Solutionbank
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

Sketch the graphs of

(a) $y = e^x + 1$

(b) $y = 4e^{-2x}$

(c) $y = 2e^x - 3$

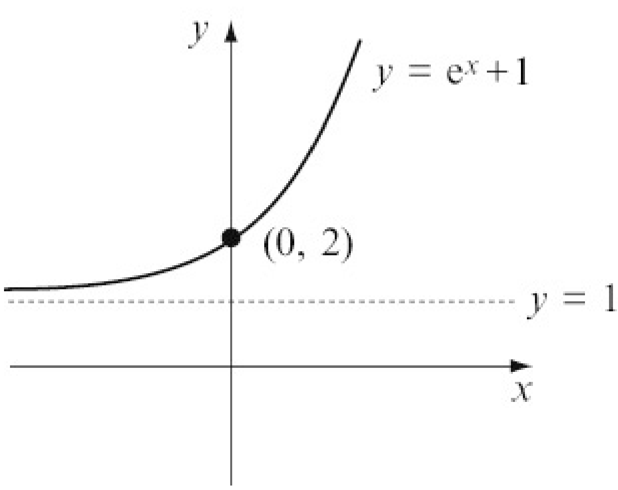
(d) $y = 4 - e^x$

(e) $y = 6 + 10e^{\frac{1}{2}x}$

(f) $y = 100e^{-x} + 10$

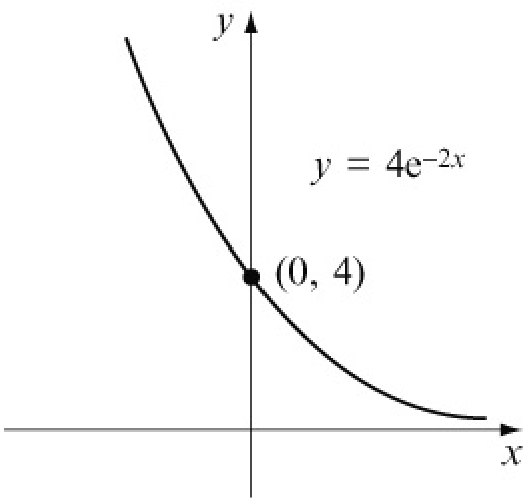
Solution:

(a) $y = e^x + 1$



This is the normal $y = e^x$ 'moved up' (translated) 1 unit.

(b) $y = 4e^{-2x}$



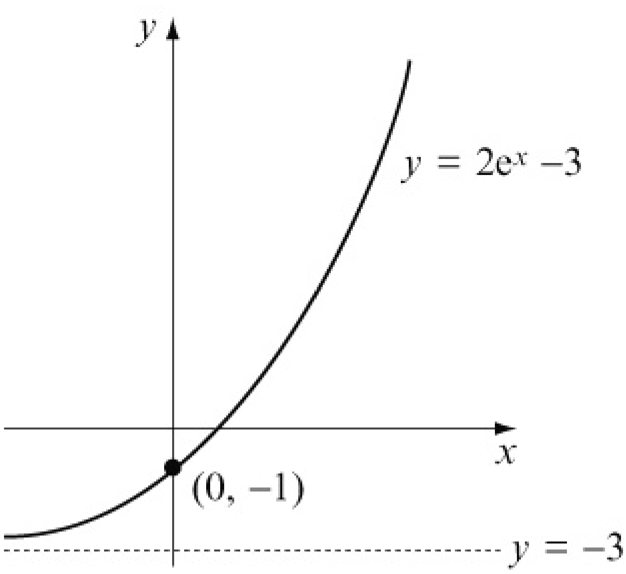
$x = 0 \quad \Rightarrow \quad y = 4$

As $x \to -\infty$, $y \to \infty$

As $x \to \infty$, $y \to 0$

This is an exponential decay type graph.

(c) $y = 2e^{x} - 3$

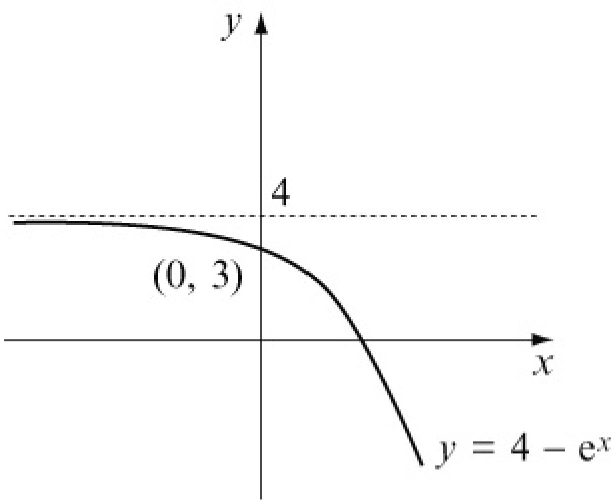


$x = 0 \quad \Rightarrow \quad y = 2 \times 1 - 3 = -1$

As $x \to \infty$, $y \to \infty$

As $x \to -\infty$, $y \to 2 \times 0 - 3 = -3$

(d) $y = 4 - e^{x}$

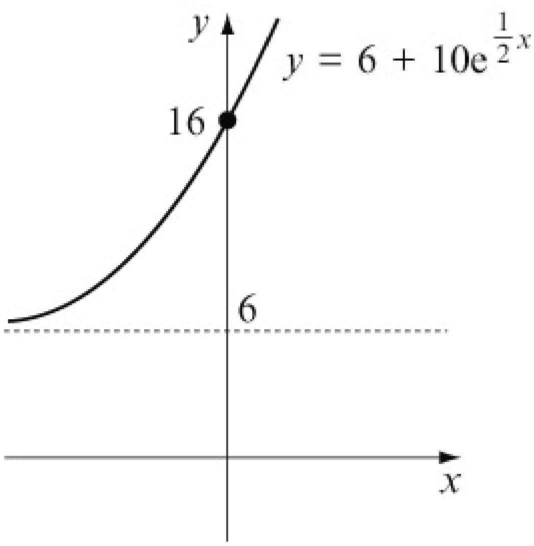


$x = 0 \quad \Rightarrow \quad y = 4 - 1 = 3$

As $x \rightarrow \infty$, $y \rightarrow 4 - \infty$, i.e. $y \rightarrow -\infty$

As $x \rightarrow -\infty$, $y \rightarrow 4 - 0 = 4$

(e) $y = 6 + 10^{\frac{1}{2}x}$

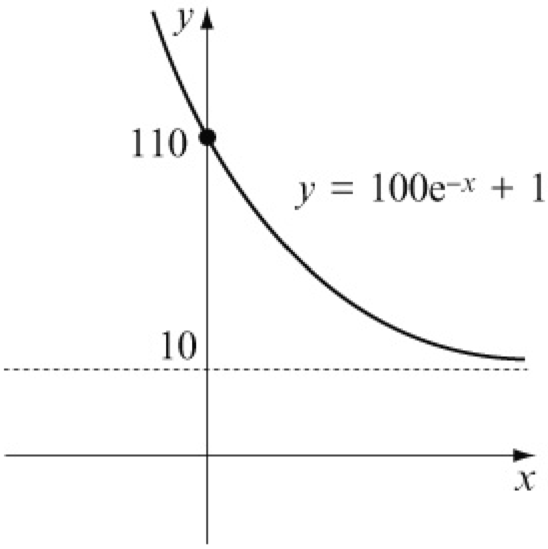


$x = 0 \quad \Rightarrow \quad y = 6 + 10 \times 1 = 16$

As $x \rightarrow \infty$, $y \rightarrow \infty$

As $x \rightarrow -\infty$, $y \rightarrow 6 + 10 \times 0 = 6$

(f) $y = 100e^{-x} + 10$



$x = 0 \quad \Rightarrow \quad y = 100 \times 1 + 10 = 110$

As $x \to \infty$, $y \to 100 \times 0 + 10 = 10$

As $x \to -\infty$, $y \to \infty$

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Exercise A, Question 2

Question:

The value of a car varies according to the formula

$V = 20\,000e^{-\frac{t}{12}}$

where V is the value in £'s and t is its age in years from new.

(a) State its value when new.

(b) Find its value (to the nearest £) after 4 years.

(c) Sketch the graph of V against t.

Solution:

$V = 20\,000e^{-\frac{t}{12}}$

(a) The new value is when $t = 0$.

$\Rightarrow \quad V = 20\,000 \times e^{-\frac{0}{12}} = 20\,000 \times 1 = 20\,000$

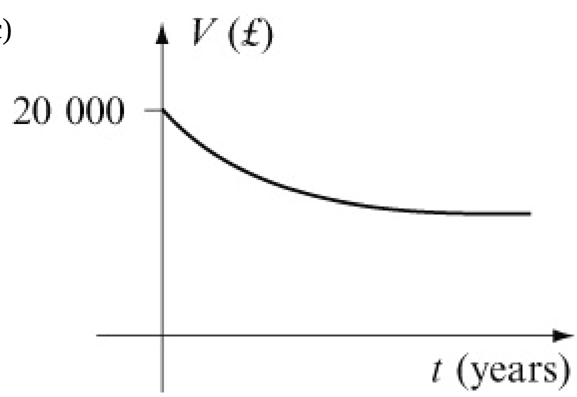
New value $= £20\,000$

(b) Value after 4 years is given when $t = 4$.

$\Rightarrow \quad V = 20\,000 \times e^{-\frac{4}{12}} = 20\,000 \times e^{-\frac{1}{3}} = 14\,330.63$

Value after 4 years is £14 331 (to nearest £)

(c)



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Exercise A, Question 3

Question:

The population of a country is increasing according to the formula

$P = 20 + 10\,e^{\frac{t}{50}}$

where P is the population in thousands and t is the time in years after the year 2000.

(a) State the population in the year 2000.

(b) Use the model to predict the population in the year 2020.

(c) Sketch the graph of P against t for the years 2000 to 2100.

Solution:

$P = 20 + 10\,e^{\frac{t}{50}}$

(a) The year 2000 corresponds to $t = 0$.

