

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise A, Question 1

Question:

Integrate the following with respect to x :

(a) $3\sec^2 x + \frac{5}{x} + \frac{2}{x^2}$

(b) $5e^x - 4\sin x + 2x^3$

(c) $2(\sin x - \cos x + x)$

(d) $3\sec x \tan x - \frac{2}{x}$

(e) $5e^x + 4\cos x - \frac{2}{x^2}$

(f) $\frac{1}{2x} + 2\operatorname{cosec}^2 x$

(g) $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$

(h) $e^x + \sin x + \cos x$

(i) $2\operatorname{cosec} x \cot x - \sec^2 x$

(j) $e^x + \frac{1}{x} - \operatorname{cosec}^2 x$

Solution:

$$\begin{aligned}
 \text{(a)} \quad & \int \left(3\sec^2 x + \frac{5}{x} + \frac{2}{x^2} \right) dx \\
 &= \int \left(3\sec^2 x + \frac{5}{x} + 2x^{-2} \right) dx \\
 &= 3\tan x + 5\ln |x| - \frac{2}{x} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \int (5e^x - 4\sin x + 2x^3) \, dx \\
 &= 5e^x + 4\cos x + \frac{2x^4}{4} + C \\
 &= 5e^x + 4\cos x + \frac{x^4}{2} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \int 2(\sin x - \cos x + x) \, dx \\
 &= \int (2\sin x - 2\cos x + 2x) \, dx \\
 &= -2\cos x - 2\sin x + x^2 + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & \int \left(3\sec x \tan x - \frac{2}{x} \right) \, dx \\
 &= 3\sec x - 2\ln|x| + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad & \int \left(5e^x + 4\cos x - \frac{2}{x^2} \right) \, dx \\
 &= \int (5e^x + 4\cos x - 2x^{-2}) \, dx \\
 &= 5e^x + 4\sin x + \frac{2}{x} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad & \int \left(\frac{1}{2x} + 2\operatorname{cosec}^2 x \right) \, dx \\
 &= \int \left(\frac{1}{2} \times \frac{1}{x} + 2\operatorname{cosec}^2 x \right) \, dx \\
 &= \frac{1}{2}\ln|x| - 2\cot x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(g)} \quad & \int \left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} \right) \, dx \\
 &= \int \left(\frac{1}{x} + x^{-2} + x^{-3} \right) \, dx \\
 &= \ln|x| + \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + C \\
 &= \ln|x| - \frac{1}{x} - \frac{1}{2x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(h)} \quad & \int (e^x + \sin x + \cos x) \, dx \\
 &= e^x - \cos x + \sin x + C
 \end{aligned}$$

$$\begin{aligned} \text{(i)} \quad \int (2 \operatorname{cosec} x \cot x - \sec^2 x) \, dx \\ = -2 \operatorname{cosec} x - \tan x + C \end{aligned}$$

$$\begin{aligned} \text{(j)} \quad \int \left(e^x + \frac{1}{x} - \operatorname{cosec}^2 x \right) \, dx \\ = e^x + \ln |x| + \cot x + C \end{aligned}$$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise A, Question 2

Question:

Find the following integrals:

$$(a) \int \left(\frac{1}{\cos^2 x} + \frac{1}{x^2} \right) dx$$

$$(b) \int \left(\frac{\sin x}{\cos^2 x} + 2e^x \right) dx$$

$$(c) \int \left(\frac{1 + \cos x}{\sin^2 x} + \frac{1 + x}{x^2} \right) dx$$

$$(d) \int \left(\frac{1}{\sin^2 x} + \frac{1}{x} \right) dx$$

$$(e) \int \sin x (1 + \sec^2 x) dx$$

$$(f) \int \cos x (1 + \operatorname{cosec}^2 x) dx$$

$$(g) \int \operatorname{cosec}^2 x (1 + \tan^2 x) dx$$

$$(h) \int \sec^2 x (1 - \cot^2 x) dx$$

$$(i) \int \sec^2 x (1 + e^x \cos^2 x) dx$$

$$(j) \int \left(\frac{1 + \sin x}{\cos^2 x} + \cos^2 x \sec x \right) dx$$

Solution:

$$\begin{aligned} (a) \int \left(\frac{1}{\cos^2 x} + \frac{1}{x^2} \right) dx \\ &= \int (\sec^2 x + x^{-2}) dx \\ &= \tan x - \frac{1}{x} + C \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \int \left(\frac{\sin x}{\cos^2 x} + 2e^x \right) dx \\
 &= \int (\tan x \sec x + 2e^x) dx \\
 &= \sec x + 2e^x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \int \left(\frac{1 + \cos x}{\sin^2 x} + \frac{1+x}{x^2} \right) dx \\
 &= \int (\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x + x^{-2} + x^{-1}) dx \\
 &= -\cot x - \operatorname{cosec} x - \frac{1}{x} + \ln |x| + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & \int \left(\frac{1}{\sin^2 x} + \frac{1}{x} \right) dx \\
 &= \int \left(\operatorname{cosec}^2 x + \frac{1}{x} \right) dx \\
 &= -\cot x + \ln |x| + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad & \int \sin x (1 + \sec^2 x) dx \\
 &= \int (\sin x + \sin x \sec^2 x) dx \\
 &= \int (\sin x + \tan x \sec x) dx \\
 &= -\cos x + \sec x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad & \int \cos x (1 + \operatorname{cosec}^2 x) dx \\
 &= \int (\cos x + \cos x \operatorname{cosec}^2 x) dx \\
 &= \int (\cos x + \cot x \operatorname{cosec} x) dx \\
 &= \sin x - \operatorname{cosec} x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(g)} \quad & \int \operatorname{cosec}^2 x (1 + \tan^2 x) dx \\
 &= \int (\operatorname{cosec}^2 x + \operatorname{cosec}^2 x \tan^2 x) dx \\
 &= \int (\operatorname{cosec}^2 x + \sec^2 x) dx \\
 &= -\cot x + \tan x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(h)} \quad & \int \sec^2 x (1 - \cot^2 x) dx \\
 &= \int (\sec^2 x - \sec^2 x \cot^2 x) dx \\
 &= \int (\sec^2 x - \operatorname{cosec}^2 x) dx \\
 &= \tan x + \cot x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(i)} \quad & \int \sec^2 x (1 + e^x \cos^2 x) dx \\
 &= \int (\sec^2 x + e^x \cos^2 x \sec^2 x) dx \\
 &= \int (\sec^2 x + e^x) dx
 \end{aligned}$$

$$= \tan x + e^x + C$$

$$\begin{aligned} \text{(j)} \quad & \int \left(\frac{1 + \sin x}{\cos^2 x} + \cos^2 x \sec x \right) dx \\ &= \int (\sec^2 x + \tan x \sec x + \cos x) dx \\ &= \tan x + \sec x + \sin x + C \end{aligned}$$

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Integration

Exercise B, Question 1

Question:

Integrate the following:

(a) $\sin (2x + 1)$

(b) $3e^{2x}$

(c) $4e^{x+5}$

(d) $\cos (1 - 2x)$

(e) $\operatorname{cosec}^2 3x$

(f) $\sec 4x \tan 4x$

(g) $3 \sin \left(\frac{1}{2}x + 1 \right)$

(h) $\sec^2 (2 - x)$

(i) $\operatorname{cosec} 2x \cot 2x$

(j) $\cos 3x - \sin 3x$

Solution:

(a) $\int \sin \left(2x + 1 \right) dx = -\frac{1}{2} \cos \left(2x + 1 \right) + C$

(b) $\int 3e^{2x} dx = \frac{3}{2} e^{2x} + C$

(c) $\int 4e^{x+5} dx = 4e^{x+5} + C$

(d) $\int \cos \left(1 - 2x \right) dx = -\frac{1}{2} \sin \left(1 - 2x \right) + C$

OR Let $y = \sin (1 - 2x)$

$$\text{then } \frac{dy}{dx} = \cos \left(1 - 2x \right) \times \left(-2 \right) \quad (\text{by chain rule})$$

$$\therefore \int \cos \left(1 - 2x \right) dx = -\frac{1}{2} \sin \left(1 - 2x \right) + C$$

$$(e) \int \operatorname{cosec}^2 3x \, dx = -\frac{1}{3} \cot 3x + C$$

$$(f) \int \sec 4x \tan 4x \, dx = \frac{1}{4} \sec 4x + C$$

$$(g) \int 3 \sin \left(\frac{1}{2}x + 1 \right) dx = -6 \cos \left(\frac{1}{2}x + 1 \right) + C$$

$$(h) \int \sec^2 (2 - x) \, dx = -\tan (2 - x) + C$$

OR Let $y = \tan (2 - x)$

$$\text{then } \frac{dy}{dx} = \sec^2 \left(2 - x \right) \times \left(-1 \right) \quad (\text{by chain rule})$$

$$\therefore \int \sec^2 (2 - x) \, dx = -\tan (2 - x) + C$$

$$(i) \int \operatorname{cosec} 2x \cot 2x \, dx = -\frac{1}{2} \operatorname{cosec} 2x + C$$

$$\begin{aligned} (j) \int (\cos 3x - \sin 3x) \, dx \\ &= \frac{1}{3} \sin 3x + \frac{1}{3} \cos 3x + C \\ &= \frac{1}{3} \left(\sin 3x + \cos 3x \right) + C \end{aligned}$$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise B, Question 2

Question:

Find the following integrals:

$$(a) \int \left(e^{2x} - \frac{1}{2} \sin \left(2x - 1 \right) \right) dx$$

$$(b) \int (e^x + 1)^2 dx$$

$$(c) \int \sec^2 2x (1 + \sin 2x) dx$$

$$(d) \int \left(\frac{3 - 2 \cos \left(\frac{1}{2}x \right)}{\sin^2 \left(\frac{1}{2}x \right)} \right) dx$$

$$(e) \int [e^{3-x} + \sin(3-x) + \cos(3-x)] dx$$

Solution:

$$(a) \int \left[e^{2x} - \frac{1}{2} \sin \left(2x - 1 \right) \right] dx = \frac{1}{2} e^{2x} + \frac{1}{4} \cos \left(2x - 1 \right) + C$$

$$\begin{aligned} (b) \int (e^x + 1)^2 dx \\ = \int (e^{2x} + 2e^x + 1) dx \\ = \frac{1}{2} e^{2x} + 2e^x + x + C \end{aligned}$$

$$\begin{aligned} (c) \int \sec^2 2x (1 + \sin 2x) dx \\ = \int (\sec^2 2x + \sec^2 2x \sin 2x) dx \\ = \int (\sec^2 2x + \sec 2x \tan 2x) dx \\ = \frac{1}{2} \tan 2x + \frac{1}{2} \sec 2x + C \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & \int \left[\frac{3 - 2 \cos \left(\frac{1}{2}x \right)}{\sin^2 \left(\frac{1}{2}x \right)} \right] dx \\
 &= \int \left(3 \operatorname{cosec}^2 \frac{1}{2}x - 2 \operatorname{cosec} \frac{1}{2}x \cot \frac{1}{2}x \right) dx \\
 &= -6 \cot \left(\frac{1}{2}x \right) + 4 \operatorname{cosec} \left(\frac{1}{2}x \right) + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad & \int [e^{3-x} + \sin(3-x) + \cos(3-x)] dx \\
 &= -e^{3-x} + \cos(3-x) - \sin(3-x) + C
 \end{aligned}$$

Note: extra minus signs from $-x$ terms and chain rule.

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Integration

Exercise B, Question 3

Question:

Integrate the following:

(a) $\frac{1}{2x + 1}$

(b) $\frac{1}{(2x + 1)^2}$

(c) $(2x + 1)^2$

(d) $\frac{3}{4x - 1}$

(e) $\frac{3}{1 - 4x}$

(f) $\frac{3}{(1 - 4x)^2}$

(g) $(3x + 2)^5$

(h) $\frac{3}{(1 - 2x)^3}$

(i) $\frac{6}{(3 - 2x)^4}$

(j) $\frac{5}{3 - 2x}$

Solution:

(a) $\int \frac{1}{2x + 1} dx = \frac{1}{2} \ln |2x + 1| + C$

$$\begin{aligned}
 \text{(b)} \quad & \int \frac{1}{(2x+1)^2} dx \\
 &= \int (2x+1)^{-2} dx \\
 &= \frac{(2x+1)^{-1}}{-1} \times \frac{1}{2} + C \\
 &= -\frac{1}{2(2x+1)} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \int (2x+1)^2 dx \\
 &= \frac{(2x+1)^3}{3} \times \frac{1}{2} + C \\
 &= \frac{(2x+1)^3}{6} + C
 \end{aligned}$$

$$\text{(d)} \quad \int \frac{3}{4x-1} dx = \frac{3}{4} \ln |4x-1| + C$$

$$\begin{aligned}
 \text{(e)} \quad & \int \frac{3}{1-4x} dx \\
 &= - \int \frac{3}{4x-1} dx \\
 &= -\frac{3}{4} \ln |4x-1| + C
 \end{aligned}$$

OR Let $y = \ln |1-4x|$

$$\text{then } \frac{dy}{dx} = \frac{1}{1-4x} \times \left(-4 \right) \quad (\text{by chain rule})$$

$$\therefore \int \frac{3}{1-4x} dx = -\frac{3}{4} \ln |1-4x| + C$$

Note: $\ln |1-4x| = \ln |4x-1|$ because of $| \quad |$ sign.

$$\begin{aligned}
 \text{(f)} \quad & \int \frac{3}{(1-4x)^2} dx \\
 &= \int 3(1-4x)^{-2} dx \\
 &= \frac{3}{-4} \times \frac{(1-4x)^{-1}}{-1} \\
 &= \frac{3}{4(1-4x)} + C
 \end{aligned}$$

$$(g) \int (3x + 2)^5 dx = \frac{(3x + 2)^6}{18} + C$$

$$(h) \int \frac{3}{(1 - 2x)^3} dx = \frac{3}{-2} \times \frac{(1 - 2x)^{-2}}{-2} + C = \frac{3}{4(1 - 2x)^2} + C$$

OR Let $y = (1 - 2x)^{-2}$

then $\frac{dy}{dx} = -2(1 - 2x)^{-3} \times (-2)$ (by chain rule)

$$\therefore \int \frac{3}{(1 - 2x)^3} dx = \frac{3}{4}(1 - 2x)^{-2} + C$$

$$(i) \int \frac{6}{(3 - 2x)^4} dx = \frac{6}{-2} \times \frac{(3 - 2x)^{-3}}{-3} + C = \frac{1}{(3 - 2x)^3} + C$$

OR Let $y = (3 - 2x)^{-3}$

then $\frac{dy}{dx} = -3(3 - 2x)^{-4} \times (-2)$

$$\therefore \int \frac{6}{(3 - 2x)^4} dx = \frac{1}{(3 - 2x)^3} + C$$

$$(j) \int \frac{5}{(3 - 2x)} dx = -\frac{5}{2} \ln |3 - 2x| + C$$

OR Let $y = \ln |3 - 2x|$

then $\frac{dy}{dx} = \frac{1}{3 - 2x} \times (-2)$ (by chain rule)

$$\therefore \int \frac{5}{3 - 2x} dx = -\frac{5}{2} \ln |3 - 2x| + C$$

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Exercise B, Question 4

Question:

Find the following integrals

$$(a) \int \left(3 \sin \left(2x + 1 \right) + \frac{4}{2x + 1} \right) dx$$

$$(b) \int [e^{5x} + (1 - x)^5] dx$$

$$(c) \int \left(\frac{1}{\sin^2 2x} + \frac{1}{1 + 2x} + \frac{1}{(1 + 2x)^2} \right) dx$$

$$(d) \int \left[(3x + 2)^2 + \frac{1}{(3x + 2)^2} \right] dx$$

Solution:

$$\begin{aligned} (a) \int \left[3 \sin \left(2x + 1 \right) + \frac{4}{2x + 1} \right] dx \\ = -\frac{3}{2} \cos \left(2x + 1 \right) + \frac{4}{2} \ln | 2x + 1 | + C \\ = -\frac{3}{2} \cos \left(2x + 1 \right) + 2 \ln | 2x + 1 | + C \end{aligned}$$

$$\begin{aligned} (b) \int [e^{5x} + (1 - x)^5] dx \\ = \int e^{5x} dx + \int (1 - x)^5 dx \\ = \frac{1}{5} e^{5x} - \frac{1}{6} (1 - x)^6 + C \quad (\text{from } \textcircled{11} \text{ and } \textcircled{10}) \end{aligned}$$

OR Let $y = (1 - x)^6$

$$\text{then } \frac{dy}{dx} = 6(1 - x)^5 \times (-1) \quad (\text{by chain rule})$$

$$\therefore \int (1 - x)^5 dx = -\frac{1}{6} (1 - x)^6 + C$$

$$(c) \int \left[\frac{1}{\sin^2 2x} + \frac{1}{1 + 2x} + \frac{1}{(1 + 2x)^2} \right] dx$$

$$\begin{aligned}
&= \int \left[\operatorname{cosec}^2 2x + \frac{1}{1+2x} + (1+2x)^{-2} \right] dx \\
&= -\frac{1}{2} \cot 2x + \frac{1}{2} \ln |1+2x| + \frac{(1+2x)^{-1}}{-1} \times \frac{1}{2} + C \\
&= -\frac{1}{2} \cot 2x + \frac{1}{2} \ln |1+2x| - \frac{1}{2(1+2x)} + C
\end{aligned}$$

$$\begin{aligned}
\text{(d)} \quad &\int \left[(3x+2)^2 + \frac{1}{(3x+2)^2} \right] dx \\
&= \int \left[(3x+2)^2 + (3x+2)^{-2} \right] dx \\
&= \frac{(3x+2)^3}{9} - \frac{(3x+2)^{-1}}{3} + C \\
&= \frac{(3x+2)^3}{9} - \frac{1}{3(3x+2)} + C
\end{aligned}$$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise C, Question 1

Question:

Integrate the following:

(a) $\cot^2 x$

(b) $\cos^2 x$

(c) $\sin 2x \cos 2x$

(d) $(1 + \sin x)^2$

(e) $\tan^2 3x$

(f) $(\cot x - \operatorname{cosec} x)^2$

(g) $(\sin x + \cos x)^2$

(h) $\sin^2 x \cos^2 x$

(i) $\frac{1}{\sin^2 x \cos^2 x}$

(j) $(\cos 2x - 1)^2$

Solution:

$$\begin{aligned} \text{(a)} \quad \int \cot^2 x \, dx &= \int (\operatorname{cosec}^2 x - 1) \, dx \\ &= -\cot x - x + C \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int \cos^2 x \, dx &= \int \frac{1}{2} (1 + \cos 2x) \, dx \\ &= \frac{1}{2}x + \frac{1}{4}\sin 2x + C \end{aligned}$$

$$\text{(c)} \quad \int \sin 2x \cos 2x \, dx = \int \frac{1}{2} \sin 4x \, dx$$

$$= -\frac{1}{8}\cos 4x + C$$

$$(d) \int (1 + \sin x)^2 dx = \int (1 + 2\sin x + \sin^2 x) dx$$

$$\text{But } \cos 2x = 1 - 2\sin^2 x$$

$$\therefore \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\therefore \int (1 + \sin x)^2 dx = \int \left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x \right) dx$$

$$= \frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x + C$$

$$(e) \int \tan^2 3x dx = \int (\sec^2 3x - 1) dx$$

$$= \frac{1}{3}\tan 3x - x + C$$

$$(f) \int (\cot x - \operatorname{cosec} x)^2 dx = \int (\cot^2 x - 2\cot x \operatorname{cosec} x + \operatorname{cosec}^2 x) dx$$

$$= \int (2\operatorname{cosec}^2 x - 1 - 2\cot x \operatorname{cosec} x) dx$$

$$= -2\cot x - x + 2\operatorname{cosec} x + C$$

$$(g) \int (\sin x + \cos x)^2 dx = \int (\sin^2 x + 2\sin x \cos x + \cos^2 x) dx$$

$$= \int (1 + \sin 2x) dx$$

$$= x - \frac{1}{2}\cos 2x + C$$

$$(h) \int \sin^2 x \cos^2 x dx = \int \left(\frac{1}{2}\sin 2x \right)^2 dx$$

$$= \int \frac{1}{4}\sin^2 2x dx$$

$$= \int \frac{1}{4} \left(\frac{1}{2} - \frac{1}{2}\cos 4x \right) dx$$

$$= \int \left(\frac{1}{8} - \frac{1}{8}\cos 4x \right) dx$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

$$(i) \frac{1}{\sin^2 x \cos^2 x} = \frac{1}{\left(\frac{1}{2}\sin 2x \right)^2} = 4\operatorname{cosec}^2 2x$$

$$\begin{aligned}\therefore \int \frac{1}{\sin^2 x \cos^2 x} dx &= \int 4 \operatorname{cosec}^2 2x dx \\ &= -2 \cot 2x + C\end{aligned}$$

$$\begin{aligned}\text{(j)} \int (\cos 2x - 1)^2 dx &= \int (\cos^2 2x - 2 \cos 2x + 1) dx \\ &= \int \left(\frac{1}{2} \cos 4x + \frac{1}{2} - 2 \cos 2x + 1 \right) dx \\ &= \int \left(\frac{1}{2} \cos 4x + \frac{3}{2} - 2 \cos 2x \right) dx \\ &= \frac{1}{8} \sin 4x + \frac{3}{2}x - \sin 2x + C\end{aligned}$$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise C, Question 2

