.

Sometimes **boundary conditions** are given in a question which enable you to find a **particular** solution to the differential equation. In this case you first find the general solution and then apply the boundary conditions to find the value of the constant of integration.

Example 24

Find the particular solution of the differential equation

$$\frac{dy}{dx} = \frac{-3(y - 2)}{(2x + 1)(x + 2)}$$

given that $x = 1$ when $y = 4$. Leave your answer in the form $y = \mathrm{f}(x)$.

$$\int \frac{1}{y - 2}\,dy = \int \frac{-3}{(2x + 1)(x + 2)}\,dx$$

First separate the variables.

$$\frac{-3}{(2x + 1)(x + 2)} \equiv \frac{A}{(2x + 1)} + \frac{B}{(x + 2)}$$

Use partial fractions for this integral.

$$-3 = A(x + 2) + B(2x + 1)$$

$x = -2$ gives $\quad -3 = -3B$ so $B = 1$

$x = -0.5$ gives $\quad -3 = \frac{3}{2}A \quad$ so $A = -2$

So

$$\int \frac{1}{y - 2}\,dy = \int \left(\frac{1}{x + 2} - \frac{2}{2x + 1}\right) dx$$

Rewrite the integral using the partial fractions.

$$\ln|y - 2| = \ln|x + 2| - \ln|2x + 1| + \ln k$$

Integrate and use $+\ln k$ instead of $+C$.

$$\ln|y - 2| = \ln\left|\frac{k(x + 2)}{(2x + 1)}\right|$$

Combine ln terms.

$$y - 2 = k\left(\frac{x + 2}{2x + 1}\right)$$

Remove ln.

$$4 - 2 = k\left(\frac{1 + 2}{2 + 1}\right) \Rightarrow k = 2$$

Use the condition $x = 1$ when $y = 4$ by substituting these values in the general solution here, and solve to find k.

So $\quad y = 2 + 2\left(\dfrac{x + 2}{2x + 1}\right)$

$$= 3 + \frac{3}{2x + 1}$$

Substitute $k = 2$ and write the answer in the form $y = \mathrm{f}(x)$ as requested.

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Exercise 6J

1 Find general solutions of the following differential equations. Leave your answer in the form $y = \mathrm{f}(x)$.

a $\dfrac{dy}{dx} = (1 + y)(1 - 2x)$

b $\dfrac{dy}{dx} = y \tan x$

c $\cos^2 x \dfrac{dy}{dx} = y^2 \sin^2 x$

d $\dfrac{dy}{dx} = 2e^{x-y}$

e $x^2 \dfrac{dy}{dx} = y + xy$

2 Find a general solution of the following differential equations. (You do not need to write the answers in the form $y = \mathrm{f}(x)$.)

a $\dfrac{dy}{dx} = \tan y \tan x$

b $\sin y \cos x \dfrac{dy}{dx} = \dfrac{x \cos y}{\cos x}$

c $(1 + x^2) \dfrac{dy}{dx} = x(1 - y^2)$

d $\cos y \sin 2x \dfrac{dy}{dx} = \cot x \operatorname{cosec} y$

e $e^{x+y} \dfrac{dy}{dx} = x(2 + e^y)$

3 Find general solutions of the following differential equations:

a $\dfrac{dy}{dx} = ye^x$

b $\dfrac{dy}{dx} = xe^y$

c $\dfrac{dy}{dx} = y \cos x$

d $\dfrac{dy}{dx} = x \cos y$

e $\dfrac{dy}{dx} = (1 + \cos 2x) \cos y$

f $\dfrac{dy}{dx} = (1 + \cos 2y) \cos x$

4 Find particular solutions of the following differential equations using the given boundary conditions.

a $\dfrac{dy}{dx} = \sin x \cos^2 x;\ y = 0,\ x = \dfrac{\pi}{3}$

b $\dfrac{dy}{dx} = \sec^2 x \sec^2 y;\ y = 0,\ x = \dfrac{\pi}{4}$

c $\dfrac{dy}{dx} = 2 \cos^2 y \cos^2 x;\ y = \dfrac{\pi}{4},\ x = 0$

d $(1 - x^2) \dfrac{dy}{dx} = xy + y;\ x = 0.5,\ y = 6$

e $2(1 + x) \dfrac{dy}{dx} = 1 - y^2;\ x = 5,\ y = \frac{1}{2}$

6.11 **Sometimes the differential equation will arise out of a context and the solution may need interpreting in terms of that context.**