Substitute $t = 0$ into $P = 20 + 10\,e^{\frac{t}{50}}$

$P = 20 + 10 \times e^{0} = 20 + 10 \times 1 = 30$

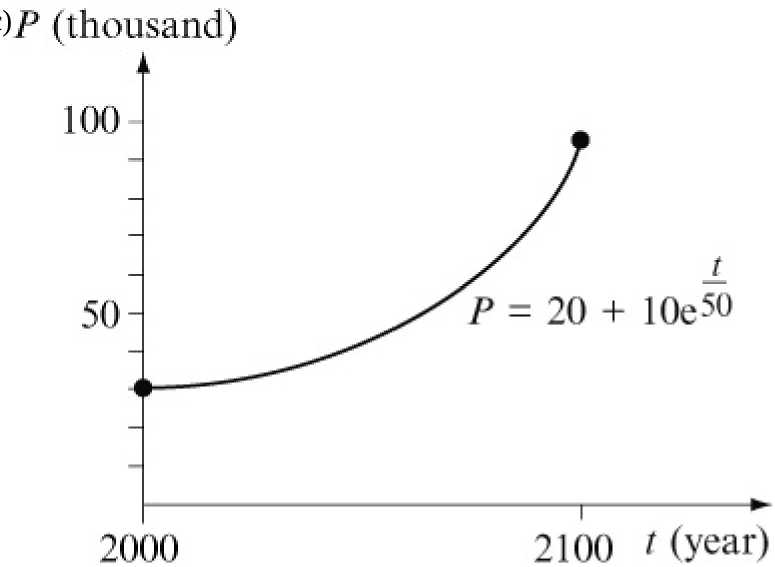
Population = 30 thousand

(b) The year 2020 corresponds to $t = 20$.

Substitute $t = 20$ into $P = 20 + 10\,e^{\frac{t}{50}}$

$P = 20 + 10\,e^{\frac{20}{50}} = 20 + 14.918 = 34.918$ thousand

Population in 2020 will be 34 918

(c)


Year 2100 is $t = 100$

$P = 20 + 10\,e^{\frac{100}{50}} = 20 + 10\,e^{2} = 93.891$ thousand

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Exercise A, Question 4

Question:

The number of people infected with a disease varies according to the formula
$N = 300 - 100e^{-0.5t}$
where N is the number of people infected with the disease and t is the time in years after detection.

(a) How many people were first diagnosed with the disease?

(b) What is the long term prediction of how this disease will spread?

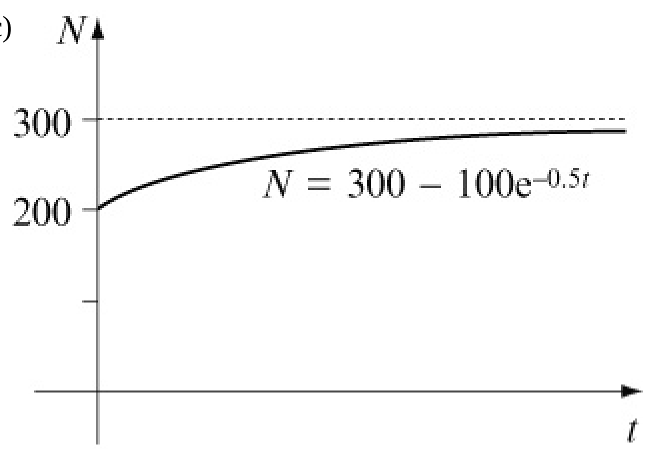
(c) Graph N against t.

Solution:

$N = 300 - 100e^{-0.5t}$
(a) The number *first* diagnosed means when $t = 0$.
Substitute $t = 0$ in $N = 300 - 100e^{-0.5t}$
$N = 300 - 100 \times e^{-0.5 \times 0} = 300 - 100 \times 1 = 200$

(b) The long term prediction suggests $t \to \infty$.
As $t \to \infty$, $e^{-0.5t} \to 0$
So $N \to 300 - 100 \times 0 = 300$

(c)



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Exercise A, Question 5

Question:

The value of an investment varies according to the formula

$V = A e^{\frac{t}{12}}$

where V is the value of the investment in £'s, A is a constant to be found and t is the time in years after the investment was made.

(a) If the investment was worth £8000 after 3 years find A to the nearest £.

(b) Find the value of the investment after 10 years.

(c) By what factor will the original investment have increased by after 20 years?

Solution:

$V = A e^{\frac{t}{12}}$

(a) We are given that $V = 8000$ when $t = 3$.
Substituting gives

$8000 = A e^{\frac{3}{12}}$

$8000 = A e^{\frac{1}{4}}$ $\quad (\div e^{\frac{1}{4}})$

$A = \frac{8000}{e^{\frac{1}{4}}}$

$A = 8000 e^{-\frac{1}{4}}$
$A = 6230.41$
$A = £6230$ (to the nearest £)

(b) Hence $V = \left(8000 \times e^{-\frac{1}{4}} \right) e^{\frac{t}{12}}$ (use real value)
After 10 years

$V = 8000 \times e^{-\frac{1}{4}} \times e^{\frac{10}{12}}$ (use laws of indices)

$= 8000 \times e^{\frac{10}{12} - \frac{3}{12}}$

$= 8000 e^{\frac{7}{12}}$

= £14 336.01
Investment is worth £14 336 (to nearest £) after 10 years.

(c) After 20 years $V = Ae^{\frac{20}{12}}$

This is $e^{\frac{20}{12}}$ times the original amount A
= 5.29 times.

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Exercise B, Question 1

Question:

Solve the following equations giving exact solutions:

(a) $e^{x} = 5$

(b) $\ln x = 4$

(c) $e^{2x} = 7$

(d) $\ln \frac{x}{2} = 4$

(e) $e^{x-1} = 8$

(f) $\ln (2x + 1) = 5$

(g) $e^{-x} = 10$

(h) $\ln (2 - x) = 4$

(i) $2e^{\text{\ }4x} - 3 = 8$

Solution:

(a) $e^{x} = 5 \quad \Rightarrow \quad x = \ln 5$

(b) $\ln x = 4 \quad \Rightarrow \quad x = e\text{\ }^{4}$

(c) $e^{2x} = 7 \quad \Rightarrow \quad 2x = \ln 7 \quad \Rightarrow \quad x = \frac{\ln 7}{2}$

(d) $\ln \left(\frac{x}{2} \right) = 4 \quad \Rightarrow \quad \frac{x}{2} = e\text{\ }^{4} \quad \Rightarrow \quad x = 2e\text{\ }^{4}$

(e) $e^{x-1} = 8 \quad \Rightarrow \quad x - 1 = \ln 8 \quad \Rightarrow \quad x = \ln 8 + 1$

(f) $\ln (2x + 1) = 5$
$\Rightarrow \quad 2x + 1 = e^{5}$

$\Rightarrow \quad 2x = e^5 - 1$

$\Rightarrow \quad x = \frac{e^5 - 1}{2}$

(g) $e^{-x} = 10$

$\Rightarrow \quad -x = \ln 10$

$\Rightarrow \quad x = -\ln 10$

$\Rightarrow \quad x = \ln 10^{-1}$

$\Rightarrow \quad x = \ln(0.1)$

(h) $\ln(2 - x) = 4$

$\Rightarrow \quad 2 - x =$ e ^{4}

$\Rightarrow \quad 2 =$ e $^{4} + x$

$\Rightarrow \quad x = 2 -$ e ^{4}

(i) 2e$^{\text{ }4x} - 3 = 8$

$\Rightarrow$ 2e$^{\text{ }4x} = 11$

$\Rightarrow$ e$^{\text{ }4x} = \frac{11}{2}$

$\Rightarrow \quad 4x = \ln\left(\frac{11}{2}\right)$

$\Rightarrow \quad x = \frac{1}{4}\ln\left(\frac{11}{2}\right)$

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Exercise B, Question 2

Question:

Solve the following giving your solution in terms of ln 2:

(a) $e^{3x} = 8$

(b) $e^{-2x} = 4$

(c) $e^{2x+1} = 0.5$

Solution:

(a) $e^{3x} = 8$

$\Rightarrow \quad 3x = \ln 8$

$\Rightarrow \quad 3x = \ln 2^3$

$\Rightarrow \quad 3x = 3\ln 2$

$\Rightarrow \quad x = \ln 2$

(b) $e^{-2x} = 4$

$\Rightarrow \quad -2x = \ln 4$

$\Rightarrow \quad -2x = \ln 2^2$

$\Rightarrow \quad -2x = 2\ln 2$

$\Rightarrow \quad x = \dfrac{2\ln 2}{-2}$

$\Rightarrow \quad x = -1\ln 2$

(c) $e^{2x+1} = 0.5$

$\Rightarrow \quad 2x + 1 = \ln(0.5)$

$\Rightarrow \quad 2x + 1 = \ln 2^{-1}$

$\Rightarrow \quad 2x + 1 = -\ln 2$

$\Rightarrow \quad 2x = -\ln 2 - 1$

$\Rightarrow \quad x = \dfrac{-\ln 2 - 1}{2}$

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Exercise B, Question 3

Question:

Sketch the following graphs stating any asymptotes and intersections with axes:

(a) $y = \ln (x + 1)$

(b) $y = 2 \ln x$

(c) $y = \ln (2x)$

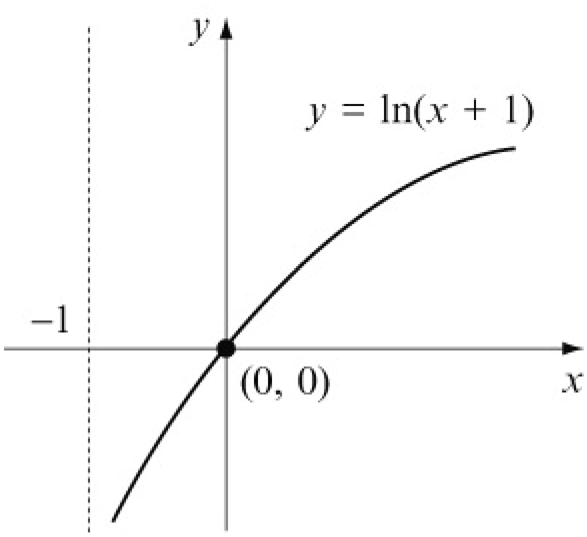
(d) $y = (\ln x)^2$

(e) $y = \ln (4 - x)$

(f) $y = 3 + \ln (x + 2)$

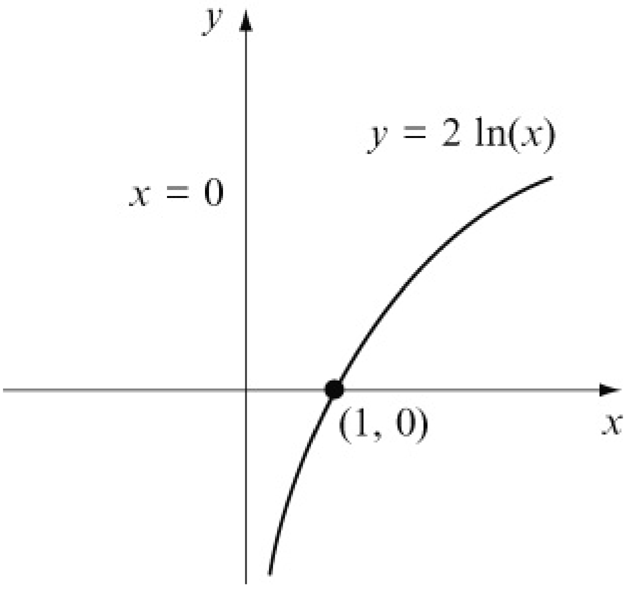
Solution:

(a) $y = \ln (x + 1)$



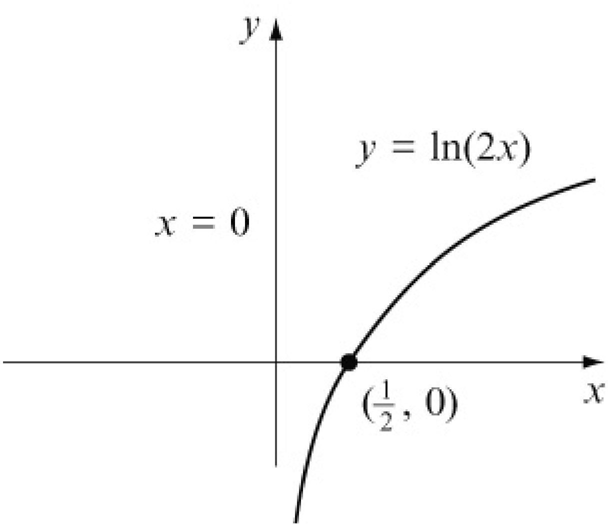
When $x = 0$, $y = \ln (1) = 0$
When $x \to -1$, $y \to -\infty$
y wouldn't exist for values of $x < -1$
When $x \to \infty$, $y \to \infty$ (slowly)

(b) $y = 2 \ln x$



When $x = 1, y = 2 \ln(1) = 0$
When $x \to 0, y \to -\infty$
y wouldn't exist for values of $x < 0$
When $x \to \infty, y \to \infty$ (slowly)

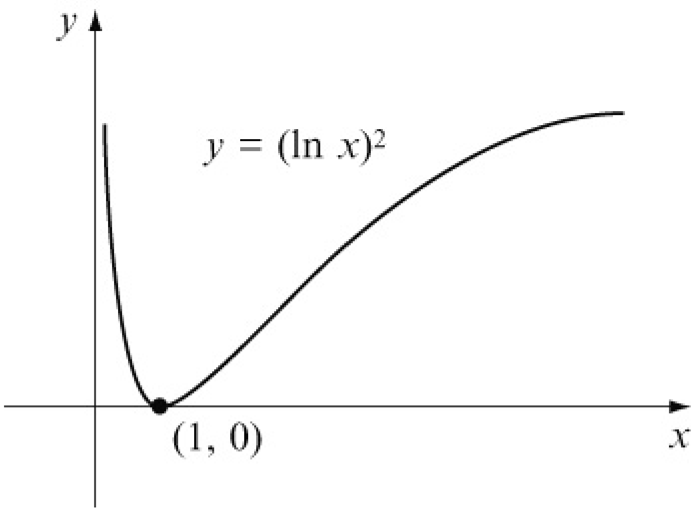
(c) $y = \ln(2x)$



When $x = \frac{1}{2}, y = \ln(1) = 0$

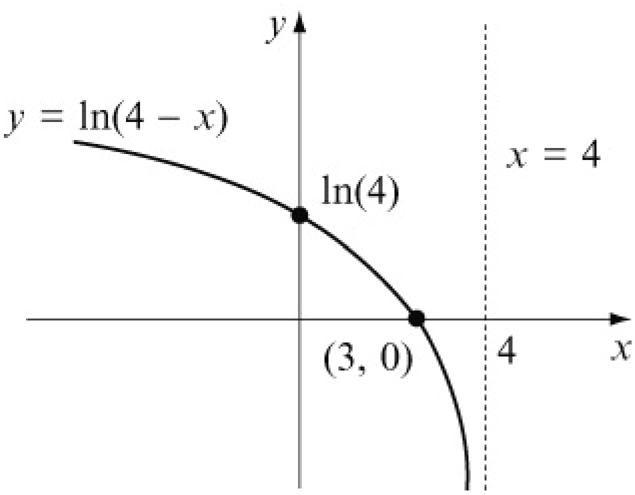
When $x \to 0, y \to -\infty$
y wouldn't exist for values of $x < 0$
When $x \to \infty, y \to \infty$ (slowly)

(d) $y = (\ln x)^2$



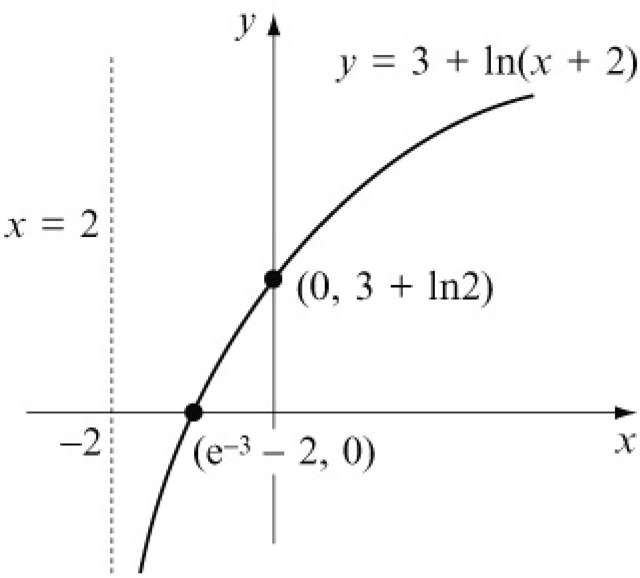
When $x = 1$, $y = (\ln 1)^2 = 0$
For $0 < x < 1$, $\ln x$ is negative, but $(\ln x)^2$ is positive.
When $x \to 0$, $y \to \infty$
When $x \to \infty$, $y \to \infty$

(e) $y = \ln(4 - x)$



When $x = 3$, $y = \ln 1 = 0$
When $x \to 4$, $y \to -\infty$
y doesn't exist for values of $x > 4$
When $x \to -\infty$, $y \to \infty$ (slowly)
When $x = 0$, $y = \ln 4$

(f) $y = 3 + \ln(x + 2)$



When $x = -1$, $y = 3 + \ln 1 = 3 + 0 = 3$
When $x \to -2$, $y \to -\infty$
y doesn't exist for values of $x < -2$
When $x \to \infty$, $y \to \infty$ slowly
When $x = 0$, $y = 3 + \ln(0 + 2) = 3 + \ln 2$
When $y = 0$,
$0 = 3 + \ln(x + 2)$
$-3 = \ln(x + 2)$
$e^{-3} = x + 2$
$x = e^{-3} - 2$

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Exercise B, Question 4

Question:

The price of a new car varies according to the formula

$P = 15\,000 \mathrm{e}^{-\frac{t}{10}}$

where P is the price in £'s and t is the age in years from new.

(a) State its new value.

(b) Calculate its value after 5 years (to the nearest £).

(c) Find its age when its price falls below £5 000.

(d) Sketch the graph showing how the price varies over time. Is this a good model?

Solution:

$P = 15\,000 \mathrm{e}^{-\frac{t}{10}}$

(a) New value is when $t = 0 \quad \Rightarrow \quad P = 15\,000 \times \mathrm{e}^{0} = 15\,000$

The new value is £15 000

(b) Value after 5 years is when $t = 5$

$\Rightarrow \quad P = 15\,000 \times \mathrm{e}^{-\frac{5}{10}} = 15\,000 \mathrm{e}^{-0.5} = 9097.96$

Value after 5 years is £9 098 (to nearest £)

(c) Find when price is £5 000

Substitute $P = 5\,000$:

$5\,000 = 15\,000 \mathrm{e}^{-\frac{t}{10}} \qquad (\div 15\,000)$

$\frac{5\,000}{15\,000} = \mathrm{e}^{-\frac{t}{10}}$

$\frac{1}{3} = \mathrm{e}^{-\frac{t}{10}}$

$\ln \left(\frac{1}{3} \right) = -\frac{t}{10}$

$t = -10 \ln \left(\frac{1}{3} \right)$

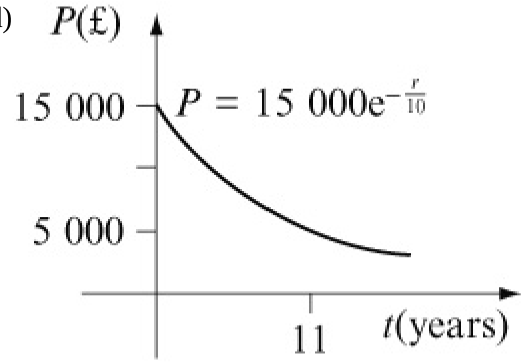
$t = 10 \ln \left(\frac{1}{3} \right)^{-1}$

$t = 10 \ln 3$

t = 10.99 years

The price falls below £5 000 after 11 years.

(d)



A fair model! Perhaps the price should be lower after 11 years.

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Exercise B, Question 5

Question:

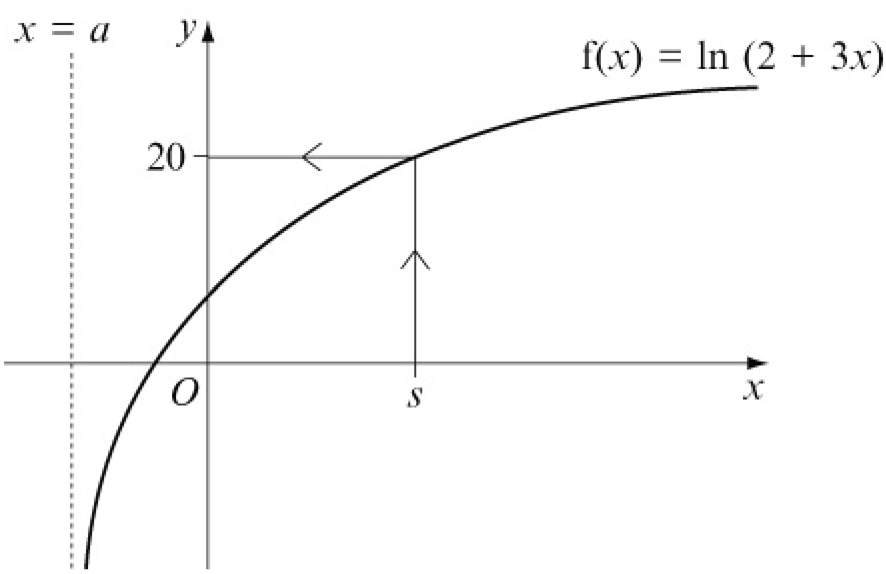
The graph below is of the function
$f(x) = \ln(2 + 3x) \quad \{ x \in \mathbb{R}, \ x > a \{$.