Question:

Find the following integrals:

$$(a) \int \left(\frac{1 - \sin x}{\cos^2 x} \right) dx$$

$$(b) \int \left(\frac{1 + \cos x}{\sin^2 x} \right) dx$$

$$(c) \int \frac{\cos 2x}{\cos^2 x} dx$$

$$(d) \int \frac{\cos^2 x}{\sin^2 x} dx$$

$$(e) \int \frac{(1 + \cos x)^2}{\sin^2 x} dx$$

$$(f) \int \frac{(1 + \sin x)^2}{\cos^2 x} dx$$

$$(g) \int (\cot x - \tan x)^2 dx$$

$$(h) \int (\cos x - \sin x)^2 dx$$

$$(i) \int (\cos x - \sec x)^2 dx$$

$$(j) \int \frac{\cos 2x}{1 - \cos^2 2x} dx$$

Solution:

$$\begin{aligned} (a) \int \left(\frac{1 - \sin x}{\cos^2 x} \right) dx &= \int \left(\sec^2 x - \tan x \sec x \right) dx \\ &= \tan x - \sec x + C \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int \left(\frac{1 + \cos x}{\sin^2 x} \right) dx &= \int \left(\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x \right) dx \\
 &= -\cot x - \operatorname{cosec} x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \int \frac{\cos 2x}{\cos^2 x} dx &= \int \frac{2\cos^2 x - 1}{\cos^2 x} dx \\
 &= \int (2 - \sec^2 x) dx \\
 &= 2x - \tan x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \int \frac{\cos^2 x}{\sin^2 x} dx &= \int \cot^2 x dx \\
 &= \int (\operatorname{cosec}^2 x - 1) dx \\
 &= -\cot x - x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad I &= \int \frac{(1 + \cos x)^2}{\sin^2 x} dx = \int \frac{1 + 2\cos x + \cos^2 x}{\sin^2 x} dx \\
 &= \int (\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + \cot^2 x) dx \\
 \text{But } \operatorname{cosec}^2 x &= 1 + \cot^2 x \Rightarrow \cot^2 x = \operatorname{cosec}^2 x - 1 \\
 \therefore I &= \int (2\operatorname{cosec}^2 x - 1 + 2\cot x \operatorname{cosec} x) dx \\
 &= -2\cot x - x - 2\operatorname{cosec} x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad I &= \int \frac{(1 + \sin x)^2}{\cos^2 x} dx = \int \frac{1 + 2\sin x + \sin^2 x}{\cos^2 x} dx \\
 &= \int (\sec^2 x + 2\tan x \sec x + \tan^2 x) dx \\
 \text{But } \sec^2 x &= 1 + \tan^2 x \Rightarrow \tan^2 x = \sec^2 x - 1 \\
 \therefore I &= \int (2\sec^2 x - 1 + 2\tan x \sec x) dx \\
 &= 2\tan x - x + 2\sec x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(g)} \quad \int (\cot x - \tan x)^2 dx &= \int (\cot^2 x - 2\cot x \tan x + \tan^2 x) dx \\
 &= \int (\operatorname{cosec}^2 x - 1 - 2 + \sec^2 x - 1) dx \\
 &= \int (\operatorname{cosec}^2 x - 4 + \sec^2 x) dx \\
 &= -\cot x - 4x + \tan x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(h)} \quad \int (\cos x - \sin x)^2 dx &= \int (\cos^2 x - 2\cos x \sin x + \sin^2 x) dx \\
 &= \int (1 - \sin 2x) dx \\
 &= x + \frac{1}{2} \cos 2x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(i)} \quad \int (\cos x - \sec x)^2 dx &= \int (\cos^2 x - 2 \cos x \sec x + \sec^2 x) dx \\
 &= \int \left(\frac{1}{2} \cos 2x + \frac{1}{2} - 2 + \sec^2 x \right) dx \\
 &= \int \left(\frac{1}{2} \cos 2x - \frac{3}{2} + \sec^2 x \right) dx \\
 &= \frac{1}{4} \sin 2x - \frac{3}{2}x + \tan x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(j)} \quad \int \frac{\cos 2x}{1 - \cos^2 2x} dx &= \int \frac{\cos 2x}{\sin^2 2x} dx \\
 &= \int \cot 2x \operatorname{cosec} 2x dx \\
 &= -\frac{1}{2} \operatorname{cosec} 2x + C
 \end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise C, Question 3

Question:

Find the following integrals:

(a) $\int \cos 2x \cos x \, dx$

(b) $\int 2 \sin 5x \cos 3x \, dx$

(c) $\int 2 \sin 3x \cos 5x \, dx$

(d) $\int 2 \sin 2x \sin 5x \, dx$

(e) $4 \int \cos 3x \cos 7x \, dx$

(f) $\int 2 \cos 4x \cos 4x \, dx$

(g) $\int 2 \cos 4x \sin 4x \, dx$

(h) $\int 2 \sin 4x \sin 4x \, dx$

Solution:

$$(a) \cos 3x + \cos x = 2 \cos \frac{3x+x}{2} \cos \frac{3x-x}{2} = 2 \cos 2x \cos x$$

$$\therefore \int \cos 2x \cos x \, dx = \frac{1}{2} \int \left(\cos 3x + \cos x \right) dx$$

$$= \frac{1}{2} \left(\frac{1}{3} \sin 3x + \sin x \right) + C$$

$$= \frac{1}{6} \sin 3x + \frac{1}{2} \sin x + C$$

$$(b) \sin 8x + \sin 2x = 2 \sin 5x \cos 3x$$

$$\therefore \int 2 \sin 5x \cos 3x \, dx = \int (\sin 8x + \sin 2x) \, dx$$

$$= -\frac{1}{8} \cos 8x - \frac{1}{2} \cos 2x + C$$

$$(c) \sin 8x - \sin 2x = 2 \sin 3x \cos 5x$$

$$\therefore \int 2 \sin 3x \cos 5x \, dx = \int (\sin 8x - \sin 2x) \, dx$$

$$= -\frac{1}{8}\cos 8x + \frac{1}{2}\cos 2x + C$$

$$(d) \cos 7x - \cos 3x = -2 \sin 5x \sin 2x$$

$$\therefore \int 2 \sin 2x \sin 5x \, dx = \int (\cos 3x - \cos 7x) \, dx$$

$$= \frac{1}{3}\sin 3x - \frac{1}{7}\sin 7x + C$$

$$(e) \cos 10x + \cos 4x = 2 \cos 7x \cos 3x$$

$$\therefore \int 4 \cos 3x \cos 7x \, dx = 2 \int (\cos 10x + \cos 4x) \, dx$$

$$= 2 \left(\frac{1}{10} \sin 10x + \frac{1}{4} \sin 4x \right) + C$$

$$= \frac{1}{5} \sin 10x + \frac{1}{2} \sin 4x + C$$

$$(f) \cos 8x + \cos 0x = 2 \cos 4x \cos 4x$$

$$\text{i.e. } \cos 8x + 1 = 2 \cos 4x \cos 4x$$

$$\therefore \int 2 \cos 4x \cos 4x \, dx = \int (1 + \cos 8x) \, dx$$

$$= x + \frac{1}{8} \sin 8x + C$$

$$(g) \sin 8x + \sin 0x = 2 \sin 4x \cos 4x$$

$$\therefore \int 2 \cos 4x \sin 4x \, dx = \int \sin 8x \, dx$$

$$= -\frac{1}{8} \cos 8x + C$$

$$(h) \cos 8x - \cos 0x = -2 \sin 4x \sin 4x$$

$$\text{i.e. } \cos 8x - 1 = -2 \sin 4x \sin 4x$$

$$\therefore \int 2 \sin 4x \sin 4x \, dx = \int (1 - \cos 8x) \, dx$$

$$= x - \frac{1}{8} \sin 8x + C$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise D, Question 1

Question:

Use partial fractions to integrate the following:

(a) $\frac{3x + 5}{(x + 1)(x + 2)}$

(b) $\frac{3x - 1}{(2x + 1)(x - 2)}$

(c) $\frac{2x - 6}{(x + 3)(x - 1)}$

(d) $\frac{3}{(2 + x)(1 - x)}$

(e) $\frac{4}{(2x + 1)(1 - 2x)}$

(f) $\frac{3(x + 1)}{9x^2 - 1}$

(g) $\frac{3 - 5x}{(1 - x)(2 - 3x)}$

(h) $\frac{x^2 - 3}{(2 + x)(1 + x)^2}$

(i) $\frac{5 + 3x}{(x + 2)(x + 1)^2}$

(j) $\frac{17 - 5x}{(3 + 2x)(2 - x)^2}$

Solution:

(a) $\frac{3x + 5}{(x + 1)(x + 2)} \equiv \frac{A}{x + 1} + \frac{B}{x + 2}$

$$\Rightarrow 3x + 5 \equiv A(x + 2) + B(x + 1)$$

$$x = -1 \Rightarrow 2 = A$$

$$x = -2 \Rightarrow -1 = -B \Rightarrow B = 1$$

$$\begin{aligned} \therefore \int \frac{3x+5}{(x+1)(x+2)} dx &= \int \left(\frac{2}{x+1} + \frac{1}{x+2} \right) dx \\ &= 2 \ln |x+1| + \ln |x+2| + C \\ &= \ln [|x+1|^2] + \ln |x+2| + C \\ &= \ln | (x+1)^2 (x+2) | + C \end{aligned}$$

$$(b) \frac{3x-1}{(2x+1)(x-2)} \equiv \frac{A}{2x+1} + \frac{B}{x-2}$$

$$\Rightarrow 3x - 1 \equiv A(x - 2) + B(2x + 1)$$

$$x = 2 \Rightarrow 5 = 5B \Rightarrow B = 1$$

$$x = -\frac{1}{2} \Rightarrow -\frac{5}{2} = -\frac{5}{2}A \Rightarrow A = 1$$

$$\begin{aligned} \therefore \int \frac{3x-1}{(2x+1)(x-2)} dx &= \int \left(\frac{1}{2x+1} + \frac{1}{x-2} \right) dx \\ &= \frac{1}{2} \ln |2x+1| + \ln |x-2| + C \\ &= \ln | (x-2) \sqrt{2x+1} | + C \end{aligned}$$

$$(c) \frac{2x-6}{(x+3)(x-1)} \equiv \frac{A}{x+3} + \frac{B}{x-1}$$

$$\Rightarrow 2x - 6 \equiv A(x - 1) + B(x + 3)$$

$$x = 1 \Rightarrow -4 = 4B \Rightarrow B = -1$$

$$x = -3 \Rightarrow -12 = -4A \Rightarrow A = 3$$

$$\begin{aligned} \therefore \int \frac{2x-6}{(x+3)(x-1)} dx &= \int \left(\frac{3}{x+3} - \frac{1}{x-1} \right) dx \\ &= 3 \ln |x+3| - \ln |x-1| + C \\ &= \ln \left| \frac{(x+3)^3}{x-1} \right| + C \end{aligned}$$

$$(d) \frac{3}{(2+x)(1-x)} \equiv \frac{A}{(2+x)} + \frac{B}{1-x}$$

$$\Rightarrow 3 \equiv A(1-x) + B(2+x)$$

$$x = 1 \Rightarrow 3 = 3B \Rightarrow B = 1$$

$$x = -2 \Rightarrow 3 = 3A \Rightarrow A = 1$$

$$\begin{aligned}
 \therefore \int \frac{3}{(2+x)(1-x)} dx &= \int \left(\frac{1}{2+x} + \frac{1}{1-x} \right) dx. \\
 &= \ln |2+x| - \ln |1-x| + C \\
 &= \ln \left| \frac{2+x}{1-x} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad \frac{4}{(2x+1)(1-2x)} &\equiv \frac{A}{2x+1} + \frac{B}{1-2x} \\
 \Rightarrow 4 &\equiv A(1-2x) + B(2x+1)
 \end{aligned}$$

$$x = \frac{1}{2} \Rightarrow 4 = 2B \Rightarrow B = 2$$

$$x = -\frac{1}{2} \Rightarrow 4 = 2A \Rightarrow A = 2$$

$$\begin{aligned}
 \therefore \int \frac{4}{(2x+1)(1-2x)} dx &= \int \left(\frac{2}{2x+1} + \frac{2}{1-2x} \right) dx \\
 &= \ln |2x+1| - \ln |1-2x| + C \\
 &= \ln \left| \frac{2x+1}{1-2x} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad \frac{3(x+1)}{9x^2-1} &\equiv \frac{3(x+1)}{(3x-1)(3x+1)} \equiv \frac{A}{3x-1} + \frac{B}{3x+1} \\
 \Rightarrow 3x+3 &\equiv A(3x+1) + B(3x-1)
 \end{aligned}$$

$$x = -\frac{1}{3} \Rightarrow 2 = -2B \Rightarrow B = -1$$

$$x = \frac{1}{3} \Rightarrow 4 = 2A \Rightarrow A = 2$$

$$\begin{aligned}
 \therefore \int \frac{3(x+1)}{9x^2-1} dx &= \int \left(\frac{2}{3x-1} - \frac{1}{3x+1} \right) dx \\
 &= \frac{2}{3} \ln |3x-1| - \frac{1}{3} \ln |3x+1| + C \\
 &= \frac{1}{3} \ln \left| \frac{(3x-1)^2}{3x+1} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(g)} \quad \frac{3-5x}{(1-x)(2-3x)} &\equiv \frac{A}{1-x} + \frac{B}{2-3x} \\
 \Rightarrow 3-5x &\equiv A(2-3x) + B(1-x)
 \end{aligned}$$

$$x = \frac{2}{3} \Rightarrow -\frac{1}{3} = \frac{1}{3}B \Rightarrow B = -1$$

$$x = 1 \Rightarrow -2 = -A \Rightarrow A = 2$$

$$\therefore \int \frac{3-5x}{(1-x)(2-3x)} dx = \int \left(\frac{2}{1-x} - \frac{1}{2-3x} \right) dx$$

$$= -2 \ln |1-x| + \frac{1}{3} \ln |2-3x| + C$$

$$= \ln \left| \frac{(2-3x)^{\frac{1}{3}}}{(1-x)^2} \right| + C$$

$$(h) \frac{x^2-3}{(2+x)(1+x)^2} \equiv \frac{A}{2+x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$$

$$\Rightarrow x^2-3 \equiv A(1+x)^2 + B(2+x)(1+x) + C(2+x)$$

$$x = -1 \Rightarrow -2 = C \Rightarrow C = -2$$

$$x = -2 \Rightarrow 1 = 1A \Rightarrow A = 1$$

$$\text{Coefficient of } x^2 \Rightarrow 1 = A + B \Rightarrow B = 0$$

$$\therefore \int \frac{x^2-3}{(2+x)(1+x)^2} dx = \int \left(\frac{1}{2+x} - \frac{2}{(1+x)^2} \right) dx$$

$$= \ln |2+x| - 2 \frac{(1+x)^{-1}}{-1} + C$$

$$= \ln |2+x| + \frac{2}{1+x} + C$$

$$(i) \frac{5+3x}{(x+2)(x+1)^2} \equiv \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\Rightarrow 5+3x \equiv A(x+1)^2 + B(x+2)(x+1) + C(x+2)$$

$$x = -1 \Rightarrow 2 = C \Rightarrow C = 2$$

$$x = -2 \Rightarrow -1 = A \Rightarrow A = -1$$

$$\text{Coefficient of } x^2 \Rightarrow 0 = A + B \Rightarrow B = 1$$

$$\therefore \int \frac{5+3x}{(x+2)(x+1)^2} dx = \int \left(-\frac{1}{x+2} + \frac{1}{x+1} + \frac{2}{(x+1)^2} \right) dx$$

$$= -\ln |x+2| + \ln |x+1| - \frac{2}{x+1} + C$$

$$= \ln \left| \frac{x+1}{x+2} \right| - \frac{2}{x+1} + C$$

$$(j) \frac{17 - 5x}{(3 + 2x)(2 - x)^2} \equiv \frac{A}{3 + 2x} + \frac{B}{2 - x} + \frac{C}{(2 - x)^2}$$

$$\Rightarrow 17 - 5x \equiv A(2 - x)^2 + B(3 + 2x)(2 - x) + C(3 + 2x)$$

$$x = 2 \Rightarrow 7 = 7C \Rightarrow C = 1$$

$$x = -\frac{3}{2} \Rightarrow \frac{49}{2} = \frac{49}{4}A \Rightarrow A = 2$$

$$\text{Coefficient of } x^2 \Rightarrow 0 = A - 2B \Rightarrow B = 1$$

$$\therefore \int \frac{17 - 5x}{(3 + 2x)(2 - x)^2} dx = \int \left(\frac{2}{3 + 2x} + \frac{1}{2 - x} + \frac{1}{(2 - x)^2} \right) dx$$

$$= \frac{2}{2} \ln |3 + 2x| - \ln |2 - x| + \frac{1}{2 - x} + C$$

$$= \ln \left| \frac{3 + 2x}{2 - x} \right| + \frac{1}{2 - x} + C$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise D, Question 2

Question:

Find the following integrals:

$$(a) \int \frac{2(x^2 + 3x - 1)}{(x + 1)(2x - 1)} dx$$

$$(b) \int \frac{x^3 + 2x^2 + 2}{x(x + 1)} dx$$

$$(c) \int \frac{x^2}{x^2 - 4} dx$$

$$(d) \int \frac{x^2 + x + 2}{3 - 2x - x^2} dx$$

$$(e) \int \frac{6 + 3x - x^2}{x^3 + 2x^2} dx$$

Solution:

$$(a) \frac{2(x^2 + 3x - 1)}{(x + 1)(2x - 1)} \equiv 1 + \frac{A}{x + 1} + \frac{B}{2x - 1}$$

$$\Rightarrow 2x^2 + 6x - 2 \equiv (x + 1)(2x - 1) + A(2x - 1) + B(x + 1)$$

$$x = -1 \Rightarrow -6 = -3A \Rightarrow A = 2$$

$$x = \frac{1}{2} \Rightarrow \frac{3}{2} = \frac{3}{2}B \Rightarrow B = 1$$

$$\therefore \int \frac{2(x^2 + 3x - 1)}{(x + 1)(2x - 1)} dx = \int \left(1 + \frac{2}{x + 1} + \frac{1}{2x - 1} \right) dx$$

$$= x + 2 \ln |x + 1| + \frac{1}{2} \ln |2x - 1| + C$$

$$= x + \ln |(x + 1)^2 \sqrt{2x - 1}| + C$$

$$(b) \frac{x^3 + 2x^2 + 2}{x(x + 1)} \Rightarrow$$

$$\begin{array}{r}
 x+1 \\
 x^2+x \overline{) x^3+2x^2+2} \\
 \underline{x^3+x^2} \\
 x^2+2 \\
 \underline{x^2+x} \\
 2-x
 \end{array}$$

$$\frac{x^3+2x^2+2}{x(x+1)} \equiv x+1 + \frac{2-x}{x(x+1)}$$

$$\equiv x+1 + \frac{A}{x} + \frac{B}{x+1}$$

$$\Rightarrow x^3+2x^2+2 \equiv (x+1)x(x+1) + A(x+1) + Bx$$

$$x=0 \Rightarrow 2=A \Rightarrow A=2$$

$$x=-1 \Rightarrow 3=-B \Rightarrow B=-3$$

$$\therefore \int \frac{x^3+2x^2+2}{x(x+1)} dx = \int \left(x+1 + \frac{2}{x} - \frac{3}{x+1} \right) dx$$

$$= \frac{x^2}{2} + x + 2 \ln|x| - 3 \ln|x+1| + C$$

$$= \frac{x^2}{2} + x + \ln \left| \frac{x^2}{(x+1)^3} \right| + C$$

$$(c) \frac{x^2}{x^2-4} \equiv 1 + \frac{A}{x-2} + \frac{B}{x+2}$$

$$\Rightarrow x^2 \equiv (x-2)(x+2) + A(x+2) + B(x-2)$$

$$x=2 \Rightarrow 4=4A \Rightarrow A=1$$

$$x=-2 \Rightarrow 4=-4B \Rightarrow B=-1$$

$$\therefore \int \frac{x^2}{x^2-4} dx = \int \left(1 + \frac{1}{x-2} - \frac{1}{x+2} \right) dx$$

$$= x + \ln|x-2| - \ln|x+2| + C$$

$$= x + \ln \left| \frac{x-2}{x+2} \right| + C$$

$$(d) \frac{x^2+x+2}{3-2x-x^2} \equiv \frac{x^2+x+2}{(3+x)(1-x)} \equiv -1 + \frac{A}{3+x} + \frac{B}{1-x}$$

$$\Rightarrow x^2+x+2 \equiv -1(3+x)(1-x) + A(1-x) + B(3+x)$$

$$x=1 \Rightarrow 4=4B \Rightarrow B=1$$

$$x=-3 \Rightarrow 8=4A \Rightarrow A=2$$

$$\therefore \int \frac{x^2+x+2}{3-2x-x^2} dx = \int \left(-1 + \frac{2}{3+x} + \frac{1}{1-x} \right) dx$$

$$= -x + 2 \ln|3+x| - \ln|1-x| + C$$

$$= -x + \ln \left| \frac{(3+x)^2}{1-x} \right| + C$$

$$\begin{aligned}
 \text{(e)} \quad \frac{6+3x-x^2}{x^3+2x^2} &\equiv \frac{6+3x-x^2}{x^2(x+2)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} \\
 \Rightarrow 6+3x-x^2 &\equiv Ax(x+2) + B(x+2) + Cx^2 \\
 x=0 \quad \Rightarrow \quad 6 &= 2B \quad \Rightarrow \quad B=3 \\
 x=-2 \quad \Rightarrow \quad -4 &= 4C \quad \Rightarrow \quad C=-1 \\
 \text{Coefficient of } x^2 \quad \Rightarrow \quad -1 &= A+C \quad \Rightarrow \quad A=0 \\
 \therefore \int \frac{6+3x-x^2}{x^3+2x^2} dx &= \int \left(\frac{3}{x^2} - \frac{1}{x+2} \right) dx \\
 &= -\frac{3}{x} - \ln|x+2| + C
 \end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise E, Question 1