Example 25

The rate of increase of a population P of microorganisms at time t is given by $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$, where k is a positive constant. Given that at $t = 0$ the population was of size 8, and at $t = 1$ the population is 56, find the size of the population at time $t = 2$.

$$\frac{dP}{dt} = kP$$

Separate the variables.

$$\int \frac{1}{P}\,dP = \int k\,dt$$

Integrate and include the $+C$.

$$\ln|P| = kt + C$$

Notice that there are 2 unknown letters, k and C. Use $t = 0$ and $P = 8$ to find C.

$t = 0$, $P = 8$ gives $\ln 8 = C$

So $\ln|P| = kt + \ln 8$

Or $\ln\left|\frac{P}{8}\right| = kt$

Use properties of logs to simplify.

$t = 1$, $P = 56$ gives $\ln 7 = k$

Now use $t = 1$ and $P = 56$ to find k.

So $\ln\left|\frac{P}{8}\right| = t\ln 7$

Now let $t = 2$.

$t = 2$ gives

$$\ln\left|\frac{P}{8}\right| = 2\ln 7$$

but $2\ln 7 = \ln 49 \Rightarrow \frac{P}{8} = 49$

So $P = 8 \times 49 = 392$

Many of the examples in the next exercise are related to differential equations met in Section 4.5.

Exercise 6K

1 The size of a certain population at time t is given by P. The rate of increase of P is given by $\frac{\mathrm{d}P}{\mathrm{d}t} = 2P$. Given that at time $t = 0$, the population was 3, find the population at time $t = 2$.

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2 The number of particles at time t of a certain radioactive substance is N. The substance is decaying in such a way that $\frac{\mathrm{d}N}{\mathrm{d}t} = -\frac{N}{3}$.
Given that at time $t = 0$ the number of particles is N_0, find the time when the number of particles remaining is $\frac{1}{2}N_0$.

3 The mass M at time t of the leaves of a certain plant varies according to the differential equation $\frac{\mathrm{d}M}{\mathrm{d}t} = M - M^2$.

a Given that at time $t = 0$, $M = 0.5$, find an expression for M in terms of t.

b Find a value for M when $t = \ln 2$.

c Explain what happens to the value of M as t increases.

4 The volume of liquid $V\,\mathrm{cm}^3$ at time t seconds satisfies

$$-15\frac{\mathrm{d}V}{\mathrm{d}t} = 2V - 450.$$

Given that initially the volume is $300\,\mathrm{cm}^3$, find to the nearest cm^3 the volume after 15 seconds.

5 The thickness of ice x mm on a pond is increasing and $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{20x^2}$, where t is measured in hours. Find how long it takes the thickness of ice to increase from 1 mm to 2 mm.

6 The depth h metres of fluid in a tank at time t minutes satisfies $\frac{\mathrm{d}h}{\mathrm{d}t} = -k\sqrt{h}$, where k is a positive constant. Find, in terms of k, how long it takes the depth to decrease from 9 m to 4 m.

7 The rate of increase of the radius r kilometres of an oil slick is given by $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r^2}$, where k is a positive constant. When the slick was first observed the radius was 3 km. Two days later it was 5 km. Find, to the nearest day when the radius will be 6.

Mixed exercise 6L

1 It is given that $y = x^{\frac{3}{2}} + \frac{48}{x}$, $x > 0$.

a Find the value of x and the value of y when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

b Show that the value of y which you found is a minimum.

The finite region R is bounded by the curve with equation $y = x^{\frac{3}{2}} + \frac{48}{x}$, the lines $x = 1$, $x = 4$ and the x-axis.

c Find, by integration, the area of R giving your answer in the form $p + q\ln r$, where the numbers p, q and r are to be found. *E*

2 The curve C has two arcs, as shown, and the equations

$$x = 3t^2,\ y = 2t^3,$$

where t is a parameter.