(a) State the value of a.

(b) Find the value of s for which $f(s) = 20$.

(c) Find the function $f^{-1}(x)$ stating its domain.

(d) Sketch the graphs f(x) and $f^{-1}(x)$ on the same axes stating the relationship between them.



Solution:

(a) $x = a$ is the asymptote to the curve. It will be where
$2 + 3x = 0$
$3x = -2$
$x = -\frac{2}{3}$

Hence $a = -\frac{2}{3}$

(b) If $f(s) = 20$ then
$\ln(2 + 3s) = 20$

$2 + 3s = e^{20}$

$3s = e^{20} - 2$

$s = \dfrac{e^{20} - 2}{3}$

(c) To find $f^{-1}(x)$, change the subject of the formula.

$y = \ln(2 + 3x)$

$e^y = 2 + 3x$

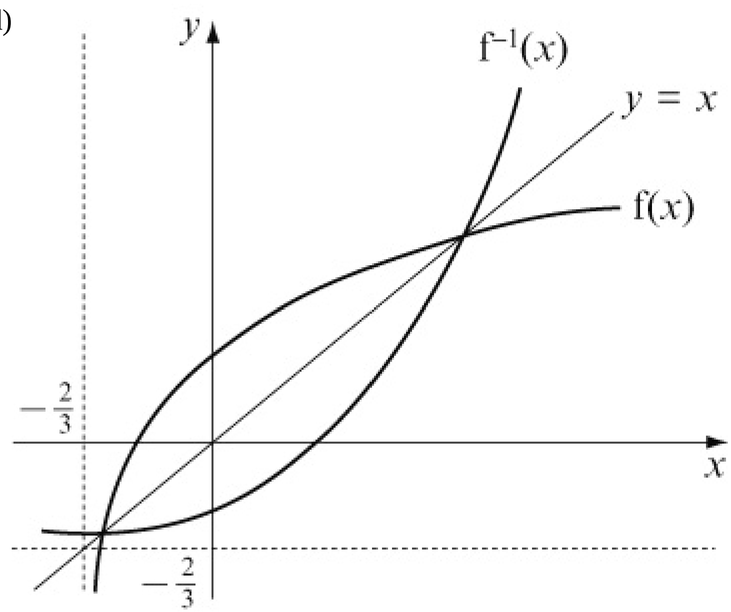
$e^y - 2 = 3x$

$x = \dfrac{e^y - 2}{3}$

Therefore $f^{-1}(x) = \dfrac{e^x - 2}{3}$

domain of $f^{-1}(x)$ = range of f(x), so $x \in \mathbb{R}$

(d)



$f^{-1}(x)$ is a reflection of f(x) in the line $y = x$.

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Exercise B, Question 6

Question:

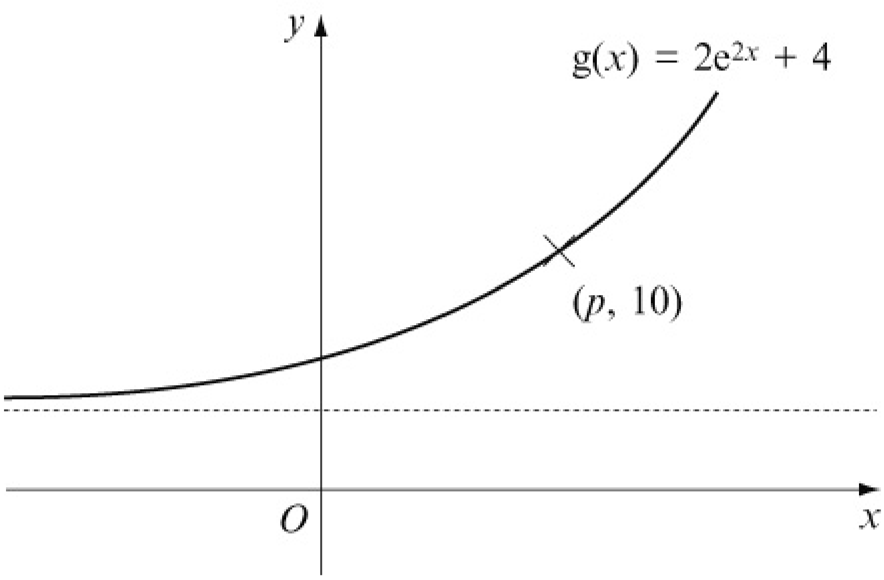
The graph below is of the function
$g(x) = 2e^{2x} + 4 \quad \{x \in \mathbb{R}\}$.

(a) Find the range of the function.

(b) Find the value of p to 2 significant figures.

(c) Find $g^{-1}(x)$ stating its domain.

(d) Sketch g(x) and $g^{-1}(x)$ on the same set of axes stating the relationship between them.



Solution:

(a) $g(x) = 2e^{2x} + 4$
As $x \to -\infty$, $g(x) \to 2 \times 0 + 4 = 4$
Therefore the range of g(x) is $g(x) > 4$

(b) If $(p, 10)$ lies on $g(x) = 2e^{2x} + 4$
$2e^{2p} + 4 = 10$
$2e^{2p} = 6$
$e^{2p} = 3$
$2p = \ln 3$

$p = \frac{1}{2}\ln 3$

$p = 0.55$ (2 s.f.)

(c) $g^{-1}(x)$ is found by changing the subject of the formula.
Let $y = 2e^{2x} + 4$
$y - 4 = 2e^{2x}$
$\frac{y-4}{2} = e^{2x}$

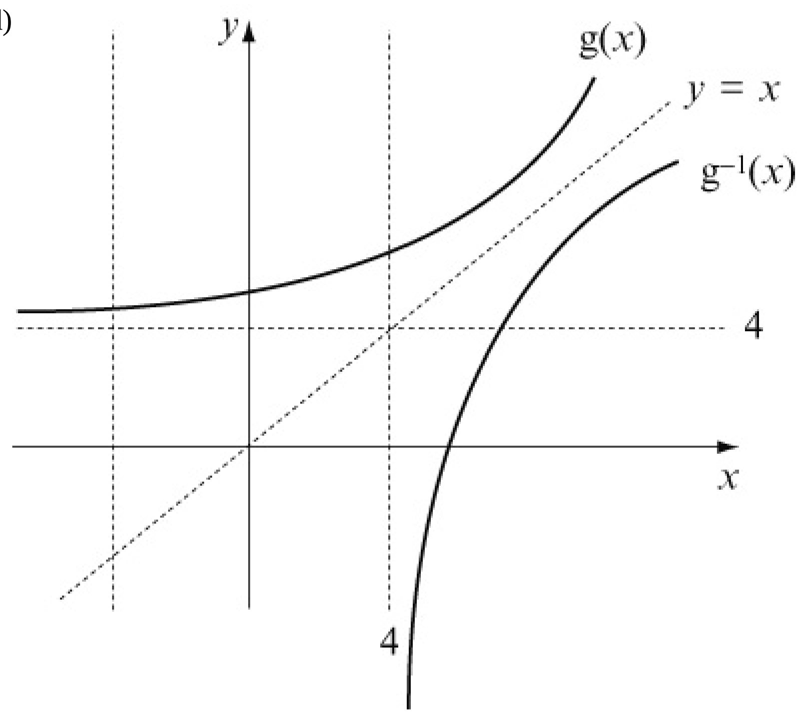
$\ln\left(\frac{y-4}{2}\right) = 2x$

$x = \frac{1}{2}\ln\left(\frac{y-4}{2}\right)$

Hence $g^{-1}(x) = \frac{1}{2}\ln\left(\frac{x-4}{2}\right)$

Its domain is the same as the range of g(x).
$g^{-1}(x)$ has a domain of $x > 4$

(d)



$g^{-1}(x)$ is a reflection of g(x) in the line $y = x$.

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Exercise B, Question 7

Question:

The number of bacteria in a culture grows according to the following equation:

$N = 100 + 50 e^{\frac{t}{30}}$

where N is the number of bacteria present and t is the time in days from the start of the experiment.

(a) State the number of bacteria present at the start of the experiment.

(b) State the number after 10 days.

(c) State the day on which the number first reaches 1 000 000.

(d) Sketch the graph showing how N varies with t.

Solution:

$N = 100 + 50 e^{\frac{t}{30}}$

(a) At the start $t = 0$

$\Rightarrow \quad N = 100 + 50 e^{\frac{0}{30}} = 100 + 50 \times 1 = 150$

There are 150 bacteria present at the start.

(b) After 10 days $t = 10$

$\Rightarrow \quad N = 100 + 50 e^{\frac{10}{30}} = 100 + 50 e^{\frac{1}{3}} = 170$

There are 170 bacteria present after 10 days.

(c) When $N = 1\,000\,000$

$1\,000\,000 = 100 + 50 e^{\frac{t}{30}} \qquad (- 100)$

$999\,900 = 50 e^{\frac{t}{30}} \qquad (\div 50)$

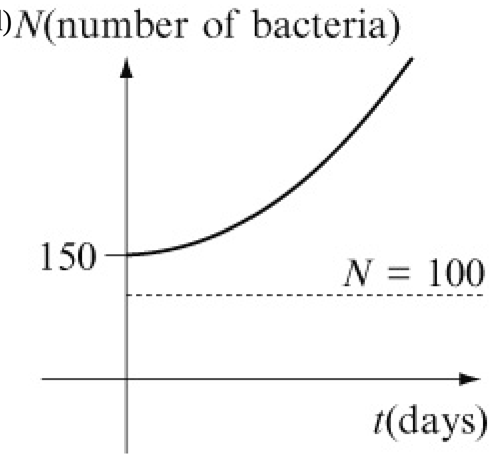
$19\,998 = e^{\frac{t}{30}}$

$\ln (19\,998) = \frac{t}{30}$

$t = 30 \ln (19\,998)$

$t = 297.10$

The number of bacteria reaches 1 000 000 on the 298th day (to the nearest day).