Question:

Integrate the following functions:

(a) $\frac{x}{x^2 + 4}$

(b) $\frac{e^{2x}}{e^{2x} + 1}$

(c) $\frac{x}{(x^2 + 4)^3}$

(d) $\frac{e^{2x}}{(e^{2x} + 1)^3}$

(e) $\frac{\cos 2x}{3 + \sin 2x}$

(f) $\frac{\sin 2x}{(3 + \cos 2x)^3}$

(g) xe^{x^2}

(h) $\cos 2x (1 + \sin 2x)^4$

(i) $\sec^2 x \tan^2 x$

(j) $\sec^2 x (1 + \tan^2 x)$

Solution:

(a) $y = \ln |x^2 + 4|$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x^2 + 4} \times 2x \quad (\text{chain rule})$$

$$\therefore \int \frac{x}{x^2 + 4} dx = \frac{1}{2} \ln |x^2 + 4| + C$$

$$(b) y = \ln |e^{2x} + 1|$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^{2x} + 1} \times e^{2x} \times 2 \quad (\text{chain rule})$$

$$\therefore \int \frac{e^{2x}}{e^{2x} + 1} dx = \frac{1}{2} \ln |e^{2x} + 1| + C$$

$$(c) y = (x^2 + 4)^{-2}$$

$$\Rightarrow \frac{dy}{dx} = -2(x^2 + 4)^{-3} \times 2x \quad (\text{chain rule})$$

$$\therefore \int \frac{x}{(x^2 + 4)^3} dx = -\frac{1}{4}(x^2 + 4)^{-2} + C$$

$$\text{or} \quad -\frac{1}{4(x^2 + 4)^2} + C$$

$$(d) y = (e^{2x} + 1)^{-2}$$

$$\Rightarrow \frac{dy}{dx} = -2(e^{2x} + 1)^{-3} \times e^{2x} \times 2 \quad (\text{chain rule})$$

$$\therefore \int \frac{e^{2x}}{(e^{2x} + 1)^3} dx = -\frac{1}{4}(e^{2x} + 1)^{-2} + C$$

$$\text{or} \quad -\frac{1}{4(e^{2x} + 1)^2} + C$$

$$(e) y = \ln |3 + \sin 2x|$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3 + \sin 2x} \times \cos 2x \times 2 \quad (\text{chain rule})$$

$$\therefore \int \frac{\cos 2x}{3 + \sin 2x} dx = \frac{1}{2} \ln |3 + \sin 2x| + C$$

$$(f) y = (3 + \cos 2x)^{-2}$$

$$\Rightarrow \frac{dy}{dx} = -2(3 + \cos 2x)^{-3} \times \left(-\sin 2x \right) \times 2 \quad (\text{chain rule})$$

$$\therefore \int \frac{\sin 2x}{(3 + \cos 2x)^3} dx = \frac{1}{4}(3 + \cos 2x)^{-2} + C$$

$$\text{or} \quad \frac{1}{4(3 + \cos 2x)^2} + C$$

$$(g) y = e^{x^2}$$

$$\Rightarrow \frac{dy}{dx} = e^{x^2} \times 2x \quad (\text{chain rule})$$

$$\therefore \int x e^{x^2} dx = \frac{1}{2} e^{x^2} + C$$

$$(h) y = (1 + \sin 2x)^5$$

$$\Rightarrow \frac{dy}{dx} = 5 (1 + \sin 2x)^4 \times \cos 2x \times 2 \quad (\text{chain rule})$$

$$\therefore \int \cos 2x (1 + \sin 2x)^4 dx = \frac{1}{10} (1 + \sin 2x)^5 + C$$

$$(i) y = \tan^3 x$$

$$\Rightarrow \frac{dy}{dx} = 3 \tan^2 x \times \sec^2 x \quad (\text{chain rule})$$

$$\therefore \int \sec^2 x \tan^2 x dx = \frac{1}{3} \tan^3 x + C$$

$$(j) \sec^2 x (1 + \tan^2 x) = \sec^2 x + \sec^2 x \tan^2 x$$

$$\therefore \int \sec^2 x (1 + \tan^2 x) dx = \int (\sec^2 x + \sec^2 x \tan^2 x) dx$$

$$= \tan x + \frac{1}{3} \tan^3 x + C$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise E, Question 2

Question:

Find the following integrals:

(a) $\int (x + 1) (x^2 + 2x + 3)^4 dx$

(b) $\int \operatorname{cosec}^2 2x \cot 2x dx$

(c) $\int \sin^5 3x \cos 3x dx$

(d) $\int \cos x e^{\sin x} dx$

(e) $\int \frac{e^{2x}}{e^{2x} + 3} dx$

(f) $\int x (x^2 + 1)^{\frac{3}{2}} dx$

(g) $\int (2x + 1) \sqrt{x^2 + x + 5} dx$

(h) $\int \frac{2x + 1}{\sqrt{x^2 + x + 5}} dx$

(i) $\int \frac{\sin x \cos x}{\sqrt{\cos 2x + 3}} dx$

(j) $\int \frac{\sin x \cos x}{\cos 2x + 3} dx$

Solution:

(a) $y = (x^2 + 2x + 3)^5$

$$\Rightarrow \frac{dy}{dx} = 5 (x^2 + 2x + 3)^4 \times (2x + 2)$$

$$= 5 (x^2 + 2x + 3)^4 \times 2 (x + 1)$$

$$\therefore \int (x + 1) (x^2 + 2x + 3)^4 dx = \frac{1}{10} (x^2 + 2x + 3)^5 + C$$

$$(b) y = \cot^2 2x$$

$$\Rightarrow \frac{dy}{dx} = 2 \cot 2x \times \left(-\operatorname{cosec}^2 2x \right) \times 2$$

$$= -4 \operatorname{cosec}^2 2x \cot 2x$$

$$\therefore \int \operatorname{cosec}^2 2x \cot 2x dx = -\frac{1}{4} \cot^2 2x + C$$

$$(c) y = \sin^6 3x$$

$$\Rightarrow \frac{dy}{dx} = 6 \sin^5 3x \times \cos 3x \times 3$$

$$\therefore \int \sin^5 3x \cos 3x dx = \frac{1}{18} \sin^6 3x + C$$

$$(d) y = e^{\sin x}$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin x} \times \cos x$$

$$\therefore \int \cos x e^{\sin x} dx = e^{\sin x} + C$$

$$(e) y = \ln |e^{2x} + 3|$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^{2x} + 3} \times e^{2x} \times 2$$

$$\therefore \int \frac{e^{2x}}{e^{2x} + 3} dx = \frac{1}{2} \ln |e^{2x} + 3| + C$$

$$(f) y = (x^2 + 1)^{\frac{5}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{5}{2} (x^2 + 1)^{\frac{3}{2}} \times 2x = 5x (x^2 + 1)^{\frac{3}{2}}$$

$$\therefore \int x (x^2 + 1)^{\frac{3}{2}} dx = \frac{1}{5} (x^2 + 1)^{\frac{5}{2}} + C$$

$$(g) y = (x^2 + x + 5)^{\frac{3}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2} (x^2 + x + 5)^{\frac{1}{2}} \times (2x + 1)$$

$$\therefore \int (2x + 1) \sqrt{x^2 + x + 5} dx = \frac{2}{3} (x^2 + x + 5)^{\frac{3}{2}} + C$$

$$(h) y = (x^2 + x + 5)^{\frac{1}{2}}$$

$$\begin{aligned}
\Rightarrow \quad \frac{dy}{dx} &= \frac{1}{2} (x^2 + x + 5)^{-\frac{1}{2}} \times (2x + 1) \\
&= \frac{1}{2} \frac{(2x + 1)}{\sqrt{x^2 + x + 5}} \\
\therefore \int \frac{2x + 1}{\sqrt{x^2 + x + 5}} dx &= 2 (x^2 + x + 5)^{\frac{1}{2}} + C
\end{aligned}$$

$$(i) y = (\cos 2x + 3)^{\frac{1}{2}}$$

$$\begin{aligned}
\Rightarrow \quad \frac{dy}{dx} &= \frac{1}{2} (\cos 2x + 3)^{-\frac{1}{2}} \times (-\sin 2x) \times 2 \\
&= -\frac{\sin 2x}{\sqrt{\cos 2x + 3}} \\
&= -\frac{2 \sin x \cos x}{\sqrt{\cos 2x + 3}} \\
\therefore \int \frac{\sin x \cos x}{\sqrt{\cos 2x + 3}} dx &= -\frac{1}{2} (\cos 2x + 3)^{\frac{1}{2}} + C
\end{aligned}$$

$$(j) y = \ln |\cos 2x + 3|$$

$$\begin{aligned}
\Rightarrow \quad \frac{dy}{dx} &= \frac{1}{\cos 2x + 3} \times (-\sin 2x) \times 2 \\
&= -\frac{2 \sin 2x}{\cos 2x + 3} \\
&= -\frac{4 \sin x \cos x}{\cos 2x + 3} \\
\therefore \int \frac{\sin x \cos x}{\cos 2x + 3} dx &= -\frac{1}{4} \ln |\cos 2x + 3| + C
\end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise F, Question 1

Question:

Use the given substitution to find the following integrals:

(a) $\int x\sqrt{1+x} \, dx; u = 1+x$

(b) $\int \frac{x}{\sqrt{1+x}} \, dx; u = 1+x$

(c) $\int \frac{1+\sin x}{\cos x} \, dx; u = \sin x$

(d) $\int x(3+2x)^5 \, dx; u = 3+2x$

(e) $\int \sin^3 x \, dx; u = \cos x$

Solution:

(a) $u = 1+x \Rightarrow du = dx \text{ and } x = u-1$

$$\therefore \int x(1+x)^{\frac{1}{2}} \, dx = \int (u-1)u^{\frac{1}{2}} \, du$$

$$= \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}} \right) \, du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}} + C$$

$$\text{OR} = \frac{2}{15}(1+x)^{\frac{3}{2}} \left[3 \left(1+x \right) - 5 \right] + C$$

$$= \frac{2}{15}(1+x)^{\frac{3}{2}} (3x-2) + C$$

(b) $u = 1+x \Rightarrow du = dx \text{ and } x = u-1$

$$\therefore \int \frac{x}{\sqrt{1+x}} \, dx = \int \frac{u-1}{u^{\frac{1}{2}}} \, du$$

$$\begin{aligned}
&= \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}} \right) du \\
&= \frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C \\
&= \frac{2}{3} (1+x)^{\frac{3}{2}} - 2(1+x)^{\frac{1}{2}} + C \\
\text{OR } &= \frac{2}{3} (1+x)^{\frac{1}{2}} \left[1+x-3 \right] + C \\
&= \frac{2}{3} (1+x)^{\frac{1}{2}} \left(x-2 \right) + C
\end{aligned}$$

$$(c) \ u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow dx = \frac{du}{\cos x}$$

$$\begin{aligned}
\therefore \int \frac{1+\sin x}{\cos x} dx &= \int \frac{1+u}{\cos x} \frac{du}{\cos x} \\
&= \int \frac{1+u}{1-\sin^2 x} du \\
&= \int \frac{1+u}{1-u^2} du \\
&= \int \frac{(1+u)}{(1-u)(1+u)} du \\
&= \int \frac{1}{1-u} du \\
&= -\ln |1-u| + C \\
&= -\ln |1-\sin x| + C
\end{aligned}$$

$$(d) \ u = 3 + 2x \Rightarrow du = 2 dx \text{ and } x = \frac{u-3}{2}$$

$$\begin{aligned}
\therefore \int x(3+2x)^5 dx &= \int \frac{u-3}{2} u^5 \frac{du}{2} \\
&= \int \left(\frac{u^6}{4} - \frac{3u^5}{4} \right) du \\
&= \frac{u^7}{28} - \frac{3u^6}{24} + C \\
&= \frac{u^7}{28} - \frac{u^6}{8} + C \\
&= \frac{(3+2x)^7}{28} - \frac{(3+2x)^6}{8} + C
\end{aligned}$$

$$(e) u = \cos x \Rightarrow du = -\sin x dx$$

$$\therefore \int \sin^3 x dx = \int -(1 - u^2) du$$

$$= \int (u^2 - 1) du$$

$$= \frac{u^3}{3} - u + C$$

$$= \frac{\cos^3 x}{3} - \cos x + C$$

$$\text{OR} = \frac{\cos x}{3} \left(\cos^2 x - 3 \right) + C$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise F, Question 2

Question:

Use the given substitution to find the following integrals:

(a) $\int x\sqrt{2+x} \, dx; u^2 = 2+x$

(b) $\int \frac{2}{\sqrt{x(x-4)}} \, dx; u = \sqrt{x}$

(c) $\int \sec^2 x \tan x \sqrt{1+\tan x} \, dx; u^2 = 1+\tan x$

(d) $\int \frac{\sqrt{x^2+4}}{x} \, dx; u^2 = x^2+4$

(e) $\int \sec^4 x \, dx; u = \tan x$

Solution:

(a) $u^2 = 2+x \Rightarrow 2u \, du = dx \text{ and } x = u^2 - 2$

$$\therefore \int x\sqrt{2+x} \, dx = \int (u^2 - 2) \times u \times 2u \, du$$

$$= \int (2u^4 - 4u^2) \, du$$

$$= \frac{2u^5}{5} - \frac{4u^3}{3} + C$$

$$= \frac{2}{5} (2+x)^{\frac{5}{2}} - \frac{4}{3} (2+x)^{\frac{3}{2}} + C$$

(b) $u = x^{\frac{1}{2}} \Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow \frac{dx}{\sqrt{x}} = 2 \, du$

and $x-4 = u^2 - 4$

$$\therefore I = \int \frac{2}{\sqrt{x(x-4)}} \, dx = \int \frac{2}{u^2-4} \times 2 \, du = \int \frac{4}{u^2-4} \, du$$

$$\frac{4}{u^2-4} = \frac{A}{u-2} + \frac{B}{u+2}$$

$$\Rightarrow 4 = A(u+2) + B(u-2)$$

$$u=2 \Rightarrow 4 = 4A \Rightarrow A = 1$$

$$u = -2 \Rightarrow 4 = -4B \Rightarrow B = -1$$

$$\begin{aligned} \therefore I &= \int \left(\frac{1}{u-2} - \frac{1}{u+2} \right) du \\ &= \ln |u-2| - \ln |u+2| + C \\ &= \ln \left| \frac{\sqrt{x-2}}{\sqrt{x+2}} \right| + C \end{aligned}$$

$$(c) u^2 = 1 + \tan x \Rightarrow 2u du = \sec^2 x dx$$

$$\begin{aligned} \therefore \int \sec^2 x \tan x \sqrt{1 + \tan x} dx \\ &= \int (u^2 - 1) \times u \times 2u du \\ &= \int (2u^4 - 2u^2) du \\ &= \frac{2u^5}{5} - \frac{2u^3}{3} + C \\ &= \frac{2}{5} (1 + \tan x)^{\frac{5}{2}} - \frac{2}{3} (1 + \tan x)^{\frac{3}{2}} + C \end{aligned}$$

$$(d) u^2 = x^2 + 4 \Rightarrow 2u du = 2x dx \Rightarrow \frac{u du}{x} = dx$$

$$\begin{aligned} \therefore \int \frac{\sqrt{x^2 + 4}}{x} dx &= \int \frac{u}{x} \times \frac{u du}{x} \\ &= \int \frac{u^2}{x^2} du \\ &= \int \frac{u^2}{u^2 - 4} du \end{aligned}$$

$$\begin{aligned} &= \int \left(1 + \frac{4}{u^2 - 4} \right) du \text{ but } \frac{4}{u^2 - 4} \equiv \frac{A}{u+2} + \frac{B}{u-2} \\ &\quad 4 \equiv A(u+2) + B(u-2) \\ &\quad u = 2 : 4 = 4A, A = 1 \\ &\quad u = -2 : 4 = -4B, B = -1 \end{aligned}$$

$$\begin{aligned} &= \int \left(1 + \frac{1}{u-2} - \frac{1}{u+2} \right) du \\ &= u + \ln |u-2| - \ln |u+2| + C \\ &= \sqrt{x^2 + 4} + \ln \left| \frac{\sqrt{x^2 + 4} - 2}{\sqrt{x^2 + 4} + 2} \right| + C \end{aligned}$$

$$(e) u = \tan x \Rightarrow du = \sec^2 x dx$$

$$\begin{aligned}\therefore \int \sec^4 x \, dx &= \int \sec^2 x \sec^2 x \, dx \\&= \int (1 + u^2) \, du \\&= u + \frac{u^3}{3} + C \\&= \tan x + \frac{\tan^3 x}{3} + C\end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise F, Question 3

Question:

Evaluate the following:

(a) $\int_0^5 x\sqrt{x+4} \, dx$

(b) $\int_0^{\frac{\pi}{3}} \sec x \tan x \sqrt{\sec x + 2} \, dx$

(c) $\int_2^5 \frac{1}{1+\sqrt{x-1}} \, dx$; use $u^2 = x - 1$

(d) $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos \theta} \, d\theta$; let $u = 1 + \cos \theta$

(e) $\int_0^1 x(2+x)^3 \, dx$

(f) $\int_1^4 \frac{1}{\sqrt{x}(4x-1)} \, dx$; let $u = \sqrt{x}$

Solution:

(a) $u^2 = x + 4 \Rightarrow 2u \, du = dx$ and $x = u^2 - 4$

Also $u = 3$ when $x = 5$

and $u = 2$ when $x = 0$.

$$\begin{aligned} \therefore \int_0^5 x\sqrt{x+4} \, dx &= \int_2^3 (u^2 - 4) \times u \times 2u \, du \\ &= \int_2^3 (2u^4 - 8u^2) \, du \\ &= \left[\frac{2}{5}u^5 - \frac{8}{3}u^3 \right]_2^3 \\ &= \left(\frac{2}{5} \times 243 - \frac{8}{3} \times 27 \right) - \left(\frac{64}{5} - \frac{64}{3} \right) \\ &= 25.2 - -8.53 \\ &= 33.73 \\ &= 33.7 \text{ (3 s.f.)} \end{aligned}$$

(b) $u^2 = \sec x + 2 \Rightarrow 2u \, du = \sec x \tan x \, dx$

Also $u = 2$ when $x = \frac{\pi}{3}$

and $u = \sqrt{3}$ when $x = 0$.

$$\begin{aligned}\therefore \int_0^{\frac{\pi}{3}} \sec x \tan x \sqrt{\sec x + 2} dx &= \int_{\sqrt{3}}^2 u \times 2u du \\ &= \int_{\sqrt{3}}^2 2u^2 du \\ &= \left[\frac{2}{3} u^3 \right]_{\sqrt{3}}^2 \\ &= \left(\frac{16}{3} \right) - \left(\frac{2}{3} \times 3 \sqrt{3} \right) \\ &= \frac{16}{3} - 2\sqrt{3}\end{aligned}$$

(c) $u^2 = x - 1 \Rightarrow 2u du = dx$

Also $u = 2$ when $x = 5$

and $u = 1$ when $x = 2$.

$$\begin{aligned}\therefore \int_2^5 \frac{1}{1 + \sqrt{x-1}} dx &= \int_1^2 \frac{1}{1+u} \times 2u du \\ &= \int_1^2 \frac{2u}{u+1} du \\ &= \int_1^2 \left(2 - \frac{2}{u+1} \right) du \\ &= [2u - 2 \ln |u+1|]_1^2 \\ &= (4 - 2 \ln 3) - (2 - 2 \ln 2) \\ &= 2 + 2 \ln \frac{2}{3}\end{aligned}$$

(d) $u = 1 + \cos \theta \Rightarrow du = -\sin \theta d\theta$ or $-du = \sin \theta d\theta$

Also $u = 1$ when $\theta = \frac{\pi}{2}$

and $u = 2$ when $\theta = 0$.