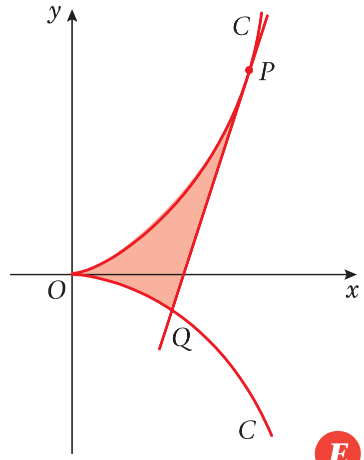
a Find an equation of the tangent to C at the point P where $t = 2$.

The tangent meets the curve again at the point Q.

b Show that the coordinates of Q are $(3, -2)$.

The shaded region R is bounded by the arcs OP and OQ of the curve C, and the line PQ, as shown.

c Find the area of R.



E

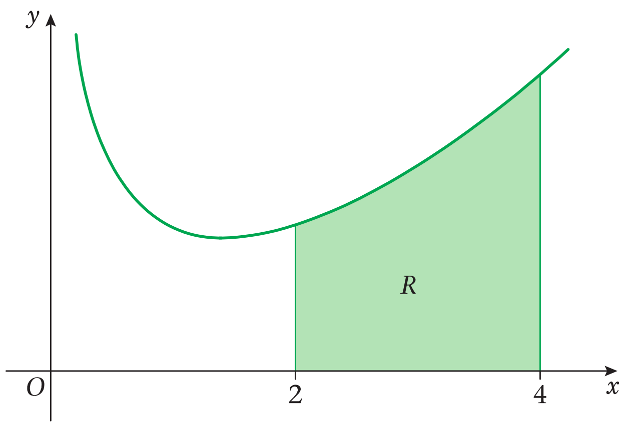
3 **a** Show that $(1 + \sin 2x)^2 \equiv \frac{1}{2}(3 + 4\sin 2x - \cos 4x)$.

b The finite region bounded by the curve with equation $y = 1 + \sin 2x$, the x-axis, the y-axis and the line with equation $x = \frac{\pi}{2}$ is rotated through 2π about the x-axis.

Using calculus, calculate the volume of the solid generated, giving your answer in terms of π.

E

4



This graph shows part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) \equiv \mathrm{e}^{0.5x} + \frac{1}{x},\ x > 0.$$

The curve has a stationary point at $x = \alpha$.

a Find $\mathrm{f}'(x)$.

b Hence calculate $\mathrm{f}'(1.05)$ and $\mathrm{f}'(1.10)$ and deduce that $1.05 < \alpha < 1.10$.

c Find $\int \mathrm{f}(x)\,\mathrm{d}x$.

The shaded region R is bounded by the curve, the x-axis and the lines $x = 2$ and $x = 4$.

d Find, to 2 decimal places, the area of R.

E

5 **a** Find $\int x\mathrm{e}^{-x}\,\mathrm{d}x$.

b Given that $y = \frac{\pi}{4}$ at $x = 0$, solve the differential equation

$$\mathrm{e}^x \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\sin 2y}.$$

E

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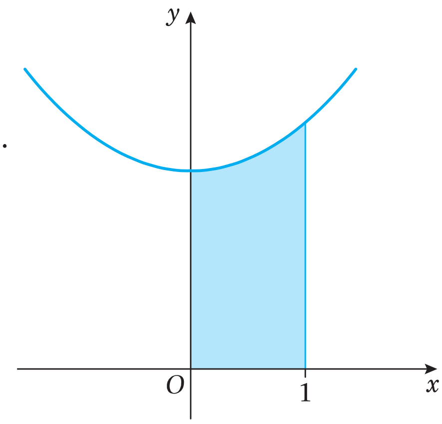
6 The diagram shows the finite shaded region bounded by the curve with equation $y = x^2 + 3$, the lines $x = 1$, $x = 0$ and the x-axis. This region is rotated through 360° about the x-axis.