(d)


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Exercise B, Question 8

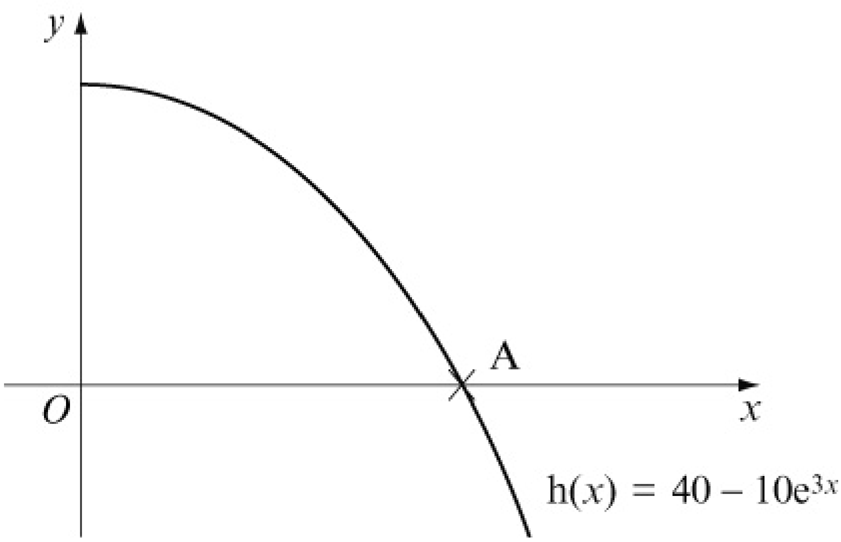
Question:

The graph below shows the function
$h(x) = 40 - 10e^{3x}$ { $x > 0$, $x \in \mathbb{R}$ { .

(a) State the range of the function.

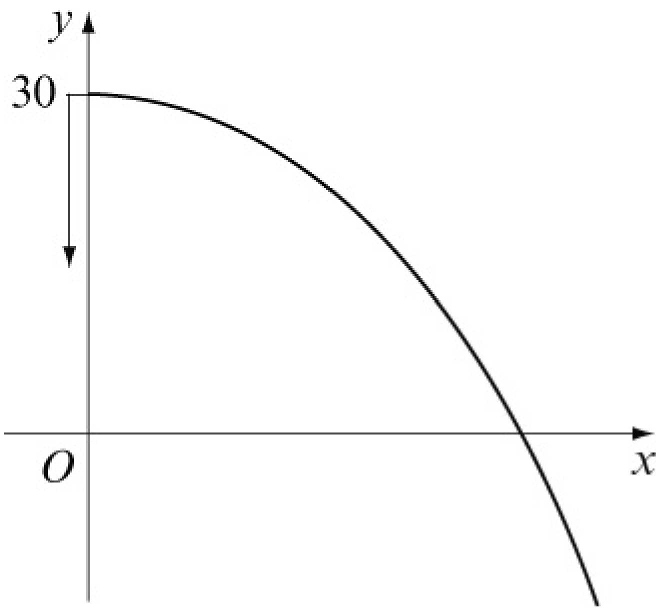
(b) Find the exact coordinates of A in terms of ln 2.

(c) Find $h^{-1}(x)$ stating its domain.



Solution:

(a) $h(x) = 40 - 10e^{3x}$
The range is the set of values that y can take.
$h(0) = 40 - 10e^{0} = 40 - 10 = 30$
Hence range is $h(x) < 30$



(b) A is where $y = 0$

Solve $40 - 10e^{3x} = 0$

$40 = 10e^{3x}$ $(\div 10)$

$4 = e^{3x}$

$\ln 4 = 3x$

$x = \frac{1}{3}\ln 4$

$x = \frac{1}{3}\ln 2^2$

$x = \frac{2}{3}\ln 2$

A is $\left(\frac{2}{3}\ln 2 , \quad 0 \right)$

(c) To find $h^{-1}(x)$ change the subject of the formula.

Let $y = 40 - 10e^{3x}$

$10e^{3x} = 40 - y$

$e^{3x} = \frac{40 - y}{10}$

$3x = \ln \left(\frac{40 - y}{10} \right)$

$x = \frac{1}{3}\ln \left(\frac{40 - y}{10} \right)$

The domain of the inverse function is the same as the range of the function.

Hence $h^{-1}(x) = \frac{1}{3}\ln \left(\frac{40 - x}{10} \right) \quad \{ x \in \mathbb{R}, \quad x < 30 \{$

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Exercise C, Question 1

Question:

Sketch the following functions stating any asymptotes and intersections with axes:

(a) $y = e^{x} + 3$

(b) $y = \ln (- x)$

(c) $y = \ln (x + 2)$

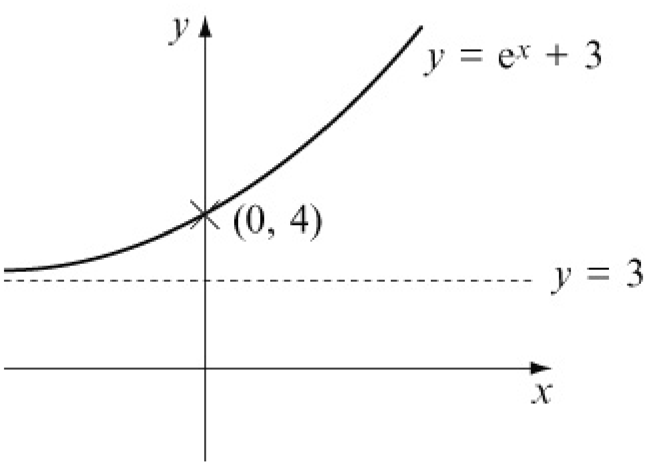
(d) $y = 3e^{-2x} + 4$

(e) $y = e^{x + 2}$

(f) $y = 4 - \ln x$

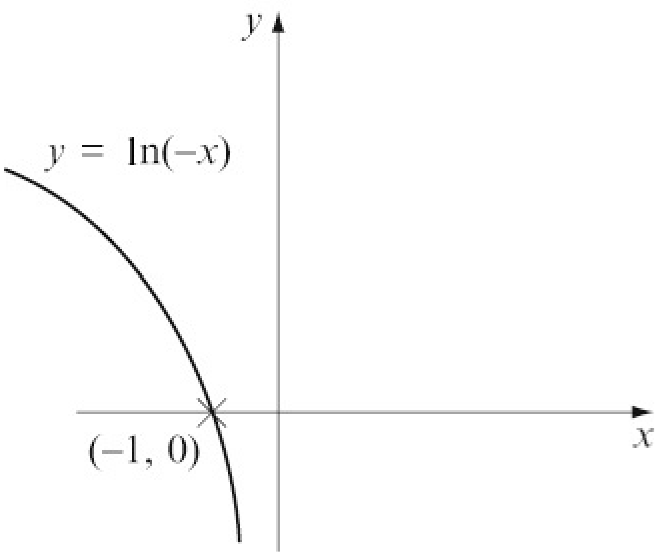
Solution:

(a) $y = e^{x} + 3$



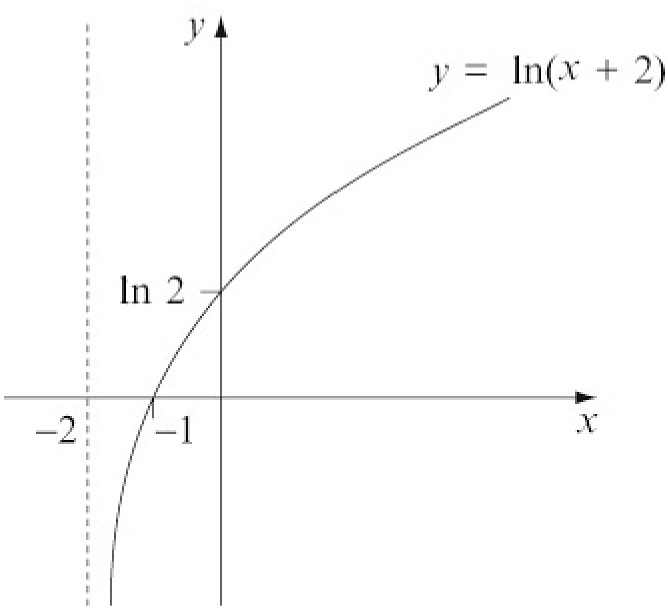
This is the graph of $y = e^{x}$ 'moved up' 3 units.
$x = 0$, $y = e^{0} + 3 = 1 + 3 = 4$

(b) $y = \ln (- x)$



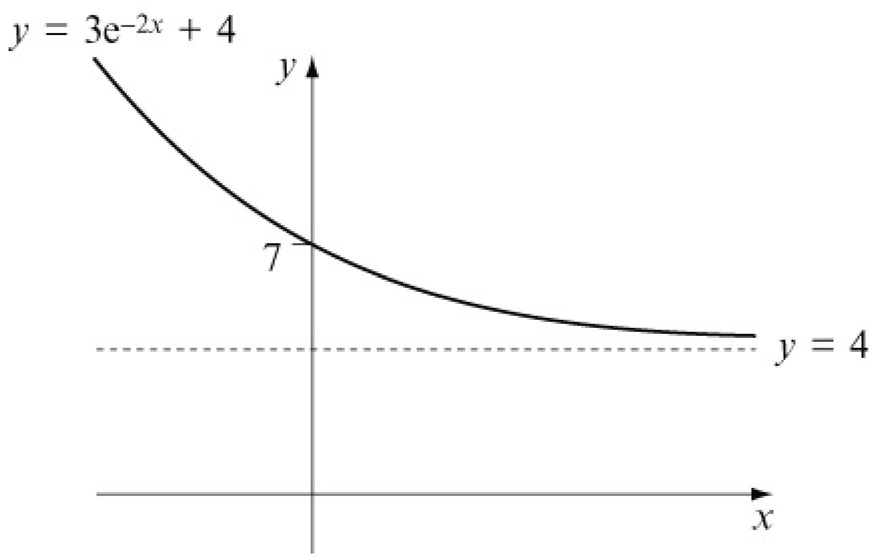
$x = -1$, $y = \ln(--1) = \ln(1) = 0$
y will not exist for values of $x > 0$
$x \to -\infty$, $y \to \infty$ (slowly)
The graph will be a reflection of $y = \ln(x)$ in the y axis.