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1 + \cos \theta} d\theta = \int_2^1 \frac{2 \sin \theta \cos \theta}{1 + \cos \theta} d\theta = \int_2^1 \frac{2(u-1)}{u} du$$

Use ‘-’ to reverse limits:

$$\begin{aligned}I &= \int_1^2 \frac{2u-2}{u} du \\ &= \int_1^2 \left(2 - \frac{2}{u} \right) du \\ &= [2u - 2 \ln |u|]_1^2\end{aligned}$$

$$= (4 - 2\ln 2) - (2 - 2\ln 1)$$

$$= 2 - 2\ln 2$$

(e) $u = 2 + x \Rightarrow du = dx$ and $x = u - 2$

Also $u = 3$ when $x = 1$

and $u = 2$ when $x = 0$.

$$\therefore \int_0^1 x(2+x)^3 dx = \int_2^3 (u-2)^3 u^3 du$$

$$= \int_2^3 (u^4 - 2u^3) du$$

$$= \left[\frac{u^5}{5} - \frac{2}{4}u^4 \right]_2^3$$

$$= \left(\frac{243}{5} - \frac{81}{2} \right) - \left(\frac{32}{5} - \frac{16}{2} \right)$$

$$= \frac{211}{5} - 32.5$$

$$= 42.2 - 32.5$$

$$= 9.7$$

(f) $u = x^{\frac{1}{2}} \Rightarrow du = \frac{1}{2}x^{-\frac{1}{2}} dx \Rightarrow \frac{dx}{\sqrt{x}} = 2 du$

and $4x - 1 = 4u^2 - 1$

Also $u = 2$ when $x = 4$

and $u = 1$ when $x = 1$.

$$\therefore I = \int_1^4 \frac{1}{\sqrt{x}(4x-1)} dx = \int_1^2 \frac{1}{4u^2-1} \times 2 du$$

$$\frac{2}{4u^2-1} = \frac{A}{2u-1} + \frac{B}{2u+1}$$

$$\Rightarrow 2 = A(2u+1) + B(2u-1)$$

$$u = \frac{1}{2} \Rightarrow 2 = 2A \Rightarrow A = 1$$

$$u = -\frac{1}{2} \Rightarrow 2 = -2B \Rightarrow B = -1$$

$$\therefore I = \int_1^2 \left(\frac{1}{2u-1} - \frac{1}{2u+1} \right) du$$

$$= \left[\frac{1}{2} \ln |2u-1| - \frac{1}{2} \ln |2u+1| \right]_1^2$$

$$= \left[\frac{1}{2} \ln \left| \frac{2u-1}{2u+1} \right| \right]_1^2$$

$$\begin{aligned} &= \left(\frac{1}{2} \ln \left| \frac{3}{5} \right| \right) - \left(\frac{1}{2} \ln \left| \frac{1}{3} \right| \right) \\ &= \frac{1}{2} \ln \frac{9}{5} \end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise G, Question 1

Question:

Find the following integrals:

(a) $\int x \sin x \, dx$

(b) $\int x e^x \, dx$

(c) $\int x \sec^2 x \, dx$

(d) $\int x \sec x \tan x \, dx$

(e) $\int \frac{x}{\sin^2 x} \, dx$

Solution:

(a) $u = x \Rightarrow \frac{du}{dx} = 1$

$$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$$

$$\begin{aligned} \therefore \int x \sin x \, dx &= -x \cos x - \int -\cos x \times 1 \, dx \\ &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \sin x + C \end{aligned}$$

(b) $u = x \Rightarrow \frac{du}{dx} = 1$

$$\frac{dv}{dx} = e^x \Rightarrow v = e^x$$

$$\begin{aligned} \therefore \int x e^x \, dx &= x e^x - \int e^x \times 1 \, dx \\ &= x e^x - e^x + C \end{aligned}$$

(c) $u = x \Rightarrow \frac{du}{dx} = 1$

$$\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$$

$$\therefore \int x \sec^2 x \, dx = x \tan x - \int \tan x \times 1 \, dx$$

$$= x \tan x - \ln |\sec x| + C$$

$$(d) \quad u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \sec x \tan x \quad \Rightarrow \quad v = \sec x$$

$$\begin{aligned} \therefore \int x \sec x \tan x \, dx &= x \sec x - \int \sec x \times 1 \, dx \\ &= x \sec x - \ln |\sec x + \tan x| + C \end{aligned}$$

$$(e) \quad u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \operatorname{cosec}^2 x \quad \Rightarrow \quad v = -\cot x$$

$$\begin{aligned} \therefore \int \frac{x}{\sin^2 x} \, dx &= \int x \operatorname{cosec}^2 x \, dx \\ &= -x \cot x - \int -\cot x \times 1 \, dx \\ &= -x \cot x + \int \cot x \, dx \\ &= -x \cot x + \ln |\sin x| + C \end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise G, Question 2

Question:

Find the following integrals:

(a) $\int x^2 \ln x \, dx$

(b) $\int 3 \ln x \, dx$

(c) $\int \frac{\ln x}{x^3} \, dx$

(d) $\int (\ln x)^2 \, dx$

(e) $\int (x^2 + 1) \ln x \, dx$

Solution:

(a) $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$$\frac{dv}{dx} = x^2 \Rightarrow v = \frac{x^3}{3}$$

$$\therefore \int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \times \frac{1}{x} \, dx$$

$$= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \, dx$$

$$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C$$

(b) $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$$\frac{dv}{dx} = 3 \Rightarrow v = 3x$$

$$\therefore \int 3 \ln x \, dx = 3x \ln x - \int 3x \times \frac{1}{x} \, dx$$

$$= 3x \ln x - \int 3 \, dx$$

$$= 3x \ln x - 3x + C$$

$$(c) u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x^{-3} \Rightarrow v = \frac{x^{-2}}{-2}$$

$$\therefore \int \frac{\ln x}{x^3} dx = -\frac{1}{2x^2} \ln x - \int -\frac{1}{2x^2} \times \frac{1}{x} dx$$

$$= -\frac{\ln x}{2x^2} + \int \frac{1}{2} x^{-3} dx$$

$$= -\frac{\ln x}{2x^2} + \frac{x^{-2}}{2 \times (-2)} + C$$

$$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$$

$$(d) u = (\ln x)^2 \Rightarrow \frac{du}{dx} = 2 \ln x \times \frac{1}{x}$$

$$\frac{dv}{dx} = 1 \Rightarrow v = x$$

$$\therefore I = \int (\ln x)^2 dx = x (\ln x)^2 - \int x \times 2 \ln x \times \frac{1}{x} dx$$

$$= x (\ln x)^2 - \int 2 \ln x dx$$

$$\text{Let } J = \int 2 \ln x dx$$

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = 2 \Rightarrow v = 2x$$

$$\therefore J = 2x \ln x - \int 2x \times \frac{1}{x} dx = 2x \ln x - 2x + C$$

$$\therefore I = x (\ln x)^2 - 2x \ln x + 2x + C$$

$$(e) u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x^2 + 1 \Rightarrow v = \frac{x^3}{3} + x$$

$$\therefore \int \left(x^2 + 1 \right) \ln x dx = \ln x \left(\frac{x^3}{3} + x \right) - \int \left(\frac{x^3}{3} + x \right) \times \frac{1}{x} dx$$

$$= \left(\frac{x^3}{3} + x \right) \ln x - \int \left(\frac{x^2}{3} + 1 \right) dx$$

$$= \left(\frac{x^3}{3} + x \right) \ln x - \frac{x^3}{9} - x + C$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise G, Question 3

Question:

Find the following integrals:

(a) $\int x^2 e^{-x} dx$

(b) $\int x^2 \cos x dx$

(c) $\int 12x^2 (3 + 2x)^5 dx$

(d) $\int 2x^2 \sin 2x dx$

(e) $\int x^2 2 \sec^2 x \tan x dx$

Solution:

(a) $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$$

$$\therefore I = \int x^2 e^{-x} dx = -x^2 e^{-x} - \int -e^{-x} \times 2x dx$$

$$= -x^2 e^{-x} + \int 2x e^{-x} dx$$

Let $J = \int 2x e^{-x} dx$

$$u = 2x \Rightarrow \frac{du}{dx} = 2$$

$$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$$

$$\therefore J = -e^{-x} 2x - \int (-e^{-x}) \times 2 dx$$

$$= -2x e^{-x} + \int 2e^{-x} dx$$

$$= -2x e^{-x} - 2e^{-x} + C$$

$$\therefore I = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C'$$

(b) $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$$\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$$

$$\therefore I = \int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$$

$$\text{Let } J = \int 2x \sin x \, dx$$

$$u = 2x \quad \Rightarrow \quad \frac{du}{dx} = 2$$

$$\frac{dv}{dx} = \sin x \quad \Rightarrow \quad v = -\cos x$$

$$\therefore J = -2x \cos x - \int (-\cos x) \times 2 \, dx$$

$$= -2x \cos x + \int 2 \cos x \, dx$$

$$= -2x \cos x + 2 \sin x + C$$

$$\therefore I = x^2 \sin x + 2x \cos x - 2 \sin x + C'$$

$$(c) \, u = 12x^2 \quad \Rightarrow \quad \frac{du}{dx} = 24x$$

$$\frac{dv}{dx} = (3 + 2x)^5 \quad \Rightarrow \quad v = \frac{(3 + 2x)^6}{12}$$

$$\therefore I = \int 12x^2 (3 + 2x)^5 \, dx = 12x^2 \frac{(3 + 2x)^6}{12} - \int 24x \frac{(3 + 2x)^6}{12} \, dx$$

$$= x^2 (3 + 2x)^6 - \int 2x (3 + 2x)^6 \, dx$$

$$\text{Let } J = \int 2x (3 + 2x)^6 \, dx$$

$$u = 2x \quad \Rightarrow \quad \frac{du}{dx} = 2$$

$$v = \frac{(3 + 2x)^7}{14} \quad \Rightarrow \quad \frac{dv}{dx} = (3 + 2x)^6$$

$$\therefore J = 2x \frac{(3 + 2x)^7}{14} - \int \frac{(3 + 2x)^7}{14} \times 2 \, dx$$

$$= x \frac{(3 + 2x)^7}{7} - \int \frac{(3 + 2x)^7}{7} \, dx$$

$$= x \frac{(3 + 2x)^7}{7} - \frac{(3 + 2x)^8}{7 \times 8} + C$$

$$\therefore I = x^2 (3 + 2x)^6 - x \frac{(3 + 2x)^7}{7} + \frac{(3 + 2x)^8}{112} + C'$$

$$(d) \, u = 2x^2 \quad \Rightarrow \quad \frac{du}{dx} = 4x$$

$$\frac{dv}{dx} = \sin 2x \quad \Rightarrow \quad v = -\frac{1}{2} \cos 2x$$

$$\begin{aligned}
 \therefore I &= \int 2x^2 \sin 2x \, dx = -\frac{2x^2}{2} \cos 2x - \int \left(-\frac{1}{2} \cos 2x \right) \times 4x \, dx \\
 &= -x^2 \cos 2x + \int 2x \cos 2x \, dx \\
 \text{Let } J &= \int 2x \cos 2x \, dx \\
 u = x &\Rightarrow \frac{du}{dx} = 1 \\
 \frac{dv}{dx} &= 2 \cos 2x \Rightarrow v = \sin 2x \\
 \therefore J &= x \sin 2x - \int \sin 2x \, dx \\
 &= x \sin 2x + \frac{1}{2} \cos 2x + C \\
 \therefore I &= -x^2 \cos 2x + x \sin 2x + \frac{1}{2} \cos 2x + C'
 \end{aligned}$$

$$\begin{aligned}
 \text{(e) } u = x^2 &\Rightarrow \frac{du}{dx} = 2x \\
 \frac{dv}{dx} &= 2 \sec x \sec x \tan x \Rightarrow v = \sec^2 x \\
 \therefore I &= \int x^2 \times 2 \sec^2 x \tan x \, dx = x^2 \sec^2 x - \int 2x \sec^2 x \, dx \\
 \text{Let } J &= \int 2x \sec^2 x \, dx \\
 u = 2x &\Rightarrow \frac{du}{dx} = 2 \\
 \frac{dv}{dx} &= \sec^2 x \Rightarrow v = \tan x \\
 \therefore J &= 2x \tan x - \int 2 \tan x \, dx \\
 &= 2x \tan x - 2 \ln |\sec x| + C \\
 \therefore I &= x^2 \sec^2 x - 2x \tan x + 2 \ln |\sec x| + C'
 \end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise G, Question 4

Question:

Evaluate the following:

(a) $\int_0^{\ln 2} x e^{2x} dx$

(b) $\int_0^{\frac{\pi}{2}} x \sin x dx$

(c) $\int_0^{\frac{\pi}{2}} x \cos x dx$

(d) $\int_1^2 \frac{\ln x}{x^2} dx$

(e) $\int_0^1 4x(1+x)^3 dx$

(f) $\int_0^{\pi} x \cos \left(\frac{1}{4}x \right) dx$

(g) $\int_0^{\frac{\pi}{3}} \sin x \ln |\sec x| dx$

Solution:

(a) $u = x \Rightarrow \frac{du}{dx} = 1$

$$\frac{dv}{dx} = e^{2x} \Rightarrow v = \frac{1}{2}e^{2x}$$

$$\begin{aligned} \therefore \int_0^{\ln 2} x e^{2x} dx &= \left[\frac{1}{2} e^{2x} \times x \right]_0^{\ln 2} - \int_0^{\ln 2} \frac{1}{2} e^{2x} dx \\ &= \left(\frac{1}{2} e^{2 \ln 2} \ln 2 \right) - \left(0 \right) - \left[\frac{1}{4} e^{2x} \right]_0^{\ln 2} \\ &= \frac{4}{2} \ln 2 - \left[\left(\frac{1}{4} e^{2 \ln 2} \right) - \left(\frac{1}{4} e^0 \right) \right] \\ &= 2 \ln 2 - \frac{4}{4} + \frac{1}{4} \end{aligned}$$

$$= 2 \ln 2 - \frac{3}{4}$$

$$(b) \ u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \sin x \quad \Rightarrow \quad v = -\cos x$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{2}} x \sin x \, dx &= \left[-x \cos x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(-\cos x \right) dx \\ &= \left(-\frac{\pi}{2} \cos \frac{\pi}{2} \right) - \left(0 \right) + \int_0^{\frac{\pi}{2}} \cos x \, dx \\ &= 0 + \left[\sin x \right]_0^{\frac{\pi}{2}} \\ &= \left(\sin \frac{\pi}{2} \right) - \left(\sin 0 \right) \\ &= 1 \end{aligned}$$

$$(c) \ u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \cos x \quad \Rightarrow \quad v = \sin x$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{2}} x \cos x \, dx &= \left[x \sin x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx \\ &= \left(\frac{\pi}{2} \sin \frac{\pi}{2} \right) - \left(0 \right) - \left[-\cos x \right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} + \left(\cos \frac{\pi}{2} \right) - \left(\cos 0 \right) \\ &= \frac{\pi}{2} - 1 \end{aligned}$$

$$(d) \ u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x^{-2} \quad \Rightarrow \quad v = -x^{-1}$$

$$\begin{aligned} \therefore \int_1^2 \frac{\ln x}{x^2} \, dx &= \left[-\frac{\ln x}{x} \right]_1^2 - \int_1^2 \frac{1}{x} \times \left(-x^{-1} \right) dx \\ &= \left(-\frac{\ln 2}{2} \right) - \left(-\frac{\ln 1}{1} \right) + \int_1^2 \frac{1}{x^2} \, dx \\ &= -\frac{1}{2} \ln 2 + \left[-x^{-1} \right]_1^2 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{2} \ln 2 + \left(-\frac{1}{2} \right) - \left(-\frac{1}{1} \right) \\
 &= \frac{1}{2} \left(1 - \ln 2 \right)
 \end{aligned}$$

$$(e) \quad u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = 4(1+x)^3 \quad \Rightarrow \quad v = (1+x)^4$$

$$\begin{aligned}
 \therefore \int_0^1 4x(1+x)^3 dx &= \left[x(1+x)^4 \right]_0^1 - \int_0^1 (1+x)^4 dx \\
 &= \left(1 \times 2^4 \right) - \left(0 \right) - \left[\frac{(1+x)^5}{5} \right]_0^1 \\
 &= 16 - \left[\left(\frac{2^5}{5} \right) - \left(\frac{1}{5} \right) \right] \\
 &= 16 - \frac{31}{5} \\
 &= 16 - 6.2 \\
 &= 9.8
 \end{aligned}$$

$$(f) \quad u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \cos \left(\frac{1}{4}x \right) \quad \Rightarrow \quad v = 4 \sin \left(\frac{1}{4}x \right)$$

$$\begin{aligned}
 \therefore \int_0^{\pi} \cos \left(\frac{1}{4}x \right) dx &= \left[4 \sin \frac{x}{4} \right]_0^{\pi} - \int_0^{\pi} 4 \sin \left(\frac{1}{4}x \right) dx \\
 &= \left(4 \sin \frac{\pi}{4} \right) - \left(0 \right) + \left[16 \cos \frac{1}{4}x \right]_0^{\pi} \\
 &= \frac{4\pi}{\sqrt{2}} + \left(16 \cos \frac{\pi}{4} \right) - \left(16 \cos 0 \right) \\
 &= \frac{4\pi}{\sqrt{2}} + \frac{16}{\sqrt{2}} - 16 \\
 \text{OR} \quad &= \frac{4\pi \sqrt{2}}{2} + \frac{16 \sqrt{2}}{2} - 16 \\
 &= 2\pi \sqrt{2} + 8 \sqrt{2} - 16
 \end{aligned}$$

$$(g) \quad u = \ln |\sec x| \quad \Rightarrow \quad \frac{du}{dx} = \tan x$$

$$\frac{dv}{dx} = \sin x \quad \Rightarrow \quad v = -\cos x$$

$$\therefore \int_0^{\frac{\pi}{3}} \sin x \ln |\sec x| \, dx = \left[-\cos x \ln |\sec x| \right]_0^{\frac{\pi}{3}} + \int_0^{\frac{\pi}{3}} \cos x \tan x \, dx$$

$$= \left(-\cos \frac{\pi}{3} \ln \left| \sec \frac{\pi}{3} \right| \right) - \left(-\cos 0 \ln \left| \sec 0 \right| \right) + \int_0^{\frac{\pi}{3}} \sin x \, dx$$

$$\begin{aligned} &= -\frac{1}{2} \ln 2 + 0 + \left[-\cos x \right]_0^{\frac{\pi}{3}} \\ &= -\frac{1}{2} \ln 2 + \left(-\frac{1}{2} \right) - \left(-1 \right) \\ &= \frac{1}{2} \left(1 - \ln 2 \right) \end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise H, Question 1

Question:

Use the trapezium rule with n strips to estimate the following:

(a) $\int_0^3 \ln(1 + x^2) dx; n = 6$

(b) $\int_0^{\frac{\pi}{3}} \sqrt{1 + \tan x} dx; n = 4$

(c) $\int_0^2 \frac{1}{\sqrt{e^x + 1}} dx; n = 4$

(d) $\int_{-1}^1 \operatorname{cosec}^2(x^2 + 1) dx; n = 4$

(e) $\int_{0.1}^{1.1} \sqrt{\cot x} dx; n = 5$

Solution:

(a)

x	0	0.5	1	1.5	2	2.5	3
$\ln(1 + x^2)$	0	0.223	0.693	1.179	1.609	1.981	2.303

$$I = \int_0^3 \ln(1 + x^2) dx$$

$$\therefore I \approx \frac{1}{2} \times 0.5 \left[0 + 2.303 + 2 \left(0.223 + 0.693 + 1.179 + 1.609 + 1.981 \right) \right]$$

$$= \frac{1}{4} \left(13.673 \right)$$

$$= 3.41825$$

$$= 3.42 \text{ (3 s.f.)}$$

(b)

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{\pi}{3}$
$\sqrt{1 + \tan x}$	1	1.126	1.256	1.414	1.653

$$I = \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan x} dx$$

$$\begin{aligned}
 \therefore I &= \frac{1}{2} \times \frac{\pi}{12} \left[1 + 1.653 + 2 \left(1.126 + 1.256 + 1.414 \right) \right] \\
 &= \frac{\pi}{24} \left(10.245 \right) \\
 &= 1.3410... \\
 &= 1.34 \text{ (3 s.f.)}
 \end{aligned}$$

(c) $\frac{1}{\sqrt{e^x + 1}}$

x	0	0.5	1	1.5	2
	0.707	0.614	0.519	0.427	0.345

$$\begin{aligned}
 I &= \int_0^2 \frac{1}{\sqrt{e^x + 1}} dx \\
 \therefore I &\simeq \frac{1}{2} \times 0.5 \left[0.707 + 0.345 + 2 \left(0.614 + 0.519 + 0.427 \right) \right] \\
 &= \frac{1}{4} \left(4.172 \right) \\
 &= 1.043 \\
 &= 1.04 \text{ (3 s.f.)}
 \end{aligned}$$

(d) $\frac{1}{\operatorname{cosec}^2(x^2 + 1)}$

x	-1	-0.5	0	0.5	1
	1.209	1.110	1.412	1.110	1.209

$$\begin{aligned}
 I &= \int_{-1}^1 \frac{1}{\operatorname{cosec}^2(x^2 + 1)} dx \\
 \therefore I &\simeq \frac{1}{2} \times 0.5 \left[1.209 \times 2 + 2 \left(1.110 + 1.412 + 1.110 \right) \right] \\
 &= \frac{1}{4} \left(9.682 \right) \\
 &= 2.42 \text{ (3 s.f.)}
 \end{aligned}$$

(e) $\sqrt{\cot x}$

x	0.1	0.3	0.5	0.7	0.9	1.1
	3.157	1.798	1.353	1.090	0.891	0.713

$$\begin{aligned}
 I &= \int_{0.1}^{1.1} \sqrt{\cot x} dx \\
 \therefore I &\simeq \frac{1}{2} \times 0.2 \left[3.157 + 0.713 + 2 \left(1.798 + 1.353 + 1.090 + 0.891 \right) \right] \\
 &= \frac{1}{10} \left(14.134 \right)
 \end{aligned}$$

$$= 1.41 \text{ (3 s.f.)}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise H, Question 2

Question:

- (a) Find the exact value of $I = \int_1^4 x \ln x \, dx$.
- (b) Find approximate values for I using the trapezium rule with
- 3 strips
 - 6 strips
- (c) Compare the percentage error for these two approximations.