Find the volume generated.



7 **a** Find $\int \frac{1}{x(x+1)}\,dx$

b Using the substitution $u = e^x$ and the answer to **a**, or otherwise, find $\int \frac{1}{1+e^x}\,dx$.

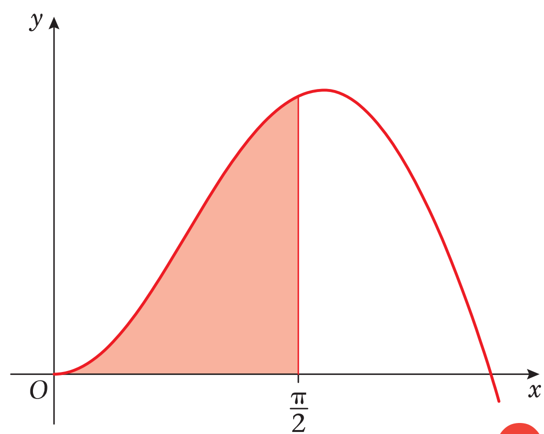
c Use integration by parts to find $\int x^2 \sin x\,dx$. *E*

8 **a** Find $\int x \sin 2x\,dx$.

b Given that $y = 0$ at $x = \frac{\pi}{4}$, solve the differential equation $\frac{dy}{dx} = x \sin 2x \cos^2 y$. *E*

9 **a** Find $\int x \cos 2x\,dx$.

b This diagram shows part of the curve with equation $y = 2x^{\frac{1}{2}} \sin x$. The shaded region in the diagram is bounded by the curve, the x-axis and the line with equation $x = \frac{\pi}{2}$. This shaded region is rotated through 2π radians about the x-axis to form a solid of revolution. Using calculus, calculate the volume of the solid of revolution formed, giving your answer in terms of π.



E

10 A curve has equation $y = f(x)$ and passes through the point with coordinates $(0, -1)$. Given that $f'(x) = \frac{1}{2}e^{2x} - 6x$,

a use integration to obtain an expression for $f(x)$,

b show that there is a root α of the equation $f'(x) = 0$, such that $1.41 < \alpha < 1.43$. *E*

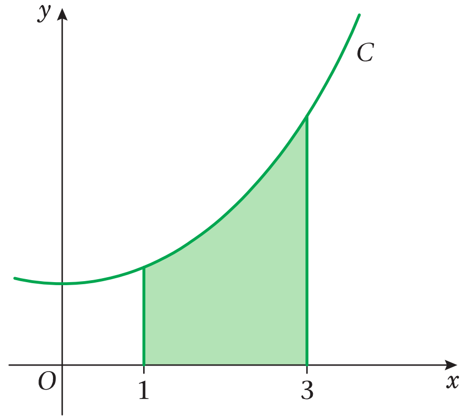
11 $f(x) = 16x^{\frac{1}{2}} - \frac{2}{x}$, $x > 0$.

a Solve the equation $f(x) = 0$.

b Find $\int f(x)\,dx$.

c Evaluate $\int_1^4 f(x)\,dx$, giving your answer in the form $p + q \ln r$, where p, q and r are rational numbers. *E*

12



Shown is part of a curve C with equation $y = x^2 + 3$. The shaded region is bounded by C, the x-axis and the lines with equations $x = 1$ and $x = 3$. The shaded region is rotated through 360° about the x-axis.

Using calculus, calculate the volume of the solid generated. Give your answer as an exact multiple of π. **E**

13 **a** Find $\int x(x^2 + 3)^5 \, dx$

b Show that $\int_1^e \frac{1}{x^2} \ln x \, dx = 1 - \frac{2}{e}$

c Given that $p > 1$, show that $\int_1^p \frac{1}{(x + 1)(2x - 1)} \, dx = \frac{1}{3} \ln \frac{4p - 2}{p + 1}$ **E**

14 $f(x) \equiv \frac{5x^2 - 8x + 1}{2x(x - 1)^2} \equiv \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}$