(c) $y = \ln(x+2)$



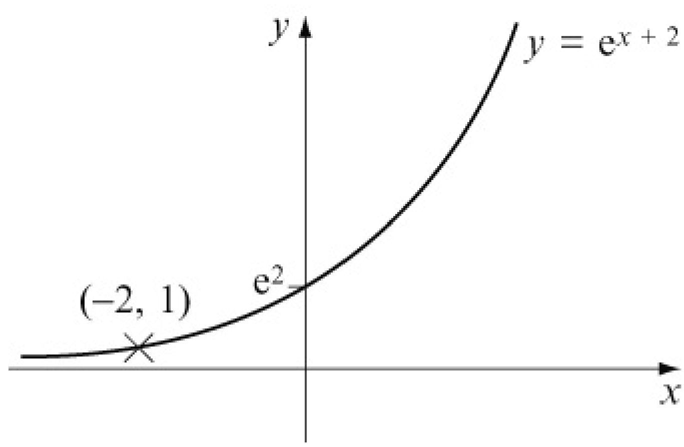
$x = -1$, $y = \ln(-1+2) = \ln(1) = 0$
y will not exist for values of $x < -2$
$x \to -2$, $y \to -\infty$
$x \to \infty$, $y \to \infty$ (slowly)
$x = 0$, $y = \ln(0+2) = \ln 2$

(d) $y = 3e^{-2x} + 4$



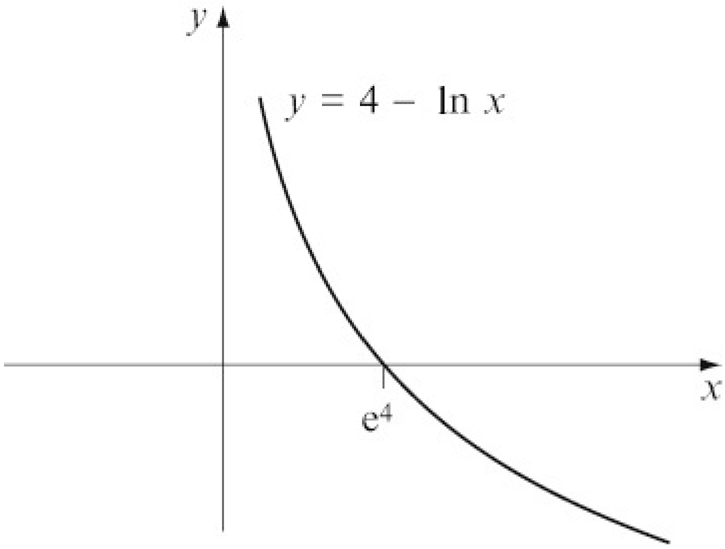
$x = 0$, $y = 3e^0 + 4 = 3 + 4 = 7$
$x \to \infty$, $y \to 3 \times 0 + 4 = 4$
$x \to -\infty$, $y \to \infty$

(e) $y = e^{x+2}$



$x = -2$, $y = e^{-2+2} = e^0 = 1$
$x \to -\infty$, $y \to 0$
$x \to \infty$, $y \to \infty$
$x = 0$, $y = e^2$

(f) $y = 4 - \ln x$



$x = 1, \quad y = 4 - \ln(1) = 4$
$x \to 0, \quad y \to 4 - (-\infty)$, so $y \to +\infty$
y will not exist for values of $x < 0$
$y = 0 \quad \Rightarrow \quad 4 - \ln x = 0 \quad \Rightarrow \quad \ln x = 4 \quad \Rightarrow \quad x =$ e ^{4}

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Exercise C, Question 2

Question:

Solve the following equations, giving exact solutions:

(a) $\ln(2x - 5) = 8$

(b) $e^{4x} = 5$

(c) $24 - e^{-2x} = 10$

(d) $\ln x + \ln(x - 3) = 0$

(e) $e^{x} + e^{-x} = 2$

(f) $\ln 2 + \ln x = 4$

Solution:

(a) $\ln(2x - 5) = 8$ (inverse of ln)

$2x - 5 = e^{8}$ $(+5)$

$2x = e^{8} + 5$ $(\div 2)$

$x = \frac{e^{8} + 5}{2}$

(b) $e^{4x} = 5$ (inverse of e)

$4x = \ln 5$ $(\div 4)$

$x = \frac{\ln 5}{4}$

(c) $24 - e^{-2x} = 10$ $(+e^{-2x})$

$24 = 10 + e^{-2x}$ (-10)

$14 = e^{-2x}$ (inverse of e)

$\ln(14) = -2x$ $(\div -2)$

$-\frac{1}{2}\ln(14) = x$

$x = -\frac{1}{2}\ln(14)$

(d) $\ln(x) + \ln(x-3) = 0$

$\ln[x(x-3)] = 0$

$x(x-3) = e^0$

$x(x-3) = 1$

$x^2 - 3x - 1 = 0$

$x = \frac{3 \pm \sqrt{9+4}}{2}$

$x = \frac{3 \pm \sqrt{13}}{2}$

$x = \frac{3 + \sqrt{13}}{2}$

(x cannot be negative because of initial equation)

(e) $e^x + e^{-x} = 2$

$e^x + \frac{1}{e^x} = 2 \quad (\times e^x)$

$(e^x)^2 + 1 = 2e^x$

$(e^x)^2 - 2e^x + 1 = 0$

$(e^x - 1)^2 = 0$

$e^x = 1$

$x = \ln 1 = 0$

(f) $\ln 2 + \ln x = 4$

$\ln 2x = 4$

$2x = e\ ^{4}$

$x = \frac{e\ ^{4}}{2}$

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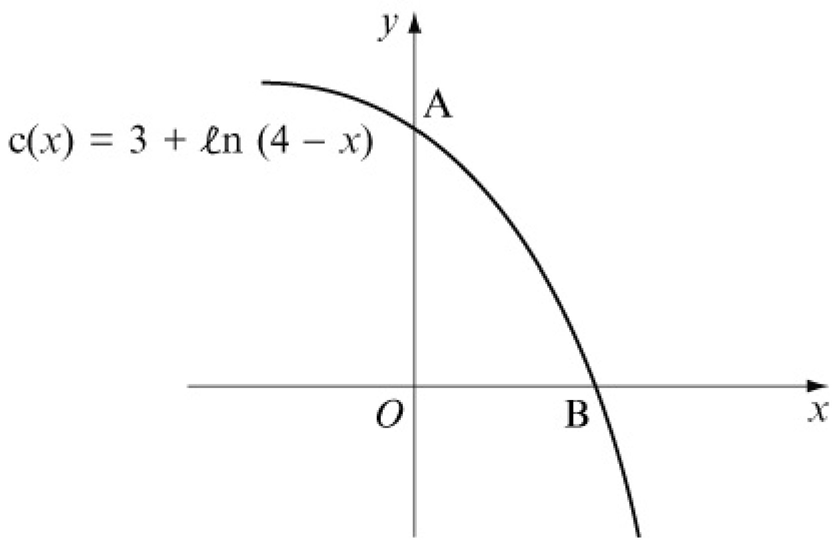
Exercise C, Question 3

Question:

The function $c(x) = 3 + \ln(4 - x)$ is shown below.



(a) State the exact coordinates of point A.

(b) Calculate the exact coordinates of point B.

(c) Find the inverse function $c^{-1}(x)$ stating its domain.

(d) Sketch c(x) and $c^{-1}(x)$ on the same set of axes stating the relationship between them.

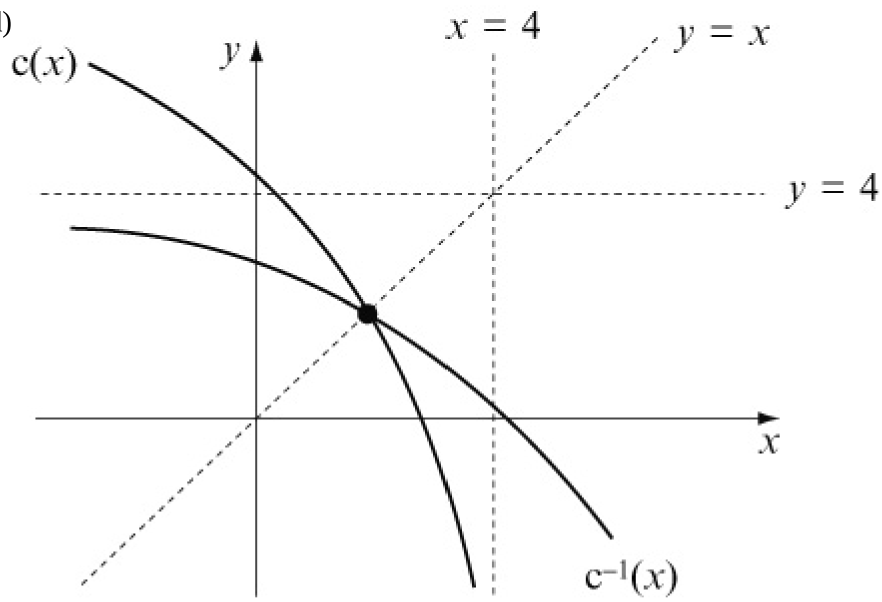
Solution:

(a) A is where $x = 0$
Substitute $x = 0$ into $y = 3 + \ln(4 - x)$ to give
$y = 3 + \ln 4$
$A = (0, 3 + \ln 4)$

(b) B is where $y = 0$
Substitute $y = 0$ into $y = 3 + \ln(4 - x)$ to give
$0 = 3 + \ln(4 - x)$
$-3 = \ln(4 - x)$
$e^{-3} = 4 - x$
$x = 4 - e^{-3}$
$B = (4 - e^{-3}, 0)$

(c) To find $c^{-1}(x)$ change the subject of the formula.
$y = 3 + \ln(4 - x)$
$y - 3 = \ln(4 - x)$
$e^{y-3} = 4 - x$
$x = 4 - e^{y-3}$
The domain of the inverse function is the range of the function. Looking at graph this is all the real numbers. So
$c^{-1}(x) = 4 - e^{x-3}$ $\{ x \in \mathbb{R} \{$

(d)



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Exercise C, Question 4

Question:

The price of a computer system can be modelled by the formula

$P = 100 + 850 e^{-\frac{t}{2}}$

where P is the price of the system in £s and t is the age of the computer in years after being purchased.

(a) Calculate the new price of the system.

(b) Calculate its price after 3 years.

(c) When will it be worth less than £200?

(d) Find its price as $t \to \infty$.

(e) Sketch the graph showing P against t.
Comment on the appropriateness of this model.

Solution:

$P = 100 + 850 e^{-\frac{t}{2}}$

(a) New price is when $t = 0$.

Substitute $t = 0$ into $P = 100 + 850 e^{-\frac{t}{2}}$ to give

$P = 100 + 850 e^{-\frac{0}{2}}$ $(e^0 = 1)$
$= 100 + 850 = 950$
The new price is £950

(b) After 3 years $t = 3$.