Solution:

(a) $I = \int_1^4 x \ln x \, dx$

$$u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x \quad \Rightarrow \quad v = \frac{1}{2}x^2$$

$$\therefore I = \left[\frac{1}{2}x^2 \ln x \right]_1^4 - \int_1^4 \frac{1}{2}x^2 \times \frac{1}{x} \, dx$$

$$= 8 \ln 4 - \left[\frac{x^2}{4} \right]_1^4$$

$$= 8 \ln 4 - \left(4 - \frac{1}{4} \right)$$

$$= 8 \ln 4 - \frac{15}{4}$$

(b) (i)

x	1	2	3	4
$x \ln x$	0	1.386	3.296	5.545

$$I \approx \frac{1}{2} \times 1 \left[5.545 + 2 \left(1.386 + 3.296 \right) \right]$$

$$= \frac{1}{2} \left(14.909 \right) = 7.4545 = 7.45 \text{ (3 s.f.)}$$

(ii)

x	1	1.5	2	2.5	3	3.5	4
$x \ln x$	0	0.608	1.386	2.291	3.296	4.385	5.545

$$I \approx \frac{1}{2} \times 0.5 \left[5.545 + 2 \left(0.608 + 1.386 + 2.291 + 3.296 + 4.385 \right) \right]$$

$$= \frac{1}{4} \left[29.477 \right] = 7.36925 = 7.37 \text{ (3 s.f.)}$$

(c) % error using 3 strips: $\frac{[7.4545 - (8 \ln 4 - 3.75)] \times 100}{8 \ln 4 - 3.75} = 1.6 \% \text{ 1 d.p.}$

% error using 6 strips: $\frac{[7.376925 - (8 \ln 4 - 3.75)] \times 100}{8 \ln 4 - 3.75} = 0.4 \% \text{ 1 d.p.}$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise H, Question 3

Question:

(a) Find an approximate value for $I = \int_0^1 e^x \tan x \, dx$ using

(i) 2 strips

(ii) 4 strips

(ii) 8 strips.

(b) Suggest a possible value for I .

Solution:

(a) (i)

x	0	0.5	1
$e^x \tan x$	0	0.901	4.233

$$I \simeq \frac{1}{2} \times 0.5 \left(0 + 4.233 + 2 \times 0.901 \right) = \frac{1}{4} \left(6.035 \right) = 1.509$$

(ii)

x	0	0.25	0.5	0.75	1
$e^x \tan x$	0	0.328	0.901	1.972	4.233

$$I \simeq \frac{1}{2} \times 0.25 \left[4.233 + 2 \left(0.328 + 0.901 + 1.972 \right) \right]$$

$$= \frac{1}{8} \left(10.635 \right) = 1.329$$

(iii)

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$e^x \tan x$	0	0.142	0.328	0.573	0.901	1.348	1.972	2.872	4.233

$$I \simeq \frac{1}{2} \times \frac{1}{8} \left[4.233 + 2 \left(0.142 + 0.328 + 0.573 + 0.901 + 1.348 + 1.972 + 2.872 \right) \right]$$

$$= \frac{1}{16} \left(20.505 \right) = 1.282$$

(b) Halving h reduces differences by about $\frac{1}{3}$:

	1.5098	→	1.329	→	1.282	→	?
Differences:		0.18		0.05		0.01/2	

So an answer in the range 1.25 – 1.27 seems likely.

(Note: Calculator gives 1.265)

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise H, Question 4

Question:

- (a) Find the exact value of $I = \int_0^2 x \sqrt{(2-x)} \, dx$.
- (b) Find an approximate value for I using the trapezium rule with
- 4 and
 - 6 strips.
- (c) Compare the percentage error for these two approximations.

Solution:

(a) $u^2 = 2 - x \Rightarrow 2u \, du = -dx$ and $x = 2 - u^2$

Also $u = 0$ when $x = 2$

and $u = \sqrt{2}$ when $x = 0$.

$$\begin{aligned} \therefore I &= \int_{\sqrt{2}}^0 (2 - u^2) u \times (-2u) \, du \\ &= \int_0^{\sqrt{2}} (2 - u^2) 2u^2 \, du \\ &= \int_0^{\sqrt{2}} (4u^2 - 2u^4) \, du \\ &= \left[\frac{4u^3}{3} - \frac{2u^5}{5} \right]_0^{\sqrt{2}} \\ &= \left(\frac{4 \times 2\sqrt{2}}{3} - \frac{2 \times 4\sqrt{2}}{5} \right) - \left(0 \right) \\ &= \frac{16\sqrt{2}}{15} \end{aligned}$$

(b) (i)

x	0	0.5	1	1.5	2
$x\sqrt{2-x}$	0	0.612	1	1.061	0

$$\begin{aligned} I &\simeq \frac{1}{2} \times 0.5 \left[0 + 2 \left(0.612 + 1 + 1.061 \right) \right] \\ &= \frac{1}{4} \left(5.346 \right) = 1.3365 = 1.34 \text{ (2 d.p.)} \end{aligned}$$

(ii)

$$\begin{array}{ccccccc}
 x & 0 & \frac{1}{3} & \frac{2}{3} & 1 & \frac{4}{3} & \frac{5}{3} & 2 \\
 x\sqrt{2-x} & 0 & 0.430 & 0.770 & 1 & 1.089 & 0.962 & 0
 \end{array}$$

$$\begin{aligned}
 I &\simeq \frac{1}{2} \times \frac{1}{3} \left[0 + 2 \left(0.430 + 0.770 + 1 + 1.089 + 0.962 \right) \right] \\
 &= \frac{1}{6} \left(8.502 \right) = 1.417 = 1.42 \text{ (2 d.p.)}
 \end{aligned}$$

$$\text{(c) (i) \% error with 4 strips} = \frac{\frac{16}{15} \sqrt{2} - 1.3365}{\frac{16}{15} \sqrt{2}} \times 100 = 11.4 \%$$

$$\text{(ii) \% error with 6 strips} = \frac{\frac{16 \sqrt{2}}{15} - 1.417}{\frac{16}{15} \sqrt{2}} \times 100 = 6.1 \%$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise I, Question 1

Question:

The region R is bounded by the curve with equation $y = f(x)$, the x -axis and the lines $x = a$ and $x = b$. In each of the following cases find the exact value of:

(i) the area of R ,

(ii) the volume of the solid of revolution formed by rotating R through 2π radians about the x -axis.

$$(a) f(x) = \frac{2}{1+x}; a = 0, b = 1$$

$$(b) f(x) = \sec x; a = 0, b = \frac{\pi}{3}$$

$$(c) f(x) = \ln x; a = 1, b = 2$$

$$(d) f(x) = \sec x \tan x; a = 0, b = \frac{\pi}{4}$$

$$(e) f(x) = x\sqrt{4-x^2}; a = 0, b = 2$$

Solution:

$$(a) (i) \text{ Area} = \int_0^1 \frac{2}{1+x} dx = \left[2 \ln |1+x| \right]_0^1 = \left(2 \ln 2 \right) - \left(2 \ln 1 \right)$$

$$\therefore \text{ Area} = 2 \ln 2$$

$$(ii) \text{ Volume} = \pi \int_0^1 \left(\frac{2}{1+x} \right)^2 dx$$

$$= \pi \int_0^1 \frac{4}{(1+x)^2} dx$$

$$= \pi \left[4 \frac{(1+x)^{-1}}{-1} \right]_0^1$$

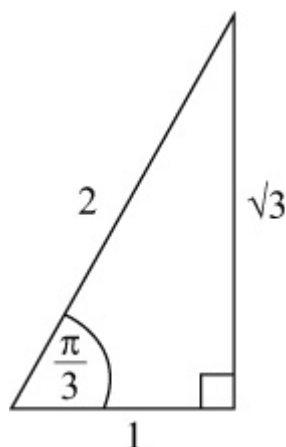
$$= \pi \left[-\frac{4}{1+x} \right]_0^1$$

$$= \pi \left[\left(-\frac{4}{2} \right) - \left(-\frac{4}{1} \right) \right]$$

$$= 2\pi$$

(b) (i) Area = $\int_0^{\frac{\pi}{3}} \sec x \, dx$

$$= \left[\ln | \sec x + \tan x | \right]_0^{\frac{\pi}{3}}$$



$$= [\ln (2 + \sqrt{3})] - [\ln (1)]$$

$$\therefore \text{Area} = \ln (2 + \sqrt{3})$$

(ii) Volume = $\pi \int_0^{\frac{\pi}{3}} \sec^2 x \, dx$

$$= \pi \left[\tan x \right]_0^{\frac{\pi}{3}}$$

$$= \pi [(\sqrt{3}) - (0)]$$

$$= \sqrt{3}\pi$$

(c) (i) Area = $\int_1^2 \ln x \, dx$

$$u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = 1 \quad \Rightarrow \quad v = x$$

$$\therefore \text{Area} = \left[x \ln x \right]_1^2 - \int_1^2 x \times \frac{1}{x} dx$$

$$= (2 \ln 2) - (0) - [x]_1^2$$

$$= 2 \ln 2 - 1$$

(ii) Volume = $\pi \int_1^2 (\ln x)^2 dx$

$$u = (\ln x)^2 \Rightarrow \frac{du}{dx} = 2 \ln x \times \frac{1}{x}$$

$$\frac{dv}{dx} = 1 \Rightarrow v = x$$

$$\therefore V = \pi \left\{ \left[x (\ln x)^2 \right]_1^2 - 2 \int_1^2 x \times \ln x \times \frac{1}{x} dx \right\}$$

$$= \pi \{ [2 (\ln 2)^2] - (0) \} - 2\pi \int_1^2 \ln x dx$$

But $\int_1^2 \ln x dx = 2 \ln 2 - 1$ from (i)

$$\therefore V = 2\pi (\ln 2)^2 - 2\pi (2 \ln 2 - 1)$$

(d) (i) Area = $\int_0^{\frac{\pi}{4}} \sec x \tan x dx$

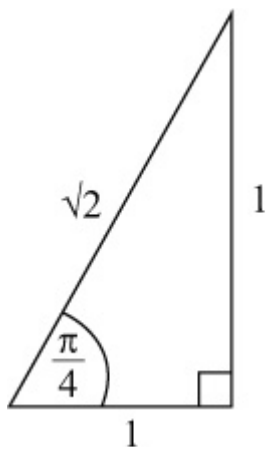
$$= \left[\sec x \right]_0^{\frac{\pi}{4}}$$

$$= (\sqrt{2}) - (1)$$

$$\therefore \text{Area} = \sqrt{2} - 1$$

(ii) Volume = $\pi \int_0^{\frac{\pi}{4}} \sec^2 x \tan^2 x dx$

$$= \pi \left[\frac{\tan^3 x}{3} \right]_0^{\frac{\pi}{4}}$$



$$= \pi \left[\left(\frac{1^3}{3} \right) - (0) \right]$$

$$= \frac{\pi}{3}$$

(e) (i) Area = $\int_0^2 x \sqrt{4 - x^2} dx$

$$\text{Let } y = (4 - x^2)^{\frac{3}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2} (4 - x^2)^{\frac{1}{2}} \times (-2x) = -3x (4 - x^2)^{\frac{1}{2}}$$

$$\therefore \text{Area} = \left[-\frac{1}{3} (4 - x^2)^{\frac{3}{2}} \right]_0^2 = \left(0 \right) - \left(-\frac{1}{3} \times 2^3 \right) =$$

$$\frac{8}{3}$$

$$\text{(ii) Volume} = \pi \int_0^2 x^2 (4 - x^2) dx$$

$$= \pi \int_0^2 (4x^2 - x^4) dx$$

$$= \pi \left[\frac{4}{3}x^3 - \frac{x^5}{5} \right]_0^2$$

$$= \pi \left[\left(\frac{32}{3} - \frac{32}{5} \right) - \left(0 \right) \right]$$

$$= \frac{64\pi}{15}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise I, Question 2

Question:

Find the exact area between the curve $y = f(x)$, the x -axis and the lines $x = a$ and $x = b$ where:

$$(a) f(x) = \frac{4x+3}{(x+2)(2x-1)}; a=1, b=2$$

$$(b) f(x) = \frac{x}{(x+1)^2}; a=0, b=2$$

$$(c) f(x) = x \sin x; a=0, b=\frac{\pi}{2}$$

$$(d) f(x) = \cos x \sqrt{2 \sin x + 1}; a=0, b=\frac{\pi}{6}$$

$$(e) f(x) = xe^{-x}; a=0, b=\ln 2$$

Solution:

$$(a) \frac{4x+3}{(x+2)(2x-1)} \equiv \frac{A}{x+2} + \frac{B}{2x-1}$$

$$\Rightarrow 4x+3 \equiv A(2x-1) + B(x+2)$$

$$x = \frac{1}{2} \Rightarrow 5 = \frac{5}{2}B \Rightarrow B = 2$$

$$x = -2 \Rightarrow -5 = -5A \Rightarrow A = 1$$

$$\therefore \text{area} = \int_1^2 \frac{4x+3}{(x+2)(2x-1)} dx$$

$$= \int_1^2 \left(\frac{1}{x+2} + \frac{2}{2x-1} \right) dx$$

$$= [\ln|x+2| + \ln|2x-1|]_1^2$$

$$= (\ln 4 + \ln 3) - (\ln 3 + \ln 1)$$

$$= \ln 4 \quad \text{or} \quad 2 \ln 2$$

$$(b) \frac{x}{(x+1)^2} \equiv \frac{A}{(x+1)^2} + \frac{B}{x+1}$$

$$\Rightarrow x \equiv A + B(x + 1)$$

Compare coefficient of x : $1 = B \Rightarrow B = 1$

Compare constants: $0 = A + B \Rightarrow A = -1$

$$\begin{aligned} \therefore \text{area} &= \int_0^2 \frac{x}{(x+1)^2} dx \\ &= \int_0^2 \left(\frac{1}{x+1} - \frac{1}{(x+1)^2} \right) dx \\ &= \left[\ln|x+1| + \frac{1}{x+1} \right]_0^2 \\ &= \left(\ln 3 + \frac{1}{3} \right) - \left(\ln 1 + 1 \right) \\ &= \ln 3 - \frac{2}{3} \end{aligned}$$

(c) Area = $\int_0^{\frac{\pi}{2}} x \sin x dx$

$$u = x \Rightarrow \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$$

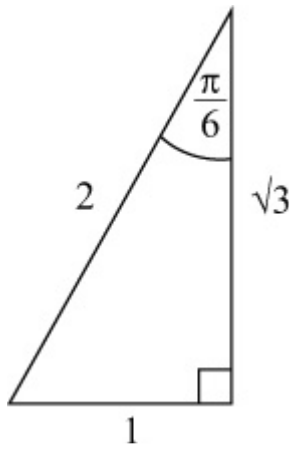
$$\begin{aligned} \therefore \text{area} &= \left[-x \cos x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x) dx \\ &= \left(-\frac{\pi}{2} \cos \frac{\pi}{2} \right) - \left(0 \right) + \int_0^{\frac{\pi}{2}} \cos x dx \\ &= 0 + \left[\sin x \right]_0^{\frac{\pi}{2}} \\ &= \left(\sin \frac{\pi}{2} - 0 \right) \\ &= 1 \end{aligned}$$

(d) Area = $\int_0^{\frac{\pi}{6}} \cos x \sqrt{2 \sin x + 1} dx$

Let $y = (2 \sin x + 1)^{\frac{3}{2}}$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2} (2 \sin x + 1)^{\frac{1}{2}} \times 2 \cos x = 3 \cos x (2 \sin x + 1)^{\frac{1}{2}}$$

$$\therefore \text{area} = \left[\frac{1}{3} (2 \sin x + 1)^{\frac{3}{2}} \right]_0^{\frac{\pi}{6}}$$



$$\begin{aligned}
 &= \left(\frac{1}{3} 2^{\frac{3}{2}} \right) - \left(\frac{1}{3} 1^{\frac{3}{2}} \right) \\
 &= \frac{2\sqrt{2}}{3} - \frac{1}{3} \\
 &= \frac{2\sqrt{2} - 1}{3}
 \end{aligned}$$

(e) Area = $\int_0^{\ln 2} x e^{-x} dx$

$$u = x \Rightarrow \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$$

$$\begin{aligned}
 \therefore \text{area} &= \left[-x e^{-x} \right]_0^{\ln 2} - \int_0^{\ln 2} (-e^{-x}) dx \\
 &= (-\ln 2 \times e^{-\ln 2}) - (0) + \int_0^{\ln 2} e^{-x} dx \\
 &= -\ln 2 \times \frac{1}{2} + \left[-e^{-x} \right]_0^{\ln 2} \\
 &= -\frac{1}{2} \ln 2 + \left(-e^{-\ln 2} \right) - \left(-e^{-0} \right) \\
 &= -\frac{1}{2} \ln 2 - \frac{1}{2} + 1 \\
 &= \frac{1}{2} \left(1 - \ln 2 \right)
 \end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise I, Question 3

Question:

The region R is bounded by the curve C , the x -axis and the lines $x = -8$ and $x = +8$.
The parametric equations for C are $x = t^3$ and $y = t^2$. Find:

(a) the area of R ,

(b) the volume of the solid of revolution formed when R is rotated through 2π radians about the x -axis.

Solution:

$$(a) \text{ Area} = \int_{x=-8}^{x=8} y \, dx$$

$$x = t^3 \Rightarrow dx = 3t^2 \, dt$$

Also $t = 2$ when $x = 8$

and $t = -2$ when $x = -8$.

$$\therefore \text{ area} = \int_{-2}^2 t^2 \times 3t^2 \, dt$$

$$= \int_{-2}^2 3t^4 \, dt$$

$$= \left[\frac{3t^5}{5} \right]_{-2}^2$$

$$= \left(\frac{96}{5} \right) - \left(-\frac{96}{5} \right)$$

$$= \frac{192}{5}$$

$$(b) V = \pi \int_{x=-8}^{x=8} y^2 \, dx$$

$$= \pi \int_{-2}^2 t^4 \times 3t^2 \, dt$$

$$= \pi \int_{-2}^2 3t^6 \, dt$$

$$= \pi \left[\frac{3t^7}{7} \right]_{-2}^2$$

$$= \pi \left[\left(\frac{3 \times 128}{7} \right) - \left(-\frac{3 \times 128}{7} \right) \right]$$

$$= \frac{768}{7} \pi$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise I, Question 4

Question:

The curve C has parametric equations $x = \sin t$, $y = \sin 2t$, $0 \leq t \leq \frac{\pi}{2}$.

(a) Find the area of the region bounded by C and the x -axis.

If this region is revolved through 2π radians about the x -axis,

(b) find the volume of the solid formed.