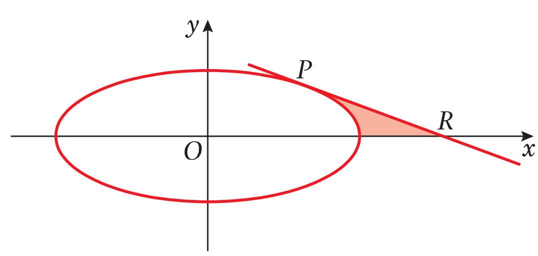
a Find the values of the constants A, B and C.

b Hence find $\int f(x) \, dx$.

c Hence show that $\int_4^9 f(x) \, dx = \ln(\frac{32}{3}) - \frac{5}{24}$ **E**

15 The curve shown has parametric equations

$$x = 5 \cos \theta, \; y = 4 \sin \theta, \; 0 \leq \theta < 2\pi.$$



a Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$.

b Find an equation of the tangent to the curve at the point P.

c Find the coordinates of the point R where this tangent meets the x-axis.

The shaded region is bounded by the tangent PR, the curve and the x-axis.

d Find the area of the shaded region, leaving your answer in terms of π. **E**

16 **a** Obtain the general solution of the differential equation

$$\frac{dy}{dx} = xy^2, \ y > 0.$$

b Given also that $y = 1$ at $x = 1$, show that

$$y = \frac{2}{3 - x^2}, \ -\sqrt{3} < x < \sqrt{3}$$

is a particular solution of the differential equation.

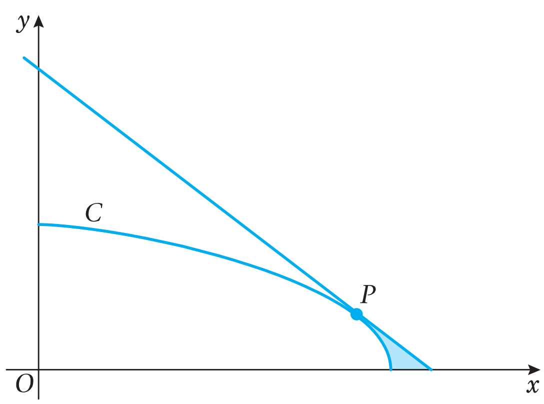
The curve C has equation $y = \dfrac{2}{3 - x^2}$, $x \neq -\sqrt{3}$, $x \neq \sqrt{3}$

c Write down the gradient of C at the point $(1, 1)$.

d Deduce that the line which is a tangent to C at the point $(1, 1)$ has equation $y = x$.

e Find the coordinates of the point where the line $y = x$ again meets the curve C. *E*

17



The diagram shows the curve C with parametric equations

$$x = a\sin^2 t, \ y = a\cos t, \ 0 \leqslant t \leqslant \tfrac{1}{2}\pi,$$

where a is a positive constant. The point P lies on C and has coordinates $(\frac{3}{4}a, \frac{1}{2}a)$.

a Find $\dfrac{dy}{dx}$, giving your answer in terms of t.

b Find an equation of the tangent to C at P.

c Show that a cartesian equation of C is $y^2 = a^2 - ax$.

The shaded region is bounded by C, the tangent at P and the x-axis. This shaded region is rotated through 2π radians about the x-axis to form a solid of revolution.

d Use calculus to calculate the volume of the solid revolution formed, giving your answer in the form $k\pi a^3$, where k is an exact fraction. *E*

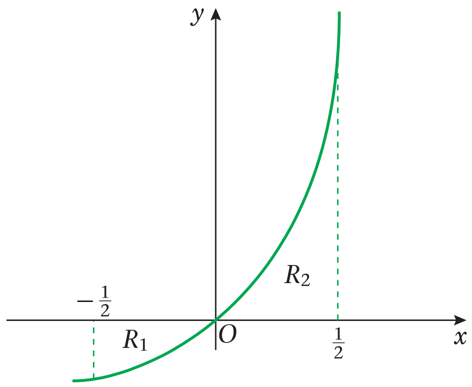
18 **a** Using the substitution $u = 1 + 2x$, or otherwise, find

$$\int \frac{4x}{(1 + 2x)^2}\,dx, \ x > -\tfrac{1}{2},$$

b Given that $y = \dfrac{\pi}{4}$ when $x = 0$, solve the differential equation

$$(1 + 2x)^2 \frac{dy}{dx} = \frac{x}{\sin^2 y}$$ *E*

19 The diagram shows the curve with equation $y = xe^{2x}$, $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$.



The finite region R_1 bounded by the curve, the x-axis and the line $x = -\frac{1}{2}$ has area A_1.
The finite region R_2 bounded by the curve, the x-axis and the line $x = \frac{1}{2}$ has area A_2.