Substitute $t = 3$ into $P = 100 + 850 e^{-\frac{t}{2}}$ to give

$P = 100 + 850 e^{-\frac{3}{2}} = 289.66$
Price after 3 years is £290 (to nearest £)

(c) It is worth less than £200 when $P < 200$

Substitute $P = 200$ into $P = 100 + 850 e^{-\frac{t}{2}}$ to give

$200 = 100 + 850e^{-\frac{t}{2}}$

$100 = 850e^{-\frac{t}{2}}$

$\frac{100}{850} = e^{-\frac{t}{2}}$

$\ln\left(\frac{100}{850}\right) = -\frac{t}{2}$

$t = -2\ln\left(\frac{100}{850}\right)$

$t = 4.28$

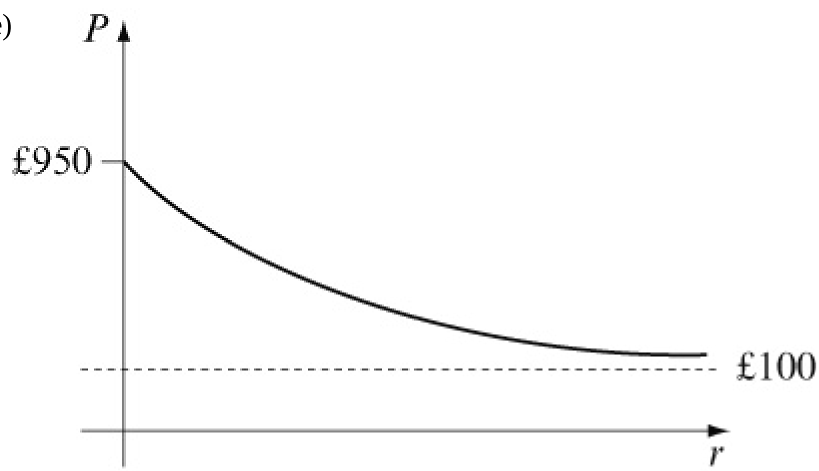
It is worth less than £200 after 4.28 years.

(d) As $t \to \infty$, $e^{-\frac{t}{2}} \to 0$

Hence $P \to 100 + 850 \times 0 = 100$

The computer will be worth £100 eventually.

(e)



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Exercise C, Question 5

Question:

The function f is defined by

$$f : x \rightarrow \ln(5x - 2) \quad \left\{ x \in \mathbb{R},\ x > \frac{2}{5} \right\}.$$

(a) Find an expression for $f^{-1}(x)$.

(b) Write down the domain of $f^{-1}(x)$.

(c) Solve, giving your answer to 3 decimal places,
$\ln(5x - 2) = 2$.

[E]

Solution:

(a) Let $y = \ln(5x - 2)$

$e^y = 5x - 2$

$e^y + 2 = 5x$

$\frac{e^y + 2}{5} = x$

The range of $y = \ln(5x - 2)$ is $y \in \mathbb{R}$

So $f^{-1}(x) = \frac{e^x + 2}{5} \quad \{ x \in \mathbb{R} \{$

(b) Domain is $x \in \mathbb{R}$

(c) $\ln(5x - 2) = 2$

$5x - 2 = e^2$

$5x = e^2 + 2$

$x = \frac{e^2 + 2}{5} = 1.8778 \ \ldots$

$x = 1.878$ (to 3d.p.)

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Exercise C, Question 6

Question:

The functions f and g are given by

$f : x \rightarrow 3x - 1 \quad \{ x \in \mathbb{R} \{$

$g : x \rightarrow e^{\frac{x}{2}} \quad \{ x \in \mathbb{R} \{$

(a) Find the value of fg(4), giving your answer to 2 decimal places.

(b) Express the inverse function $f^{-1}(x)$ in the form $f^{-1} : x \rightarrow \quad \ldots \quad .$

(c) Using the same axes, sketch the graphs of the functions f and gf.
Write on your sketch the value of each function at $x = 0$.

(d) Find the values of x for which $f^{-1}(x) = \frac{5}{f(x)}$.

[E]

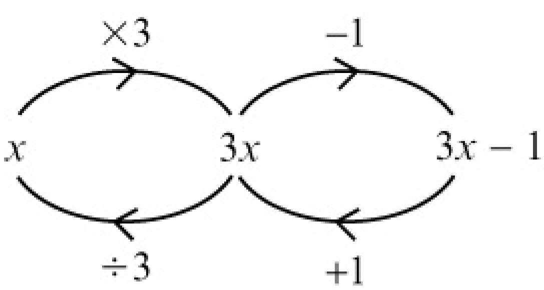
Solution:

(a) $fg(4) = f(e^{\frac{4}{2}}) = f(e^2) = 3e^2 - 1$
$= 21.17$ (2d.p.)

(b) If $f : x \rightarrow 3x - 1 \quad \{ x \in \mathbb{R} \{$

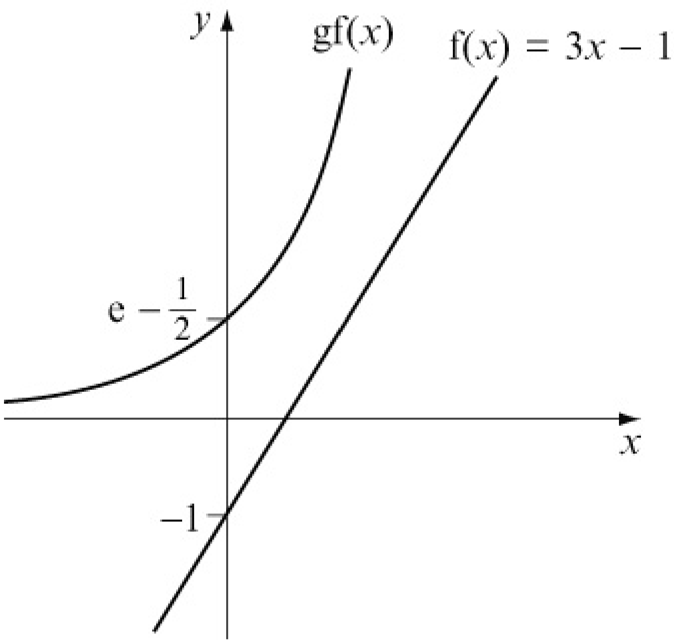
then $f^{-1} : x \rightarrow \frac{x+1}{3} \quad \left\{ x \in \mathbb{R} \right\}$

by using flow diagram method:



(c) $gf(x) = g(3x - 1) = e^{\frac{3x-1}{2}} \qquad f(x) = 3x - 1$

At $x = 0$, $\text{gf}(x) = e^{\frac{0-1}{2}} = e^{-\frac{1}{2}}$ and $f(x) = 3 \times 0 - 1 = -1$



(d) $f^{-1}(x) = \frac{5}{f(x)}$

$\frac{x+1}{3} = \frac{5}{3x-1}$ (cross multiply)

$(x+1)(3x-1) = 5 \times 3$

$3x^2 + 2x - 1 = 15$

$3x^2 + 2x - 16 = 0$

$(3x+8)(x-2) = 0$

$x = 2, -\frac{8}{3}$

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Exercise C, Question 7

Question:

The points P and Q lie on the curve with equation $y = e^{\frac{1}{2}x}$.
The x-coordinates of P and Q are ln 4 and ln 16 respectively.

(a) Find an equation for the line PQ.

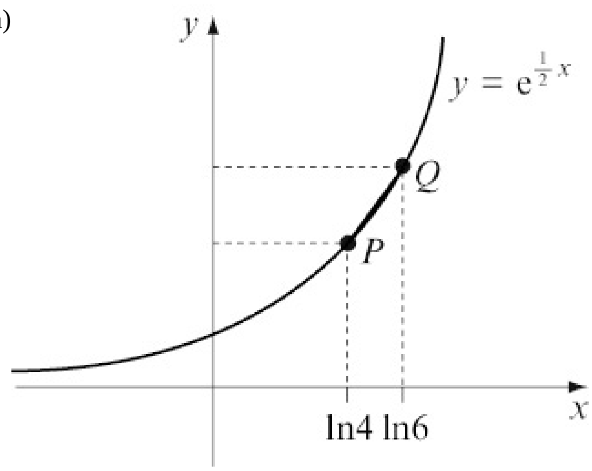
(b) Show that this line passes through the origin O.

(c) Calculate the length, to 3 significant figures, of the line segment PQ.

[E]

Solution:

(a)



Q has y coordinate $e^{\frac{1}{2}\ln 16} = e^{\ln 16^{\frac{1}{2}}} = 16^{\frac{1}{2}} = 4$

P has y coordinate $e^{\frac{1}{2}\ln 4} = e^{\ln 4^{\frac{1}{2}}} = 4^{\frac{1}{2}} = 2$

Gradient of the line PQ $= \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{4-2}{\ln 16 - \ln 4} = \dfrac{2}{\ln \frac{16}{4}} = \dfrac{2}{\ln 4}$

Using $y = mx + c$ the equation of the line PQ is

$y = \dfrac{2}{\ln 4}x + c$

(ln 4, 2) lies on line so

$2 = \dfrac{2}{\cancel{\ln 4}} \times \cancel{\ln 4} + c$

$2 = 2 + c$
$c = 0$

Equation of PQ is $y = \frac{2x}{\ln 4}$

(b) The line passes through the origin as $c = 0$.

(c) Length from (ln 4, 2) to (ln 16, 4) is

$$\sqrt{(\ln 16 - \ln 4)^2 + 4 - 2)^2} = \sqrt{\left(\ln \frac{16}{4}\right)^2 + 2^2} = \sqrt{(\ln 4) + 4} = 2.43$$

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Exercise C, Question 8

Question:

The functions f and g are defined over the set of real numbers by
$f : x \rightarrow 3x - 5$
$g : x \rightarrow e^{-2x}$

(a) State the range of g(x).

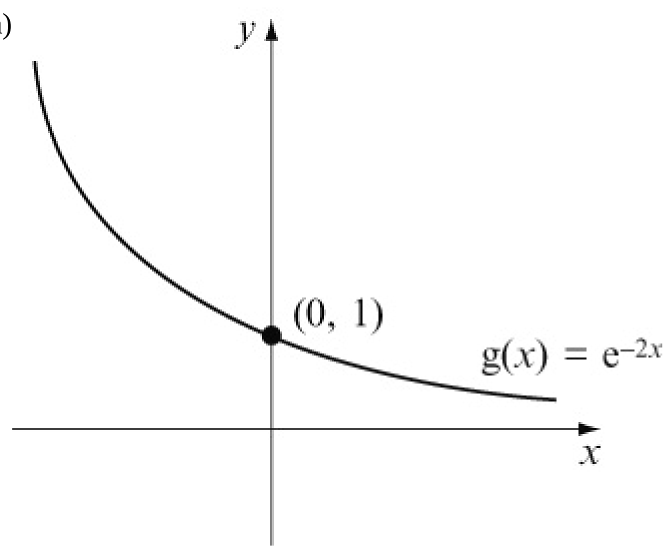
(b) Sketch the graphs of the inverse functions f^{-1} and g^{-1} and write on your sketches the coordinates of any points at which a graph meets the coordinate axes.

(c) State, giving a reason, the number of roots of the equation
$f^{-1}(x) = g^{-1}(x)$.

(d) Evaluate $fg\left(-\frac{1}{3}\right)$, giving your answer to 2 decimal places.