Solution:

$$(a) \text{ Area} = \int_{t=0}^{t=\frac{\pi}{2}} y \, dx$$

$$x = \sin t \Rightarrow dx = \cos t \, dt$$

$$\therefore \text{ area} = \int_0^{\frac{\pi}{2}} \sin 2t \times \cos t \, dt$$

$$= \int_0^{\frac{\pi}{2}} 2 \cos^2 t \sin t \, dt$$

$$= \left[-\frac{2}{3} \cos^3 t \right]_0^{\frac{\pi}{2}}$$

$$= \left(0 \right) - \left(-\frac{2}{3} \right)$$

$$= \frac{2}{3}$$

$$(b) V = \pi \int_{t=0}^{t=\frac{\pi}{2}} y^2 \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \sin^2 2t \cos t \, dt$$

$$= \pi \int_0^{\frac{\pi}{2}} 4 \cos^3 t \sin t \times \sin t \, dt$$

$$u = \sin t \Rightarrow \frac{du}{dt} = \cos t$$

$$\frac{dv}{dt} = 4 \cos^3 t \sin t \Rightarrow v = -\cos^4 t$$

$$\begin{aligned}
\therefore V &= \pi \left\{ \left[-\sin t \cos^4 t \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos^5 t \, dt \right\} \\
&= \pi \int_0^{\frac{\pi}{2}} \cos^5 t \, dt \\
&= \pi \int_0^{\frac{\pi}{2}} (\cos^2 t)^2 \times \cos t \, dt \quad \text{Let } y = \sin t \Rightarrow dy = \cos t \, dt \\
&= \pi \int_0^1 (1 - y^2)^2 \, dy \\
&= \pi \int_0^1 (1 - 2y^2 + y^4) \, dy \\
&= \pi \left[y - \frac{2}{3}y^3 + \frac{y^5}{5} \right]_0^1 \\
&= \pi \left(1 - \frac{2}{3} + \frac{1}{5} \right) - \left(0 \right) \\
&= \frac{8\pi}{15}
\end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise J, Question 1

Question:

Find general solutions of the following differential equations. Leave your answer in the form $y = f(x)$.

$$(a) \frac{dy}{dx} = (1 + y)(1 - 2x)$$

$$(b) \frac{dy}{dx} = y \tan x$$

$$(c) \cos^2 x \frac{dy}{dx} = y^2 \sin^2 x$$

$$(d) \frac{dy}{dx} = 2e^{x-y}$$

$$(e) x^2 \frac{dy}{dx} = y + xy$$

Solution:

$$\begin{aligned} (a) \frac{dy}{dx} &= (1 + y)(1 - 2x) \\ \Rightarrow \int \frac{1}{1+y} dy &= \int (1 - 2x) dx \\ \Rightarrow \ln |1 + y| &= x - x^2 + C \\ \Rightarrow 1 + y &= e^{(x - x^2 + C)} \\ \Rightarrow 1 + y &= A e^{x - x^2}, \quad (A = e^C) \\ \Rightarrow y &= A e^{x - x^2} - 1 \end{aligned}$$

$$\begin{aligned} (b) \frac{dy}{dx} &= y \tan x \\ \Rightarrow \int \frac{1}{y} dy &= \int \tan x dx \\ \Rightarrow \ln |y| &= \ln |\sec x| + C \end{aligned}$$

$$\Rightarrow \ln |y| = \ln |k \sec x|, \quad (C = \ln k)$$

$$\Rightarrow y = k \sec x$$

$$(c) \cos^2 x \frac{dy}{dx} = y^2 \sin^2 x$$

$$\Rightarrow \int \frac{1}{y^2} dy = \int \frac{\sin^2 x}{\cos^2 x} dx$$

$$\Rightarrow \int \frac{1}{y^2} dy = \int \tan^2 x dx = \int \left(\sec^2 x - 1 \right) dx$$

$$\Rightarrow -\frac{1}{y} = \tan x - x + C$$

$$\Rightarrow y = \frac{-1}{\tan x - x + C}$$

$$(d) \frac{dy}{dx} = 2e^{x-y} = 2e^x e^{-y}$$

$$\Rightarrow \int \frac{1}{e^{-y}} dy = \int 2e^x dx$$

$$\text{i.e. } \Rightarrow \int e^y dy = \int 2e^x dx$$

$$\Rightarrow e^y = 2e^x + C$$

$$\Rightarrow y = \ln (2e^x + C)$$

$$(e) x^2 \frac{dy}{dx} = y + xy = y \left(1 + x \right)$$

$$\Rightarrow \int \frac{1}{y} dy = \int x^{-2} + \frac{1}{x} dx$$

$$\Rightarrow \ln |y| = -x^{-1} + \ln |x| + C$$

$$\Rightarrow \ln |y| - \ln |x| = C - \frac{1}{x}$$

$$\Rightarrow \ln \left| \frac{y}{x} \right| = C - \frac{1}{x}$$

$$\Rightarrow \frac{y}{x} = e^{C - \frac{1}{x}}$$

$$\Rightarrow \frac{y}{x} = Ae^{-\frac{1}{x}}, \quad \left(e^C = A \right)$$

$$\Rightarrow y = Axe^{-\frac{1}{x}}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise J, Question 2

Question:

Find a general solution of the following differential equations. (You do not need to write the answers in the form $y = f(x)$.)

(a) $\frac{dy}{dx} = \tan y \tan x$

(b) $\sin y \cos x \frac{dy}{dx} = \frac{x \cos y}{\cos x}$

(c) $\left(1 + x^2\right) \frac{dy}{dx} = x \left(1 - y^2\right)$

(d) $\cos y \sin 2x \frac{dy}{dx} = \cot x \operatorname{cosec} y$

(e) $e^{x+y} \frac{dy}{dx} = x \left(2 + e^y\right)$

Solution:

(a) $\frac{dy}{dx} = \tan y \tan x$

$$\Rightarrow \int \frac{1}{\tan y} dy = \int \tan x dx$$

$$\Rightarrow \int \cot y dy = \int \tan x dx$$

$$\Rightarrow \ln |\sin y| = \ln |\sec x| + C = \ln |k \sec x| \quad (\ln k = C)$$

$$\Rightarrow \sin y = k \sec x$$

(b) $\sin y \cos x \frac{dy}{dx} = \frac{x \cos y}{\cos x}$

$$\Rightarrow \int \frac{\sin y}{\cos y} dy = \int \frac{x}{\cos^2 x} dx$$

$$\Rightarrow \int \tan y dy = \int x \sec^2 x dx$$

$$\Rightarrow \ln |\sec y| = \int x \sec^2 x dx$$

$$u = x \Rightarrow \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$$

$$\Rightarrow \ln |\sec y| = x \tan x - \int \tan x dx$$

$$\Rightarrow \ln |\sec y| = x \tan x - \ln |\sec x| + C$$

$$(c) \left(1 + x^2 \right) \frac{dy}{dx} = x \left(1 - y^2 \right)$$

$$\Rightarrow \int \frac{1}{1 - y^2} dy = \int \frac{x}{1 + x^2} dx$$

$$\frac{1}{1 - y^2} \equiv \frac{A}{1 - y} + \frac{B}{1 + y}$$

$$\Rightarrow 1 \equiv A(1 + y) + B(1 - y)$$

$$y = 1 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$y = -1 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$$

$$\therefore \int \left(\frac{\left(\frac{1}{2} \right)}{1 - y} + \frac{\left(\frac{1}{2} \right)}{1 + y} \right) dy = \int \frac{x}{1 + x^2} dx$$

$$\Rightarrow \frac{1}{2} \ln |1 + y| - \frac{1}{2} \ln |1 - y| = \frac{1}{2} \ln |1 + x^2| + C$$

$$(\text{using } \int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C)$$

$$\Rightarrow \ln \left| \frac{1 + y}{1 - y} \right| = \ln |1 + x^2| + 2C$$

$$\Rightarrow \left| \frac{1 + y}{1 - y} \right| = k \left(1 + x^2 \right) \quad \left(\ln k = 2C \right)$$

$$(d) \cos y \sin 2x \frac{dy}{dx} = \cot x \operatorname{cosec} y$$

$$\Rightarrow \int \frac{\cos y}{\operatorname{cosec} y} dy = \int \frac{\cot x}{\sin 2x} dx$$

$$\Rightarrow \int \sin y \cos y dy = \int \frac{\cos x}{\sin x \cdot 2 \sin x \cos x} dx$$

$$\Rightarrow \int \frac{1}{2} \sin 2y \, dy = \int \frac{1}{2} \operatorname{cosec}^2 x \, dx$$

$$\Rightarrow -\frac{1}{4} \cos 2y = -\frac{1}{2} \cot x + C$$

$$\text{or } \cos 2y = 2 \cot x + k$$

$$(e) \, e^{x+y} \frac{dy}{dx} = x \left(2 + e^y \right)$$

$$\Rightarrow e^x e^y \frac{dy}{dx} = x \left(2 + e^y \right)$$

$$\Rightarrow \int \frac{e^y}{2 + e^y} \, dy = \int x e^{-x} \, dx$$

$$u = x \Rightarrow \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$$

$$\therefore \ln |2 + e^y| = -xe^{-x} + \int e^{-x} \, dx$$

$$\Rightarrow \ln |2 + e^y| = -xe^{-x} - e^{-x} + C$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise J, Question 3

Question:

Find general solutions of the following differential equations:

(a) $\frac{dy}{dx} = ye^x$

(b) $\frac{dy}{dx} = xe^y$

(c) $\frac{dy}{dx} = y \cos x$

(d) $\frac{dy}{dx} = x \cos y$

(e) $\frac{dy}{dx} = \left(1 + \cos 2x \right) \cos y$

(f) $\frac{dy}{dx} = \left(1 + \cos 2y \right) \cos x$

Solution:

(a) $\frac{dy}{dx} = ye^x$

$$\Rightarrow \int \frac{1}{y} dy = \int e^x dx$$

$$\Rightarrow \ln |y| = e^x + C$$

(b) $\frac{dy}{dx} = xe^y$

$$\Rightarrow \int \frac{1}{e^y} dy = \int x dx$$

$$\Rightarrow \int e^{-y} dy = \int x dx$$

$$\Rightarrow -e^{-y} = \frac{1}{2}x^2 + C$$

$$(c) \frac{dy}{dx} = y \cos x$$

$$\Rightarrow \int \frac{1}{y} dy = \int \cos x dx$$

$$\Rightarrow \ln |y| = \sin x + C$$

$$\text{or } y = Ae^{\sin x}$$

$$(d) \frac{dy}{dx} = x \cos y$$

$$\Rightarrow \int \frac{1}{\cos y} dy = \int x dx$$

$$\Rightarrow \int \sec y dy = \int x dx$$

$$\Rightarrow \ln |\sec y + \tan y| = \frac{x^2}{2} + C$$

$$(e) \frac{dy}{dx} = \left(1 + \cos 2x \right) \cos y$$

$$\Rightarrow \int \frac{1}{\cos y} dy = \int \left(1 + \cos 2x \right) dx$$

$$\Rightarrow \int \sec y dy = \int (1 + \cos 2x) dx$$

$$\Rightarrow \ln |\sec y + \tan y| = x + \frac{1}{2} \sin 2x + C$$

$$(f) \frac{dy}{dx} = \left(1 + \cos 2y \right) \cos x$$

$$\Rightarrow \int \frac{1}{1 + \cos 2y} dy = \int \cos x dx$$

$$\Rightarrow \int \frac{1}{2 \cos^2 y} dy = \int \cos x dx$$

$$\Rightarrow \int \frac{1}{2} \sec^2 y dy = \int \cos x dx$$

$$\Rightarrow \frac{1}{2} \tan y = \sin x + C$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise J, Question 4

Question:

Find particular solutions of the following differential equations using the given boundary conditions.

(a) $\frac{dy}{dx} = \sin x \cos^2 x$; $y = 0$, $x = \frac{\pi}{3}$

(b) $\frac{dy}{dx} = \sec^2 x \sec^2 y$; $y = 0$, $x = \frac{\pi}{4}$

(c) $\frac{dy}{dx} = 2 \cos^2 y \cos^2 x$; $y = \frac{\pi}{4}$, $x = 0$

(d) $\left(1 - x^2\right) \frac{dy}{dx} = xy + y$; $x = 0.5$, $y = 6$

(e) $2 \left(1 + x\right) \frac{dy}{dx} = 1 - y^2$; $x = 5$, $y = \frac{1}{2}$

Solution:

(a) $\frac{dy}{dx} = \sin x \cos^2 x$

$$\Rightarrow \int dy = \int \sin x \cos^2 x dx$$

$$\Rightarrow y = -\frac{\cos^3 x}{3} + C$$

$$y = 0, x = \frac{\pi}{3} \Rightarrow 0 = -\frac{\left(\frac{1}{8}\right)}{3} + C \Rightarrow C = \frac{1}{24}$$

$$\therefore y = \frac{1}{24} - \frac{1}{3} \cos^3 x$$

(b) $\frac{dy}{dx} = \sec^2 x \sec^2 y$

$$\Rightarrow \int \frac{1}{\sec^2 y} dy = \int \sec^2 x dx$$

$$\Rightarrow \int \cos^2 y dy = \int \sec^2 x dx$$

$$\Rightarrow \int \left(\frac{1}{2} + \frac{1}{2} \cos 2y \right) dy = \int \sec^2 x dx$$

$$\Rightarrow \frac{1}{2}y + \frac{1}{4} \sin 2y = \tan x + C$$

$$\text{or } \sin 2y + 2y = 4 \tan x + k$$

$$y = 0, x = \frac{\pi}{4} \Rightarrow 0 = 4 + k \Rightarrow k = -4$$

$$\therefore \sin 2y + 2y = 4 \tan x - 4$$

$$(c) \frac{dy}{dx} = 2 \cos^2 y \cos^2 x$$

$$\Rightarrow \int \frac{1}{\cos^2 y} dy = \int 2 \cos^2 x dx$$

$$\Rightarrow \int \sec^2 y dy = \int (1 + \cos 2x) dx$$

$$\Rightarrow \tan y = x + \frac{1}{2} \sin 2x + C$$

$$x = 0, y = \frac{\pi}{4} \Rightarrow 1 = 0 + C$$

$$\therefore \tan y = x + \frac{1}{2} \sin 2x + 1$$

$$(d) \left(1 - x^2 \right) \frac{dy}{dx} = xy + y$$

$$\Rightarrow \left(1 - x^2 \right) \frac{dy}{dx} = \left(x + 1 \right) y$$

$$\Rightarrow \int \frac{1}{y} dy = \int \frac{1+x}{1-x^2} dx$$

$$\Rightarrow \int \frac{1}{y} dy = \int \frac{1+x}{(1-x)(1+x)} dx$$

$$\Rightarrow \int \frac{1}{y} dy = \int \frac{1}{1-x} dx$$

$$\Rightarrow \ln |y| = -\ln |1-x| + C$$

$$x = 0.5, y = 6 \Rightarrow \ln 6 = -\ln \frac{1}{2} + C \Rightarrow C = \ln 3$$

$$\therefore \ln |y| = \ln 3 - \ln |1-x|$$

$$\text{or } y = \frac{3}{1-x}$$

$$(e) 2 \left(1+x \right) \frac{dy}{dx} = 1-y^2$$

$$\Rightarrow \int \frac{2}{1-y^2} dy = \int \frac{1}{1+x} dx$$

$$\frac{2}{1-y^2} \equiv \frac{A}{1-y} + \frac{B}{1+y}$$

$$\Rightarrow 2 \equiv A(1+y) + B(1-y)$$

$$y = 1 \Rightarrow 2 = 2A \Rightarrow A = 1$$

$$y = -1 \Rightarrow 2 = 2B \Rightarrow B = 1$$

$$\therefore \int \left(\frac{1}{1+y} + \frac{1}{1-y} \right) dy = \int \frac{1}{1+x} dx$$

$$\Rightarrow \ln |1+y| - \ln |1-y| = \ln |1+x| + C$$

$$\Rightarrow \ln \left| \frac{1+y}{1-y} \right| = \ln |k(1+x)| \quad \left(C = \ln k \right)$$

$$\Rightarrow \frac{1+y}{1-y} = k \left(1+x \right)$$

$$x = 5, y = \frac{1}{2} \Rightarrow \frac{\frac{3}{2}}{\frac{1}{2}} = 6k \Rightarrow k = \frac{1}{2}$$

$$\therefore \frac{1+y}{1-y} = \frac{1+x}{2}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise K, Question 1

Question:

The size of a certain population at time t is given by P . The rate of increase of P is given by $\frac{dP}{dt} = 2P$. Given that at time $t = 0$, the population was 3, find the population at time $t = 2$.

Solution:

$$\frac{dP}{dt} = 2P$$

$$\Rightarrow \int \frac{1}{P} dP = \int 2 dt$$

$$\Rightarrow \ln |P| = 2t + C$$

$$\Rightarrow P = Ae^{2t}$$

$$t = 0, P = 3 \Rightarrow 3 = Ae^0 \Rightarrow A = 3$$

$$\therefore P = 3e^{2t}$$

$$\text{When } t = 2, P = 3e^4 = 164$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise K, Question 2

Question:

The number of particles at time t of a certain radioactive substance is N . The substance is decaying in such a way that $\frac{dN}{dt} = -\frac{N}{3}$.

Given that at time $t = 0$ the number of particles is N_0 , find the time when the number of particles remaining is $\frac{1}{2}N_0$.

Solution:

$$\frac{dN}{dt} = -\frac{N}{3}$$

$$\Rightarrow \int \frac{1}{N} dN = \int -\frac{1}{3} dt$$

$$\Rightarrow \ln |N| = -\frac{1}{3}t + C$$

$$\Rightarrow N = Ae^{-\frac{1}{3}t}$$

$$t = 0, N = N_0 \Rightarrow N_0 = Ae^0 \Rightarrow A = N_0$$

$$\therefore N = N_0 e^{-\frac{1}{3}t}$$

$$N = \frac{1}{2} N_0 \Rightarrow \frac{1}{2} = e^{-\frac{1}{3}t}$$

$$\Rightarrow -\ln 2 = -\frac{1}{3}t$$

$$\Rightarrow t = 3 \ln 2 \text{ or } 2.08$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise K, Question 3

Question:

The mass M at time t of the leaves of a certain plant varies according to the differential equation $\frac{dM}{dt} = M - M^2$.

(a) Given that at time $t = 0$, $M = 0.5$, find an expression for M in terms of t .

(b) Find a value for M when $t = \ln 2$.

(c) Explain what happens to the value of M as t increases.

Solution:

$$\frac{dM}{dt} = M - M^2$$

$$\Rightarrow \int \frac{1}{M(1-M)} dM = \int 1 dt \text{ but } \frac{1}{M(1-M)} \equiv \frac{A}{M} + \frac{B}{1-M}$$

$$\therefore 1 \equiv A(1-M) + BM$$

$$M = 0 : 1 = 1A, A = 1$$

$$M = 1 : 1 = 1B, B = 1$$

$$\Rightarrow \int \left(\frac{1}{M} + \frac{1}{1-M} \right) dM = \int 1 dt$$

$$\Rightarrow \ln |M| - \ln |1-M| = t + C$$

$$\Rightarrow \ln \left| \frac{M}{1-M} \right| = t + C$$

$$\Rightarrow \frac{M}{1-M} = Ae^t$$

$$(a) t = 0, M = 0.5 \Rightarrow \frac{0.5}{0.5} = Ae^0 \Rightarrow A = 1$$

$$\therefore M = e^t - e^t M \Rightarrow M = \frac{e^t}{1 + e^t}$$

$$(b) \ t = \ln 2 \quad \Rightarrow \quad M = \frac{e^{\ln 2}}{1 + e^{\ln 2}} = \frac{2}{1 + 2} = \frac{2}{3}$$

$$(c) \ t \rightarrow \infty \quad \Rightarrow \quad M = \frac{1}{e^{-t} + 1} \rightarrow \frac{1}{1} = 1$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise K, Question 4

Question:

The volume of liquid $V\text{cm}^3$ at time t seconds satisfies

$$-15 \frac{dV}{dt} = 2V - 450.$$

Given that initially the volume is 300cm^3 , find to the nearest cm^3 the volume after 15 seconds.

Solution:

$$-15 \frac{dV}{dt} = 2V - 450$$

$$\Rightarrow \int \frac{1}{2V - 450} dV = \int -\frac{1}{15} dt$$

$$\Rightarrow \frac{1}{2} \ln |2V - 450| = -\frac{1}{15}t + C$$

$$\Rightarrow 2V - 450 = Ae^{-\frac{2}{15}t}$$

$$t = 0, V = 300 \Rightarrow 150 = Ae^0 \Rightarrow A = 150$$

$$\therefore 2V = 150e^{-\frac{2}{15}t} + 450$$

$$t = 15 \Rightarrow 2V = 150e^{-2} + 450$$

$$\Rightarrow V = \frac{150}{2} \left(e^{-2} + 3 \right)$$

$$\Rightarrow V = 75 (3 + e^{-2}) = 235$$

Solutionbank

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Integration

Exercise K, Question 5

Question:

The thickness of ice x mm on a pond is increasing and $\frac{dx}{dt} = \frac{1}{20x^2}$, where t is measured in hours. Find how long it takes the thickness of ice to increase from 1 mm to 2 mm.

Solution:

$$\frac{dx}{dt} = \frac{1}{20x^2}$$

$$\Rightarrow \int x^2 dx = \int \frac{1}{20} dt$$

$$\Rightarrow \frac{1}{3}x^3 = \frac{t}{20} + C$$

$$t = 0, x = 1 \Rightarrow \frac{1}{3} = C$$

$$\therefore \frac{20(x^3 - 1)}{3} = t$$

$$x = 2 \Rightarrow t = \frac{20}{3} (8 - 1)$$

$$\Rightarrow t = \frac{140}{3} \text{ or } 46 \frac{2}{3}$$

Solutionbank

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Integration

Exercise K, Question 6

Question:

The depth h metres of fluid in a tank at time t minutes satisfies $\frac{dh}{dt} = -k\sqrt{h}$, where k is a positive constant. Find, in terms of k , how long it takes the depth to decrease from 9 m to 4 m.

Solution:

$$\frac{dh}{dt} = -k\sqrt{h}$$

$$\Rightarrow \int \frac{1}{h^{\frac{1}{2}}} dh = \int -k dt$$

$$\Rightarrow \int h^{-\frac{1}{2}} dh = \int -k dt$$

$$\Rightarrow 2h^{\frac{1}{2}} = -kt + C$$

$$t = 0, h = 9 \Rightarrow 2 \times 3 = 0 + C \Rightarrow C = 6$$

$$\therefore 2h^{\frac{1}{2}} - 6 = -kt$$

$$\text{or } t = \frac{6 - 2\sqrt{h}}{k}$$

$$h = 4 \Rightarrow t = \frac{6 - 2 \times 2}{k} = \frac{2}{k}$$

Solutionbank

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Integration

Exercise K, Question 7

Question:

The rate of increase of the radius r kilometres of an oil slick is given by $\frac{dr}{dt} =$

$\frac{k}{r^2}$, where k is a positive constant. When the slick was first observed the radius was 3 km. Two days later it was 5 km. Find, to the nearest day when the radius will be 6.

Solution:

$$\frac{dr}{dt} = \frac{k}{r^2}$$

$$\Rightarrow \int r^2 dr = \int k dt$$

$$\Rightarrow \frac{1}{3}r^3 = kt + C$$

$$t = 0, r = 3 \Rightarrow \frac{27}{3} = C \Rightarrow C = 9$$

$$\therefore kt = \frac{1}{3}r^3 - 9$$

$$t = 2, r = 5 \Rightarrow 2k = \frac{125}{3} - 9 \Rightarrow k = 16\frac{1}{3}$$

$$\therefore \frac{49}{3}t = \frac{1}{3}r^3 - 9$$

$$\text{or } t = \frac{r^3 - 27}{49}$$

$$r = 6 \Rightarrow t = \frac{6^3 - 27}{49} = 3.85... = 4 \text{ days}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 1

Question:

It is given that $y = x^{\frac{3}{2}} + \frac{48}{x}$, $x > 0$.