a Find the exact values of A_1 and A_2 by integration.

b Show that $A_1 : A_2 = (e - 2) : e$. *E*

20 Find $\int x^2 e^{-x}\,dx$.

Given that $y = 0$ at $x = 0$, solve the differential equation $\frac{dy}{dx} = x^2 e^{3y - x}$. *E*

21 The curve with equation $y = e^{3x} + 1$ meets the line $y = 8$ at the point $(h, 8)$.

a Find h, giving your answer in terms of natural logarithms.

b Show that the area of the finite region enclosed by the curve with equation $y = e^{3x} + 1$, the x-axis, the y-axis and the line $x = h$ is $2 + \frac{1}{3}\ln 7$. *E*

22 **a** Given that

$$\frac{x^2}{x^2 - 1} \equiv A + \frac{B}{x - 1} + \frac{C}{x + 1},$$

find the values of the constants A, B and C.

b Given that $x = 2$ at $t = 1$, solve the differential equation

$$\frac{dx}{dt} = 2 - \frac{2}{x^2},\ x > 1.$$

You need not simplify your final answer. *E*

23 The curve C is given by the equations

$$x = 2t,\ y = t^2,$$

where t is a parameter.

a Find an equation of the normal to C at the point P on C where $t = 3$.

The normal meets the y-axis at the point B. The finite region R is bounded by the part of the curve C between the origin O and P, and the lines OB and BP.

b Show the region R, together with its boundaries, in a sketch.

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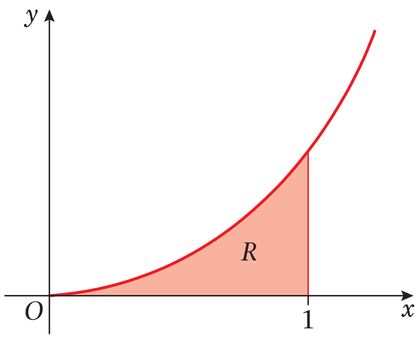
24



Shown is part of the curve with equation $y = e^{2x} - e^{-x}$. The shaded region R is bounded by the curve, the x-axis and the line with equation $x = 1$.

Use calculus to find the area of R, giving your answer in terms of e. **E**

25 **a** Given that $2y = x - \sin x \cos x$, show that $\frac{dy}{dx} = \sin^2 x$.

b Hence find $\int \sin^2 x \, dx$.

c Hence, using integration by parts, find $\int x \sin^2 x \, dx$. **E**

26 The rate, in $cm^3 s^{-1}$, at which oil is leaking from an engine sump at any time t seconds is proportional to the volume of oil, V cm^3, in the sump at that instant. At time $t = 0$, $V = A$.

a By forming and integrating a differential equation, show that

$$V = Ae^{-kt}$$

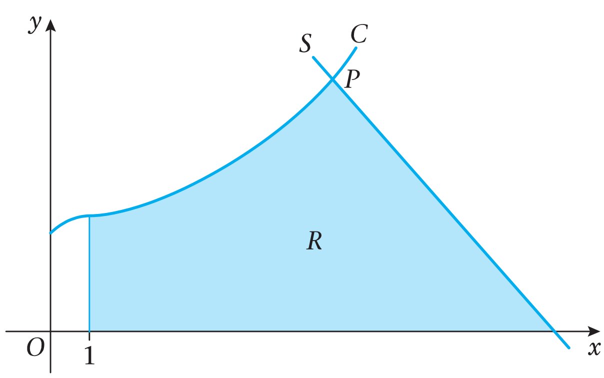
where k is a positive constant.

b Sketch a graph to show the relation between V and t.