Solution:

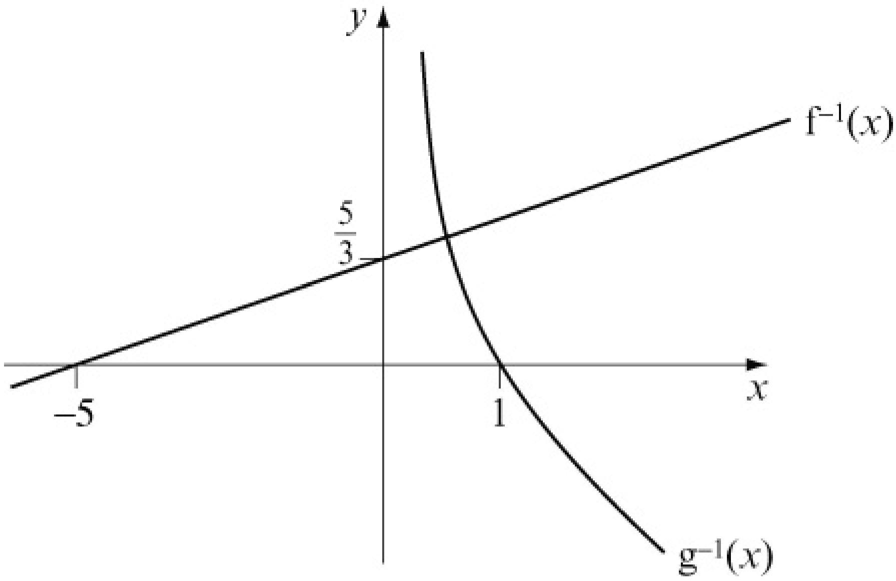
(a)



$g(x) > 0$

(b) $f^{-1}(x) = \frac{x+5}{3}$

$g^{-1}(x) = -\frac{1}{2}\ln x$



(c) $f^{-1}(x) = g^{-1}(x)$ would have 1 root because there is 1 point of intersection.

(d) $fg\left(-\frac{1}{3}\right) = f\left(e^{-2\times-\frac{1}{3}}\right) = f\left(e^{\frac{2}{3}}\right) = 3 \times e^{\frac{2}{3}} - 5 = 0.84$

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Exercise C, Question 9

Question:

The function f is defined by $f : x \to e^x + k, x \in \mathbb{R}$ and k is a positive constant.

(a) State the range of f(x).

(b) Find f(ln k), simplifying your answer.

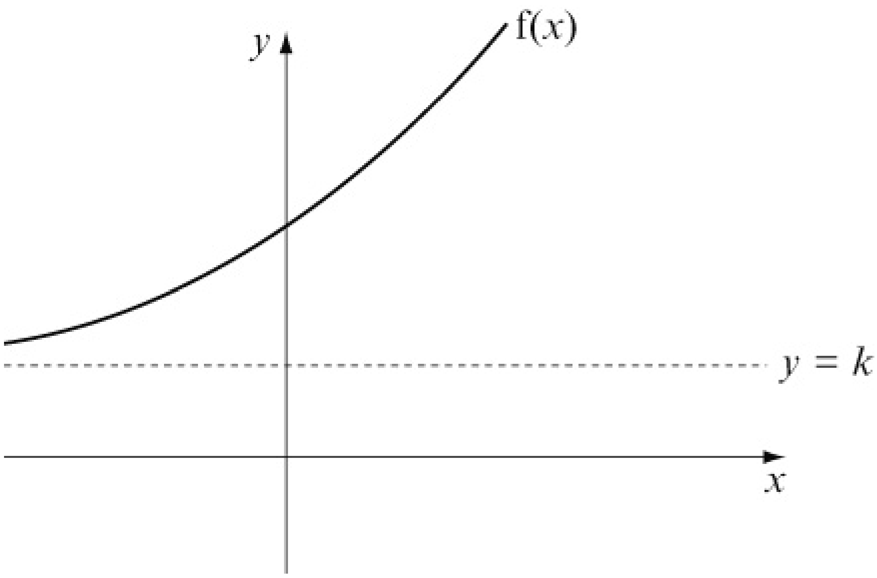
(c) Find f^{-1}, the inverse function of f, in the form $f^{-1} : x \to \ \ldots$, stating its domain.

(d) On the same axes, sketch the curves with equations $y = f(x)$ and $y = f^{-1}(x)$, giving the coordinates of all points where the graphs cut the axes.

[E]

Solution:

(a) $f : x \to e^x + k$
As $x \to -\infty$, $f(x) \to 0 + k = k$



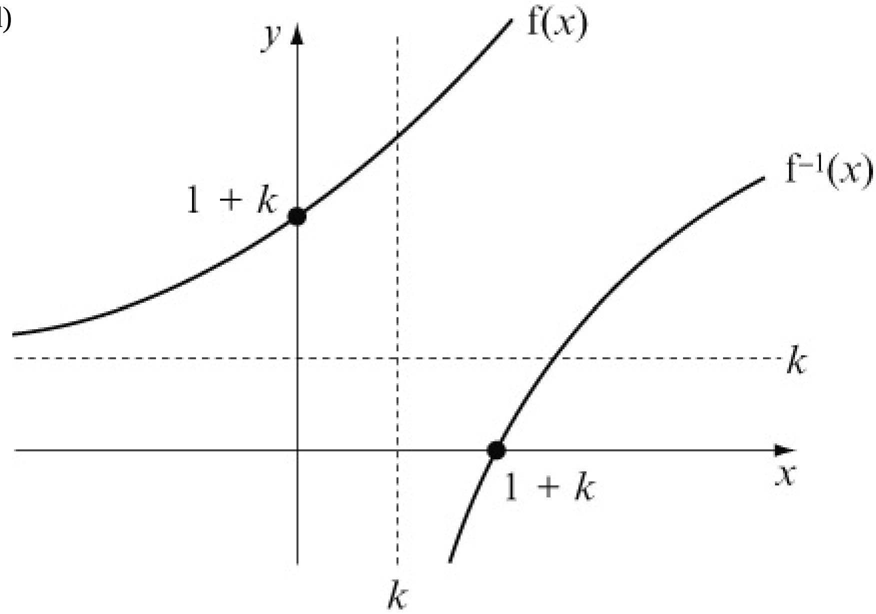
Range of f(x) is $f(x) > k$

(b) $f(\ln k) = e^{\ln k} + k = k + k = 2k$

(c) Let $y = e^x + k$

$y - k = e^x$
$\ln(y - k) = x$
Hence $f^{-1} : x \to \ln(x - k), x > k$

(d)



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Exercise C, Question 10

Question:

The function f is given by

$f : x \to \ln (4 - 2x) \quad \{ x \in \mathbb{R} , \quad x < 2 \{$

(a) Find an expression for $f^{-1} (x)$.

(b) Sketch the curve with equation $y = f^{-1} (x)$, showing the coordinates of the points where the curve meets the axes.

(c) State the range of $f^{-1} (x)$.

The function g is given by

$g : x \to e^{x} \quad \{ x \in \mathbb{R} \{$

(d) Find the value of gf(0.5).

[E]

Solution:

$f (x) = \ln (4 - 2x) \quad \{ x \in \mathbb{R} , \quad x < 2 \{$

(a) Let $y = \ln (4 - 2x)$ and change the subject of the formula.

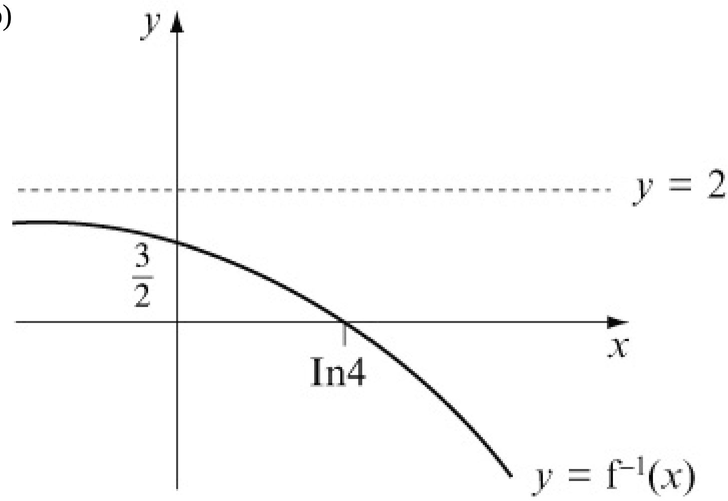
$e^{y} = 4 - 2x$

$2x = 4 - e^{y}$

$x = \frac{4 - e^{y}}{2}$

$f^{-1} : x \to \frac{4 - e^{x}}{2} \quad \{ x \in \mathbb{R} \{$

(b)



$x = 0 \quad \Rightarrow \quad f^{-1}(x) = \frac{4-1}{2} = \frac{3}{2}$

$y = 0 \quad \Rightarrow \quad \frac{4-e^x}{2} = 0 \quad \Rightarrow \quad e^x = 4 \quad \Rightarrow \quad x = \ln 4$

$x \to \infty, y \to -\infty$

$x \to -\infty, y \to \frac{4-0}{2} = 2$

(c) Range of $f^{-1}(x)$ is $f^{-1}(x) < 2$

(d) $gf(0.5) = g[\ln(4 - 2 \times 0.5)] = g(\ln 3) = e^{\ln 3} = 3$

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Exercise C, Question 11

Question:

The function f(x) is defined by
$f(x) = 3x^3 - 4x^2 - 5x + 2$

(a) Show that $(x + 1)$ is a factor of f(x).

(b) Factorise f(x) completely.

(c) Solve, giving your answers to 2 decimal places, the equation
$3[\ln(2x)]^3 - 4[\ln(2x)]^2 - 5\ln(2x) + 2 = 0 \quad x > 0$

[E]

Solution:

$f(x) = 3x^3 - 4x^2 - 5x + 2$
(a) $f(-1) = 3 \times (-1)^3 - 4 \times (-1)^2 - 5 \times (-1) + 2 = -3 - 4 + 5 + 2 = 0$
As $f(-1) = 0$ then $(x + 1)$ is a factor.

(b) $f(x) = 3x^3 - 4x^2 - 5x + 2$
$f(x) = (x + 1)(3x^2 - 7x + 2)$ (by inspection)
$f(x) = (x + 1)(3x - 1)(x - 2)$

(c) If $3[\ln(2x)]^3 - 4[\ln(2x)]^2 - 5[\ln(2x)] + 2 = 0$
$\Rightarrow [\ln(2x) + 1][3\ln(2x) - 1][\ln(2x) - 2] = 0$

$\Rightarrow \ln(2x) = -1, \frac{1}{3}, 2$

$\Rightarrow 2x = e^{-1}, e^{\frac{1}{3}}, e^2$

$\Rightarrow x = \frac{1}{2}e^{-1}, \frac{1}{2}e^{\frac{1}{3}}, \frac{1}{2}e^2$

$\Rightarrow x = 0.18, 0.70, 3.69$