(a) Find the value of x and the value of y when $\frac{dy}{dx} = 0$.

(b) Show that the value of y which you found is a minimum.

The finite region R is bounded by the curve with equation $y = x^{\frac{3}{2}} + \frac{48}{x}$, the lines $x = 1$, $x = 4$ and the x -axis.

(c) Find, by integration, the area of R giving your answer in the form $p + q \ln r$, where the numbers p , q and r are to be found.

E

Solution:

$$(a) y = x^{\frac{3}{2}} + 48x^{-1} \Rightarrow \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - 48x^{-2}$$

$$\frac{dy}{dx} = 0 \Rightarrow \frac{3}{2}x^{\frac{1}{2}} = \frac{48}{x^2}$$

$$\Rightarrow x^{\frac{5}{2}} = \frac{2}{3} \times 48 = 32$$

$$\Rightarrow x = 4, y = 2^3 + 12 = 20$$

$$\Rightarrow x = 4, y = 20$$

$$(b) \frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{1}{2}} + 96x^{-3} > 0 \text{ for all } x > 0$$

$\therefore 20$ is a minimum value of y

$$(c) \text{Area} = \int_1^4 \left(x^{\frac{3}{2}} + \frac{48}{x} \right) dx$$

$$\begin{aligned} &= \left[\frac{2}{5}x^{\frac{5}{2}} + 48\ln|x| \right]_1^4 \\ &= \left(\frac{2}{5} \times 32 + 48\ln 4 \right) - \left(\frac{2}{5} + 0 \right) \\ &= \frac{62}{5} + 48\ln 4 \end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

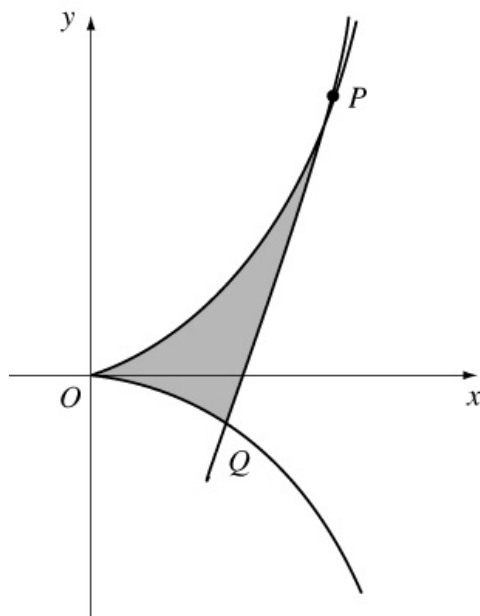
Exercise L, Question 2

Question:

The curve C has two arcs, as shown, and the equations

$$x = 3t^2, y = 2t^3,$$

where t is a parameter.



(a) Find an equation of the tangent to C at the point P where $t = 2$.
The tangent meets the curve again at the point Q .

(b) Show that the coordinates of Q are $(3, -2)$.

The shaded region R is bounded by the arcs OP and OQ of the curve C , and the line PQ , as shown.

(c) Find the area of R .

E

Solution:

$$(a) \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{6t^2}{6t} = t$$

P is $(12, 16)$

$$\therefore \text{ tangent is } y - 16 = 2(x - 12) \quad \text{or} \quad y = 2x - 8$$

(b) Substitute $x = 3t^2, y = 2t^3$ into the equation for the tangent

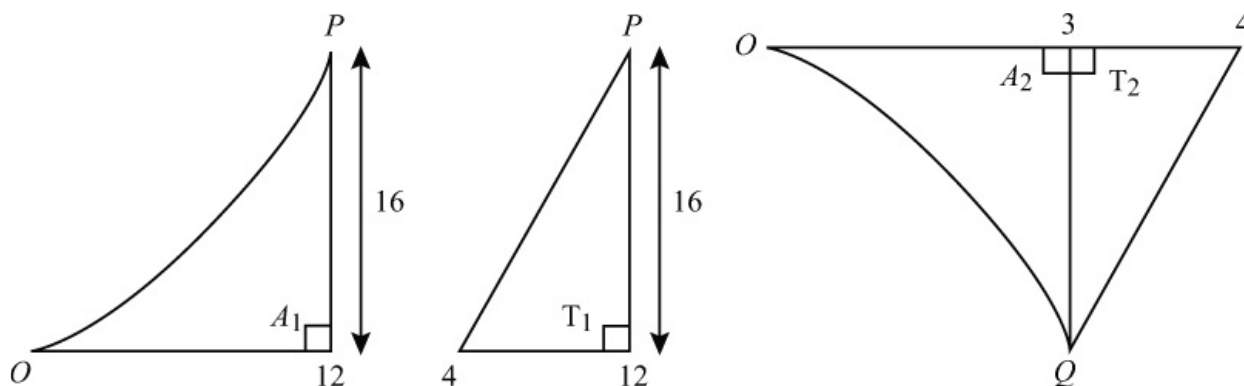
$$\Rightarrow 2t^3 = 6t^2 - 8$$

$$\Rightarrow t^3 - 3t^2 + 4 = 0$$

$$\Rightarrow (t - 2)^2(t + 1) = 0$$

$$\Rightarrow t = -1 \text{ at } Q(3, -2)$$

(c)



$$\text{Area of } R = A_1 - T_1 + A_2 + T_2$$

$$A_1 + A_2 = \int y dx = \int_{t=-1}^{t=2} 2t^3 \times 6t dt = \int_{-1}^2 12t^4 dt$$

$$= \left[\frac{12}{5} t^5 \right]_{-1}^2 = \left(\frac{12 \times 32}{5} \right) - \left(-\frac{12}{5} \right) = 79.2$$

$$T_1 = \frac{1}{2} \times 16 \times 12 = 96$$

$$T_2 = \frac{1}{2} \times 1 \times 2 = 1$$

$$\therefore \text{ area of } R = 79.2 - 96 + 1 = -15.8$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 3

Question:

(a) Show that $(1 + \sin 2x)^2 \equiv \frac{1}{2} (3 + 4\sin 2x - \cos 4x)$.

(b) The finite region bounded by the curve with equation $y = 1 + \sin 2x$, the x -axis, the y -axis and the line with equation $x = \frac{\pi}{2}$ is rotated through 2π about the x -axis.

Using calculus, calculate the volume of the solid generated, giving your answer in terms of π .

E

Solution:

$$\begin{aligned} \text{(a)} \quad (1 + \sin 2x)^2 &= 1 + 2\sin 2x + \sin^2 2x \\ &= 1 + 2\sin 2x + \frac{1}{2} (1 - \cos 4x) \\ &= \frac{3}{2} + 2\sin 2x - \frac{1}{2} \cos 4x \\ &= \frac{1}{2} (3 + 4\sin 2x - \cos 4x) \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad V &= \pi \int y^2 dx = \pi \int_0^{\frac{\pi}{2}} (1 + \sin 2x)^2 dx \\ &= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (3 + 4\sin 2x - \cos 4x) dx \\ &= \frac{\pi}{2} \left[3x - 2\cos 2x - \frac{1}{4}\sin 4x \right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} \left[\left(\frac{3\pi}{2} - 2\cos \pi - \frac{1}{4}\sin 2\pi \right) - \left(0 - 2 - 0 \right) \right] \\ &= \frac{\pi}{2} \left(\frac{3\pi}{2} + 2 + 2 \right) \\ &= \frac{\pi}{4} (3\pi + 8) \end{aligned}$$

Solutionbank

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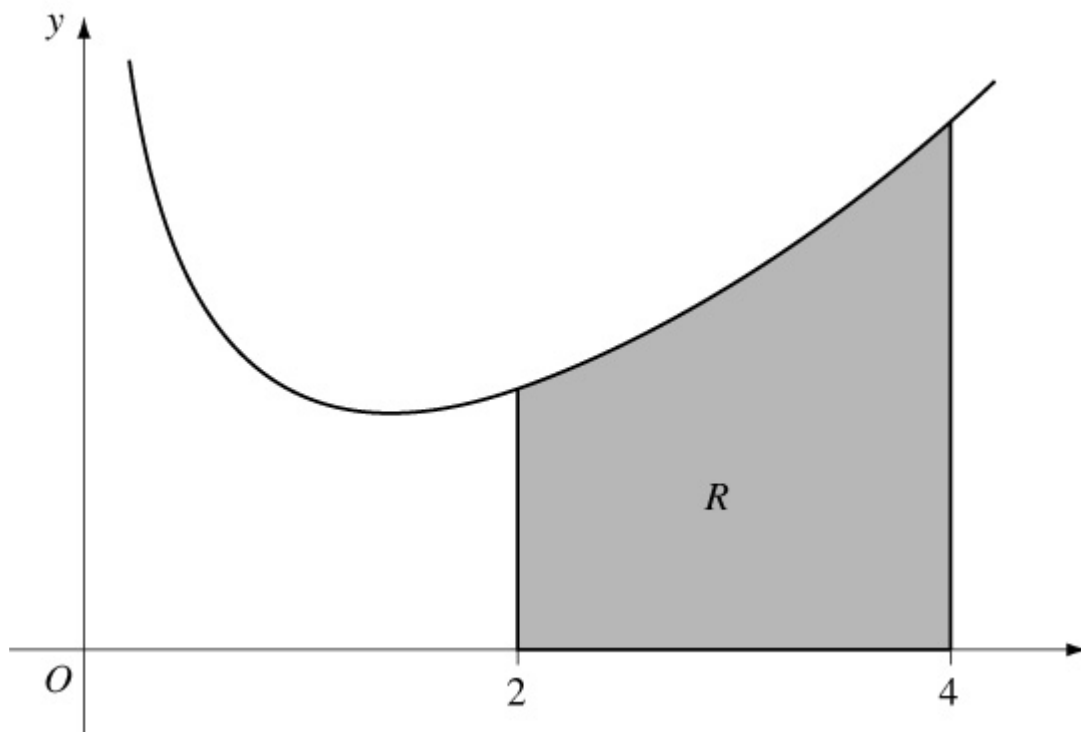
Integration

Exercise L, Question 4

Question:

This graph shows part of the curve with equation $y = f(x)$ where

$$f(x) \equiv e^{0.5x} + \frac{1}{x}, x > 0.$$



The curve has a stationary point at $x = \alpha$.

(a) Find $f'(x)$.

(b) Hence calculate $f'(1.05)$ and $f'(1.10)$ and deduce that $1.05 < \alpha < 1.10$.

(c) Find $\int f(x) dx$.

The shaded region R is bounded by the curve, the x-axis and the lines $x = 2$ and $x = 4$.

(d) Find, to 2 decimal places, the area of R.

E

Solution:

$$(a) f' \left(x \right) = \frac{1}{2} e^{\frac{1}{2}x} - \frac{1}{x^2}$$

$$(b) f' (1.05) = - 0.061... < 0$$

$$f' (1.10) = + 0.040... > 0$$

Change of sign $\therefore$ root α in interval (1.05, 1.10)

$$(c) \int \left(e^{0.5x} + \frac{1}{x} \right) dx = 2e^{0.5x} + \ln | x | + C$$

$$\begin{aligned} (d) \text{Area} &= \int_2^4 y dx \\ &= \left[2e^{0.5x} + \ln | x | \right]_2^4 \\ &= (2e^2 + \ln 4) - (2e^1 + \ln 2) \\ &= 2e^2 - 2e^1 + \ln 2 \\ &= 10.03 \text{ (2 d.p.)} \end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 5

Question:

(a) Find $\int xe^{-x} dx$.

(b) Given that $y = \frac{\pi}{4}$ at $x = 0$, solve the differential equation

$$e^x \frac{dy}{dx} = \frac{x}{\sin 2y}$$



Solution:

(a) $I = \int xe^{-x} dx$

$$u = x \Rightarrow \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$$

$$\therefore I = -xe^{-x} - \int (-e^{-x}) dx$$

$$\text{i.e. } I = -xe^{-x} - e^{-x} + C$$

(b) $e^x \frac{dy}{dx} = \frac{x}{\sin 2y}$

$$\Rightarrow \int \sin 2y dy = \int xe^{-x} dx$$

$$\Rightarrow -\frac{1}{2} \cos 2y = -xe^{-x} - e^{-x} + C$$

$$x = 0, y = \frac{\pi}{4} \Rightarrow 0 = 0 - 1 + C \Rightarrow C = 1$$

$$\therefore \frac{1}{2} \cos 2y = xe^{-x} + e^{-x} - 1$$

$$\text{or } \cos 2y = 2(xe^{-x} + e^{-x} - 1)$$

Solutionbank

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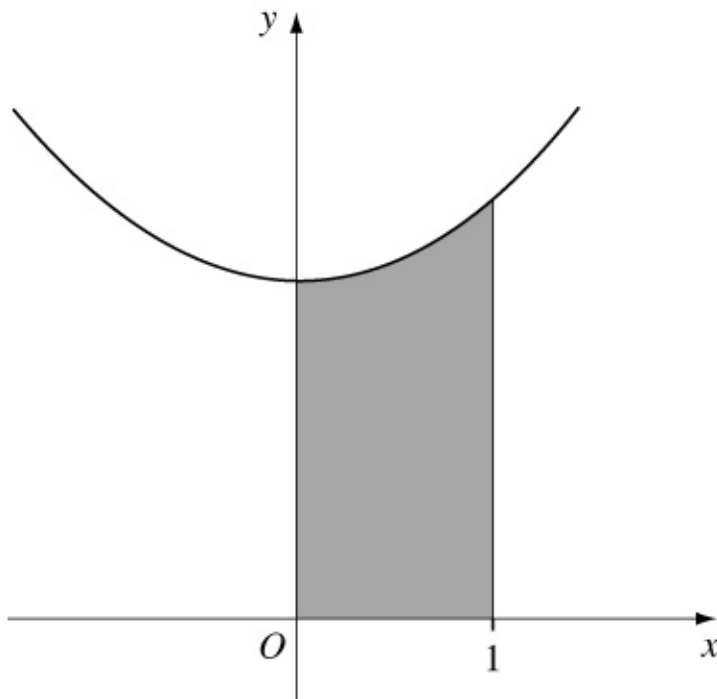
Integration

Exercise L, Question 6

Question:

The diagram shows the finite shaded region bounded by the curve with equation $y = x^2 + 3$, the lines $x = 1$, $x = 0$ and the x -axis. This region is rotated through 360° about the x -axis.

Find the volume generated.



Solution:

$$\begin{aligned}
 V &= \pi \int_0^1 y^2 dx = \pi \int_0^1 (x^2 + 3)^2 dx \\
 &= \pi \int_0^1 (x^4 + 6x^2 + 9) dx \\
 &= \pi \left[\frac{1}{5}x^5 + 2x^3 + 9x \right]_0^1 \\
 &= \pi \left[\left(\frac{1}{5} + 2 + 9 \right) - \left(0 \right) \right] \\
 &= \frac{56\pi}{5}
 \end{aligned}$$

Solutionbank

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Integration

Exercise L, Question 7

Question:

(a) Find $\int \frac{1}{x(x+1)} dx$

(b) Using the substitution $u = e^x$ and the answer to **a**, or otherwise, find $\int \frac{1}{1+e^x} dx$.

(c) Use integration by parts to find $\int x^2 \sin x dx$.



Solution:

(a) $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$

$$\begin{aligned} \therefore \int \frac{1}{x(x+1)} dx &= \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx \\ &= \ln |x| - \ln |x+1| + C \\ &= \ln \left| \frac{x}{x+1} \right| + C \end{aligned}$$

(b) $I = \int \frac{1}{1+e^x} dx \quad u = e^x \Rightarrow du = e^x dx$

$$\begin{aligned} \therefore I &= \int \frac{1}{(1+u)} \times \frac{1}{u} du = \ln \left| \frac{u}{1+u} \right| + C \quad \text{or} \quad \ln \left| \frac{e^x}{1+e^x} \right| \\ &+ C \end{aligned}$$

(c) $I = \int x^2 \sin x dx$

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x$$

$$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$$

$$\begin{aligned} \therefore I &= -x^2 \cos x - \int (-\cos x) \times 2x dx \\ &= -x^2 \cos x + \int 2x \cos x dx \end{aligned}$$

$$\text{Let } J = \int 2x \cos x \, dx$$

$$u = 2x \quad \Rightarrow \quad \frac{du}{dx} = 2$$

$$\frac{dv}{dx} = \cos x \quad \Rightarrow \quad v = \sin x$$

$$\begin{aligned} \therefore J &= 2x \sin x - \int 2 \sin x \, dx \\ &= 2x \sin x + 2 \cos x + C \end{aligned}$$

$$\therefore I = -x^2 \cos x + 2x \sin x + 2 \cos x + k$$

Solutionbank

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Integration

Exercise L, Question 8

Question:

(a) Find $\int x \sin 2x \, dx$.

(b) Given that $y = 0$ at $x = \frac{\pi}{4}$, solve the differential equation $\frac{dy}{dx} = x \sin 2x \cos^2 y$.



Solution:

(a) $I = \int x \sin 2x \, dx$

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \sin 2x \quad \Rightarrow \quad v = -\frac{1}{2} \cos 2x$$

$$\therefore I = -\frac{1}{2}x \cos 2x - \int -\frac{1}{2} \cos 2x \, dx$$

$$= -\frac{1}{2}x \cos 2x + \frac{1}{4} \sin 2x + C$$

(b) $\frac{dy}{dx} = x \sin 2x \cos^2 y$

$$\Rightarrow \int \sec^2 y \, dy = \int x \sin 2x \, dx$$

$$\Rightarrow \tan y = -\frac{1}{2}x \cos 2x + \frac{1}{4} \sin 2x + C$$

$$y = 0, x = \frac{\pi}{4} \quad \Rightarrow \quad 0 = 0 + \frac{1}{4} + C \quad \Rightarrow \quad C = -\frac{1}{4}$$

$$\therefore \tan y = -\frac{1}{2}x \cos 2x + \frac{1}{4} \sin 2x - \frac{1}{4}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

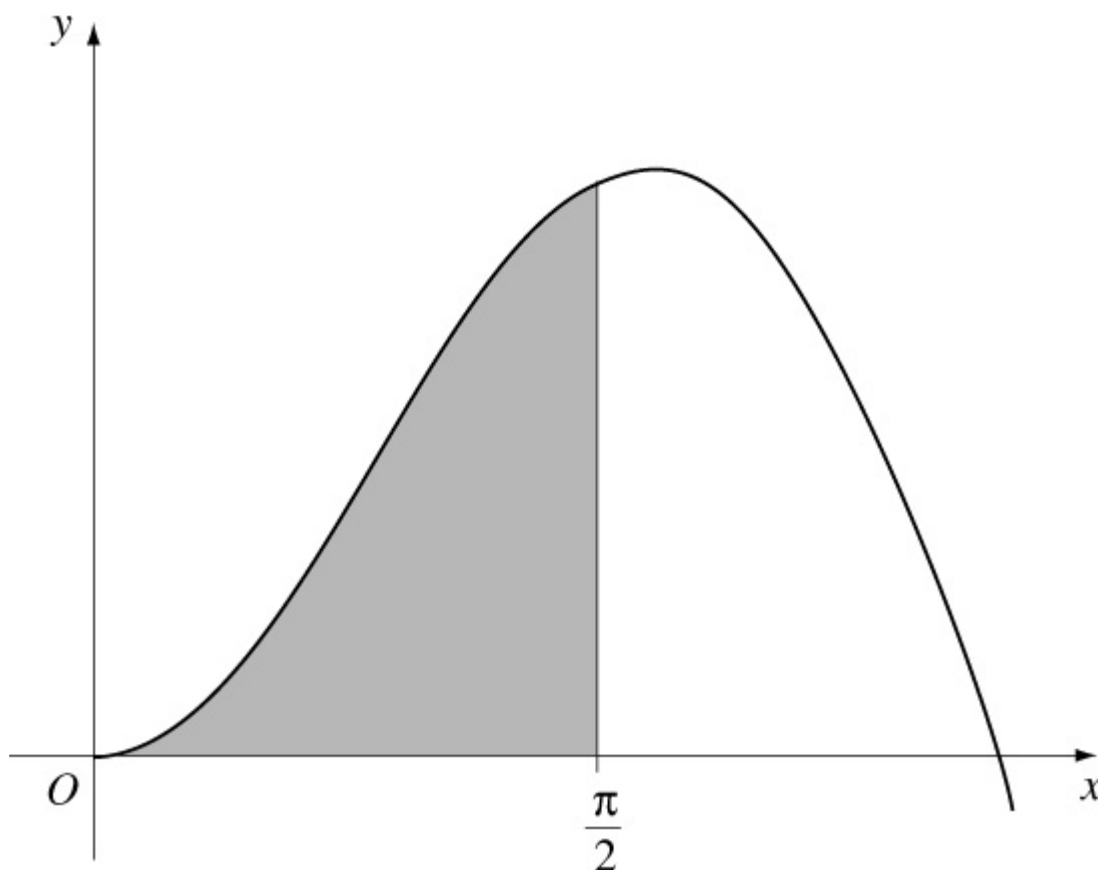
Exercise L, Question 9

Question:

(a) Find $\int x \cos 2x \, dx$.