Given further that $V = \frac{1}{2}A$ at $t = T$,

c show that $kT = \ln 2$. **E**

27



This graph shows part of the curve C with parametric equations

$$x = (t + 1)^2, \; y = \frac{1}{2}t^3 + 3, \; t \geqslant -1.$$

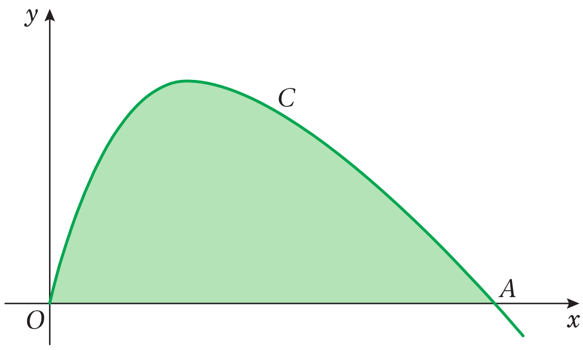
P is the point on the curve where $t = 2$. The line S is the normal to C at P.

a Find an equation of S.

The shaded region R is bounded by C, S, the x-axis and the line with equation $x = 1$.

b Using integration and showing all your working, find the area of R. **E**

28



Shown is part of the curve C with parametric equations

$$x = t^2,\ y = \sin 2t,\ t \geqslant 0.$$

The point A is an intersection of C with the x-axis.

a Find, in terms of π, the x-coordinate of A.

b Find $\dfrac{dy}{dx}$ in terms of t, $t > 0$.

c Show that an equation of the tangent to C at A is $4x + 2\pi y = \pi^2$.

The shaded region is bounded by C and the x-axis.

d Use calculus to find, in terms of π, the area of the shaded region. *E*

29 Showing your method clearly in each case, find

a $\displaystyle\int \sin^2 x \cos x \, dx$,

b $\displaystyle\int x \ln x \, dx$.

Using the substitution $t^2 = x + 1$, where $x > -1$, $t > 0$,

c Find $\displaystyle\int \frac{x}{\sqrt{x+1}} \, dx$.

d Hence evaluate $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}} \, dx$. *E*

30 **a** Using the substitution $u = 1 + 2x^2$, find $\displaystyle\int x(1 + 2x^2)^5 \, dx$.

b Given that $y = \dfrac{\pi}{8}$ at $x = 0$, solve the differential equation

$$\frac{dy}{dx} = x(1 + 2x^2)^5 \cos^2 2y.$$ *E*

31 Find $\displaystyle\int x^2 \ln 2x \, dx$. *E*

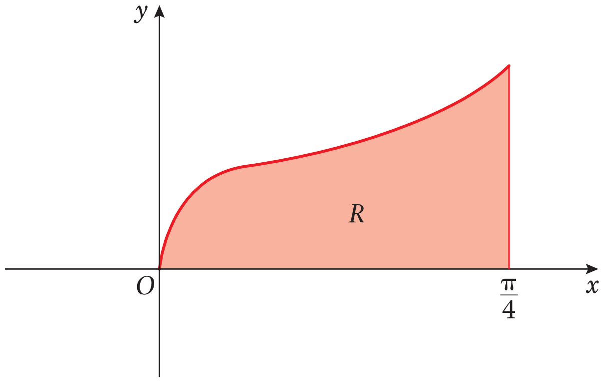
32 Obtain the solution of

$$x(x+2)\frac{dy}{dx} = y,\ y > 0,\ x > 0,$$

for which $y = 2$ at $x = 2$, giving your answer in the form $y^2 = f(x)$. *E*

33 **a** Use integration by parts to show that

$$\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx = \tfrac{1}{4}\pi - \tfrac{1}{2}\ln 2.$$

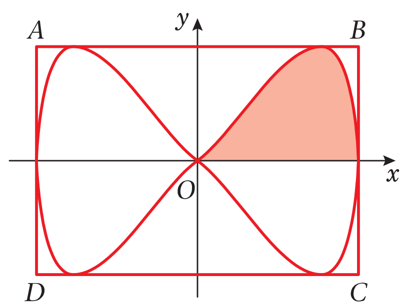


The finite region R, bounded by the curve with equation $y = x^{\frac{1}{2}} \sec x$, the line $x = \dfrac{\pi}{4}$ and the x-axis is shown. The region R is rotated through 2π radians about the x-axis.

b Find the volume of the solid of revolution generated.

c Find the gradient of the curve with equation $y = x^{\frac{1}{2}} \sec x$ at the point where $x = \dfrac{\pi}{4}$. **E**

34 Part of the design of a stained glass window is shown. The two loops enclose an area of blue glass. The remaining area within the rectangle $ABCD$ is red glass.



The loops are described by the curve with parametric equations

$$x = 3\cos t, \; y = 9\sin 2t, \; 0 \leqslant t < 2\pi.$$

a Find the Cartesian equation of the curve in the form $y^2 = \mathrm{f}(x)$.

b Show that the shaded area enclosed by the curve and the positive x-axis, is given by

$$\int_0^{\frac{\pi}{2}} A \sin 2t \sin t \, dt,$$ stating the value of the constant A.

c Find the value of this integral.

The sides of the rectangle $ABCD$ are the tangents to the curve that are parallel to the coordinate axes. Given that 1 unit on each axis represents 1 cm,

d find the total area of the red glass. **E**

Summary of key points

1 You should be familiar with the following integrals.

$$\int x^n = \frac{x^{n+1}}{n+1} + C$$

$$\int e^x = e^x + C$$

$$\int \frac{1}{x} = \ln|x| + C$$

$$\int \cos x = \sin x + C$$

$$\int \sin x = -\cos x + C$$

$$\int \sec^2 x = \tan x + C$$

$$\int \operatorname{cosec} x \cot x = -\operatorname{cosec} x + C$$

$$\int \operatorname{cosec}^2 x = -\cot x + C$$

$$\int \sec x \tan x = \sec x + C$$

2 Using the chain rule in reverse you can obtain generalisations of the above formulae.

$$\int f'(ax+b)\,dx = \frac{1}{a}f(ax+b) + C$$

$$\int (ax+b)^n\,dx = \frac{1}{a}\,\frac{(ax+b)^{n+1}}{n+1} + C$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

$$\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + C$$

$$\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + C$$

$$\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$$

3 Sometimes trigonometric identities can be useful to help change the expression into one you know how to integrate.
e.g. To integrate $\sin^2 x$ or $\cos^2 x$ use formulae for $\cos 2x$, so

$$\int \sin^2 x \, dx = \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) dx$$

4 You can use partial fractions to integrate expressions of the type $\dfrac{x-5}{(x+1)(x-2)}$.

5 You should remember the following general patterns:

$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$$

$$\int f'(x)[f(x)]^n \, dx = \frac{1}{n+1}[f(x)]^{n+1};\ n \neq -1$$

6 Sometimes you can simplify an integral by changing the variable. This process is similar to using the chain rule in differentiation and is called **integration by substitution**.

7 **Integration by parts:**

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$$

8 $\int \tan x \, dx = \ln|\sec x| + C$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

$$\int \operatorname{cosec} x \, dx = -\ln|\operatorname{cosec} x + \cot x| + C$$

9 Remember: the trapezium rule is

$$\int_a^b y \, dx \approx \tfrac{1}{2}h[y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n]$$

where $h = \dfrac{b-a}{n}$ and $y_i = f(a + ih)$

10 Area of region between $y = f(x)$, the x-axis and $x = a$ and $x = b$ is given by:

$$\text{Area} = \int_a^b y \, dx$$

11 Volume of revolution formed by rotating y about the x-axis between $x = a$ and $x = b$ is given by:

$$\text{Volume} = \pi \int_a^b y^2 \, dx$$

12 When $\dfrac{dy}{dx} = f(x)g(y)$ you can write

$$\int \frac{1}{g(y)} \, dy = \int f(x) \, dx$$

This is called separating the variables.