(b) This diagram shows part of the curve with equation $y = 2x \frac{1}{2} \sin x$. The shaded region in the diagram is bounded by the curve, the x -axis and the line with equation $x = \frac{\pi}{2}$. This shaded region is rotated through 2π radians about the x -axis to form a solid of revolution. Using calculus, calculate the volume of the solid of revolution formed, giving your answer in terms of π .

E



Solution:

(a) $I = \int x \cos 2x \, dx$

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \cos 2x \Rightarrow v = \frac{1}{2} \sin 2x$$

$$\therefore I = \frac{x}{2} \sin 2x - \int \frac{1}{2} \sin 2x dx$$

$$= \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x + C$$

$$(b) V = \pi \int_0^{\frac{\pi}{2}} y^2 dx = \pi \int_0^{\frac{\pi}{2}} 4x \sin^2 x dx$$

$$\cos 2A = 1 - 2 \sin^2 A \Rightarrow 2 \sin^2 x = 1 - \cos 2x$$

$$\therefore V = \pi \int_0^{\frac{\pi}{2}} 2x \left(1 - \cos 2x \right) dx$$

$$= \pi \int_0^{\frac{\pi}{2}} 2x dx - 2\pi \int_0^{\frac{\pi}{2}} x \cos 2x dx$$

$$= \left[\pi x^2 \right]_0^{\frac{\pi}{2}} - 2\pi \left[\frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^3}{4} - 2\pi \left[\left(\frac{\pi}{4} \sin \pi + \frac{1}{4} \cos \pi \right) - \left(0 + \frac{1}{4} \right) \right]$$

$$= \frac{\pi^3}{4} + \pi$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 10

Question:

A curve has equation $y = f(x)$ and passes through the point with coordinates $(0, -1)$. Given that $f'(x) = \frac{1}{2}e^{2x} - 6x$,

(a) use integration to obtain an expression for $f(x)$,

(b) show that there is a root α of the equation $f'(x) = 0$, such that $1.41 < \alpha < 1.43$. **E**

Solution:

$$(a) f'(x) = \frac{1}{2}e^{2x} - 6x$$

$$\Rightarrow f(x) = \frac{1}{4}e^{2x} - 3x^2 + C$$

$$f(0) = -1 \Rightarrow -1 = \frac{1}{4} - 0 + C \Rightarrow C = -\frac{5}{4}$$

$$\therefore f(x) = \frac{1}{4}e^{2x} - 3x^2 - \frac{5}{4}$$

$$(b) f'(1.41) = -0.07... < 0$$

$$f'(1.43) = +0.15... > 0$$

Change of sign $\therefore$ root in interval $(1.41, 1.43)$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 11

Question:

$$f\left(x\right) = 16x^{\frac{1}{2}} - \frac{2}{x}, x > 0.$$

(a) Solve the equation $f(x) = 0$.

(b) Find $\int f(x) dx$.

(c) Evaluate $\int_1^4 f(x) dx$, giving your answer in the form $p + q \ln r$, where p , q and r are rational numbers.

E

Solution:

$$(a) f\left(x\right) = 0 \Rightarrow 16x^{\frac{1}{2}} = \frac{2}{x}$$

$$\Rightarrow 16x^{\frac{3}{2}} = 2$$

$$\Rightarrow x^{\frac{3}{2}} = \frac{1}{8}$$

$$\Rightarrow x = \left(3\sqrt{\frac{1}{8}}\right)^2 = \frac{1}{4}$$

$$(b) \int \left(16x^{\frac{1}{2}} - \frac{2}{x}\right) dx = \frac{16x^{\frac{3}{2}}}{\frac{3}{2}} - 2 \ln |x| + C$$

$$= \frac{32}{3}x^{\frac{3}{2}} - 2 \ln |x| + C$$

$$(c) \int_1^4 f(x) dx = \left[\frac{32}{3}x^{\frac{3}{2}} - 2 \ln |x| \right]_1^4$$

$$= \left(\frac{32}{3} \times 2^3 - 2 \ln 4 \right) - \left(\frac{32}{3} - 0 \right)$$

$$= \frac{224}{3} - 2 \ln 4$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

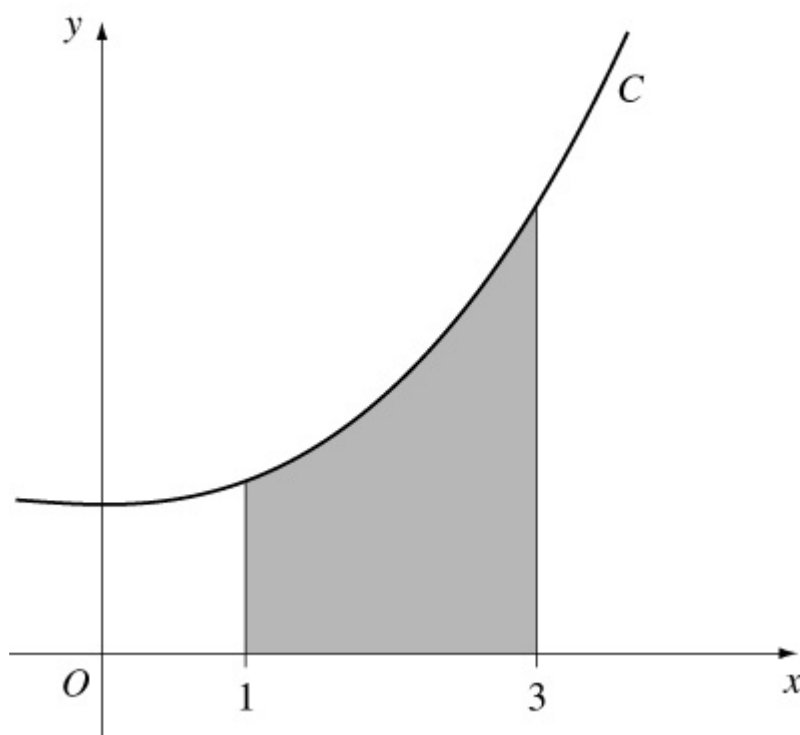
Exercise L, Question 12

Question:

Shown is part of a curve C with equation $y = x^2 + 3$. The shaded region is bounded by C , the x -axis and the lines with equations $x = 1$ and $x = 3$. The shaded region is rotated through 360° about the x -axis.

Using calculus, calculate the volume of the solid generated. Give your answer as an exact multiple of π .

E



Solution:

$$\begin{aligned}
 V &= \pi \int_1^3 y^2 dx = \pi \int_1^3 (x^2 + 3)^2 dx \\
 &= \pi \int_1^3 (x^4 + 6x^2 + 9) dx \\
 &= \pi \left[\frac{1}{5}x^5 + 2x^3 + 9x \right]_1^3 \\
 &= \pi \left[\left(\frac{243}{5} + 54 + 27 \right) - \left(\frac{1}{5} + 2 + 9 \right) \right] \\
 &= \pi \left(\frac{242}{5} + 81 - 11 \right)
 \end{aligned}$$

$$= 118.4\pi$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 13

Question:

(a) Find $\int x (x^2 + 3)^5 dx$

(b) Show that $\int_1^e \frac{1}{x^2} \ln x dx = 1 - \frac{2}{e}$

(c) Given that $p > 1$, show that $\int_1^p \frac{1}{(x+1)(2x-1)} dx = \frac{1}{3} \ln \frac{4p-2}{p+1}$

E

Solution:

(a) Let $y = (x^2 + 3)^6$

$$\Rightarrow \frac{dy}{dx} = 6(x^2 + 3)^5 \times 2x$$

$$\therefore \int x(x^2 + 3)^5 dx = \frac{1}{12} (x^2 + 3)^6 + C$$

(b) $I = \int_1^e \frac{1}{x^2} \ln x dx$

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = \frac{1}{x^2} \Rightarrow v = -\frac{1}{x}$$

$$\therefore I = \left[-\frac{1}{x} \ln x \right]_1^e - \int_1^e \left(-\frac{1}{x^2} \right) dx$$

$$= \left(-\frac{1}{e} \right) - \left(0 \right) + \left[-\frac{1}{x} \right]_1^e$$

$$= -\frac{1}{e} + \left(-\frac{1}{e} \right) - \left(-1 \right)$$

$$= 1 - \frac{2}{e}$$

$$(c) \frac{1}{(x+1)(2x-1)} \equiv \frac{A}{x+1} + \frac{B}{2x-1}$$

$$\Rightarrow 1 \equiv A(2x-1) + B(x+1)$$

$$x = \frac{1}{2} \Rightarrow 1 = \frac{3}{2}B \Rightarrow B = \frac{2}{3}$$

$$x = -1 \Rightarrow 1 = -3A \Rightarrow A = -\frac{1}{3}$$

$$\therefore \int_1^p \frac{1}{(x+1)(2x-1)} dx = \int_1^p \left(\frac{\frac{2}{3}}{2x-1} + \frac{-\frac{1}{3}}{x+1} \right) dx$$

$$= \left[\frac{1}{3} \ln |2x-1| - \frac{1}{3} \ln |x+1| \right]_1^p$$

$$= \left[\frac{1}{3} \ln \left| \frac{2x-1}{x+1} \right| \right]_1^p$$

$$= \left[\frac{1}{3} \ln \left(\frac{2p-1}{p+1} \right) \right] - \left(\frac{1}{3} \ln \frac{1}{2} \right)$$

$$= \frac{1}{3} \ln \left(\frac{4p-2}{p+1} \right)$$

Solutionbank

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Integration

Exercise L, Question 14

Question:

$$f\left(\frac{1}{x}\right) \equiv \frac{5x^2 - 8x + 1}{2x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

(a) Find the values of the constants A , B and C .

(b) Hence find $\int f\left(\frac{1}{x}\right) dx$.

(c) Hence show that $\int_4^9 f\left(\frac{1}{x}\right) dx = \ln\left(\frac{32}{3}\right) - \frac{5}{24}$

E

Solution:

$$(a) f\left(\frac{1}{x}\right) \equiv \frac{5x^2 - 8x + 1}{2x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\Rightarrow 5x^2 - 8x + 1 \equiv 2A(x-1)^2 + 2Bx(x-1) + 2Cx$$

$$x = 0 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$x = 1 \Rightarrow -2 = 2C \Rightarrow C = -1$$

$$\text{Coefficients of } x^2: 5 = 2A + 2B \Rightarrow B = 2$$

$$(b) \int f\left(\frac{1}{x}\right) dx = \int \left(\frac{\frac{1}{2}}{x} + \frac{2}{x-1} - \frac{1}{(x-1)^2} \right) dx$$

$$= \frac{1}{2} \ln|x| + 2 \ln|x-1| + \frac{1}{x-1} + C$$

$$(c) \int_4^9 f\left(\frac{1}{x}\right) dx = \left[\frac{1}{2} \ln|x| + 2 \ln|x-1| + \frac{1}{x-1} \right]_4^9$$

$$= \left[\ln|\sqrt{x(x-1)^2}| + \frac{1}{x-1} \right]_4^9$$

$$\begin{aligned} &= \left[\ln \left(3 \times 64 \right) + \frac{1}{8} \right] - \left[\ln \left(2 \times 9 \right) + \frac{1}{3} \right] \\ &= \ln \left(\frac{3 \times 64}{2 \times 9} \right) + \frac{1}{8} - \frac{1}{3} \\ &= \ln \frac{32}{3} - \frac{5}{24} \end{aligned}$$

Solutionbank

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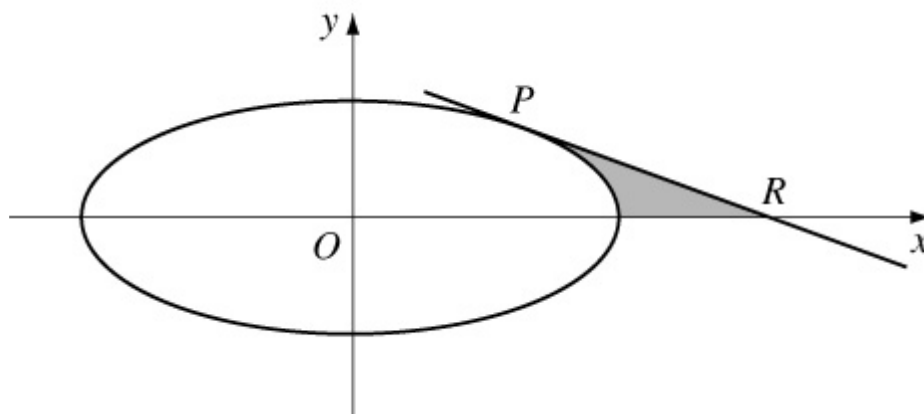
Integration

Exercise L, Question 15

Question:

The curve shown has parametric equations

$$x = 5 \cos \theta, y = 4 \sin \theta, 0 \leq \theta < 2\pi.$$



- Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$.
- Find an equation of the tangent to the curve at the point P .
- Find the coordinates of the point R where this tangent meets the x -axis. The shaded region is bounded by the tangent PR , the curve and the x -axis.
- Find the area of the shaded region, leaving your answer in terms of π .

E

Solution:

$$(a) \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = - \frac{4 \cos \theta}{5 \sin \theta}$$

$$\therefore \text{gradient of tangent at } P = - \frac{4}{5}$$

$$(b) P = \left(\frac{5}{\sqrt{2}}, \frac{4}{\sqrt{2}} \right)$$

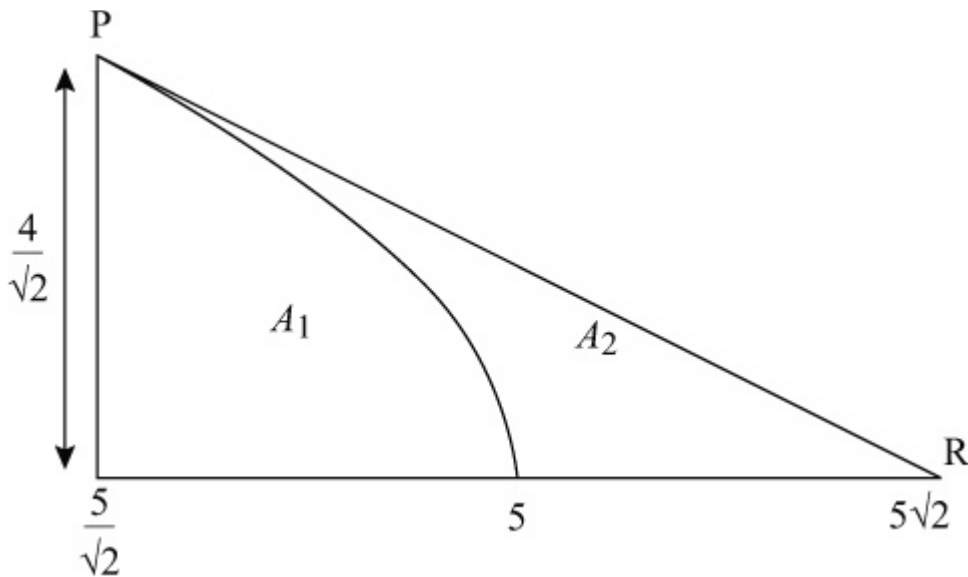
$\therefore$ equation of tangent is

$$y - \frac{4}{\sqrt{2}} = -\frac{4}{5} \left(x - \frac{5}{\sqrt{2}} \right) \quad \text{or} \quad y - 2\sqrt{2} = -\frac{4}{5} \left(x - \frac{5}{\sqrt{2}} \right)$$

(c) At R, $y = 0 \Rightarrow x = \frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2} = 5\sqrt{2}$

$\therefore R$ is $(5\sqrt{2}, 0)$

(d)



$$A_1 + A_2 = \frac{1}{2} \times \left(5\sqrt{2} - \frac{5}{\sqrt{2}} \right) \times \frac{4}{\sqrt{2}} = \frac{1}{2} \times \frac{5}{\sqrt{2}} \times \frac{4}{\sqrt{2}} = 5$$

$$A_1 = \int y dx = \int_{\frac{\pi}{4}}^0 4 \sin \theta \times \left(-5 \sin \theta \right) d\theta$$

$$= 10 \int_0^{\frac{\pi}{4}} \left(1 - \cos 2\theta \right) d\theta$$

$$= \left[10\theta - 5 \sin 2\theta \right]_0^{\frac{\pi}{4}}$$

$$= \frac{5\pi}{2} - 5$$

$$\therefore A_2 = 5 - A_1 = 5 - \left(\frac{5\pi}{2} - 5 \right) = 10 - 2.5\pi$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 16

Question:

(a) Obtain the general solution of the differential equation

$$\frac{dy}{dx} = xy^2, y > 0.$$

(b) Given also that $y = 1$ at $x = 1$, show that

$$y = \frac{2}{3 - x^2}, \quad -\sqrt{3} < x < \sqrt{3}$$

is a particular solution of the differential equation.

The curve C has equation $y = \frac{2}{3 - x^2}, x \neq -\sqrt{3}, x \neq \sqrt{3}$

(c) Write down the gradient of C at the point $(1, 1)$.

(d) Deduce that the line which is a tangent to C at the point $(1, 1)$ has equation $y = x$.

(e) Find the coordinates of the point where the line $y = x$ again meets the curve C .

E

Solution:

(a) $\frac{dy}{dx} = xy^2$

$$\Rightarrow \int \frac{1}{y^2} dy = \int x dx$$

$$\Rightarrow -\frac{1}{y} = \frac{x^2}{2} + C$$

$$\text{or } y = \frac{-2}{x^2 + k} \quad \left(k = 2C \right)$$

(b) $y = 1, x = 1 \Rightarrow 1 = \frac{-2}{1 + k} \Rightarrow k = -3$

$$\therefore y = \frac{2}{3-x^2}$$

for $x^2 \neq 3$ and $y > 0$, i.e. $-\sqrt{3} < x < \sqrt{3}$

(c) When $x = 1$, $y = 1$ $\frac{dy}{dx}$ is 1

(d) Equation of tangent is $y - 1 = 1 (x - 1)$, i.e. $y = x$.

$$(e) x = \frac{2}{3-x^2} \Rightarrow -x^3 + 3x = 2 \text{ or } x^3 - 3x + 2 = 0$$

$$\Rightarrow (x - 1)^2 (x + 2) = 0$$

$\therefore y = x$ meets curve at $(-2, -2)$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

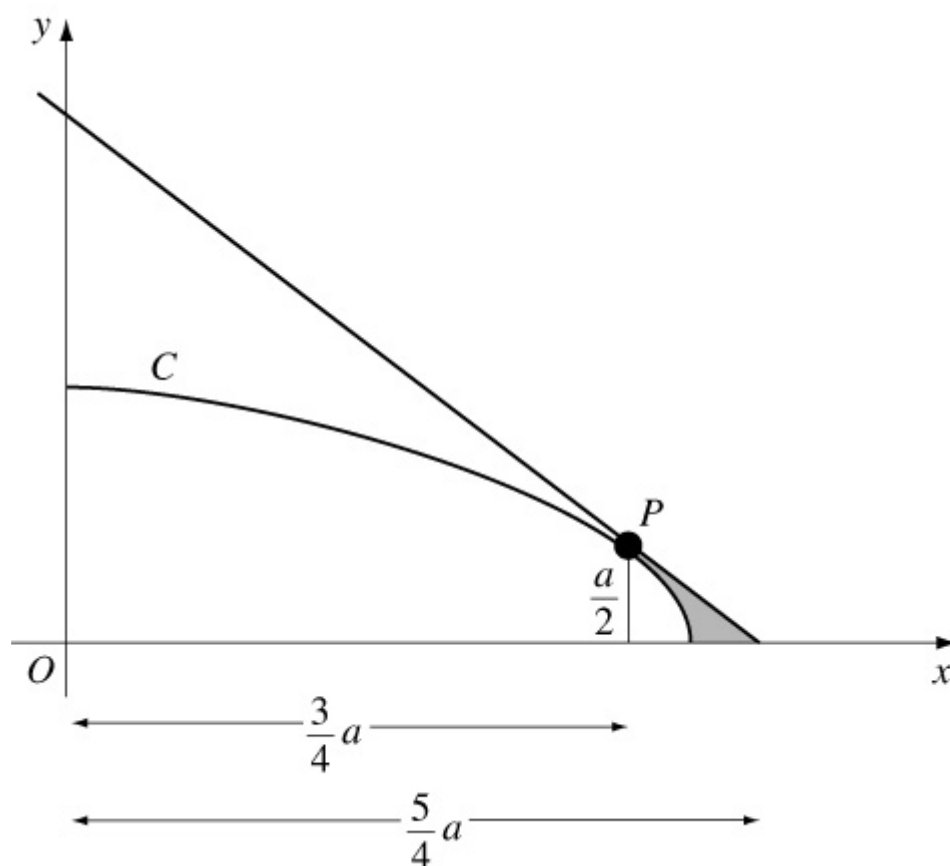
Exercise L, Question 17

Question:

The diagram shows the curve C with parametric equations

$$x = a \sin^2 t, y = a \cos t, 0 \leq t \leq \frac{1}{2}\pi,$$

where a is a positive constant. The point P lies on C and has coordinates $\left(\frac{3}{4}a, \frac{1}{2}a \right)$.



(a) Find $\frac{dy}{dx}$, giving your answer in terms of t .

(b) Find an equation of the tangent to C at P .

(c) Show that a cartesian equation of C is $y^2 = a^2 - ax$.

The shaded region is bounded by C , the tangent at P and the x -axis. This shaded region is rotated through 2π radians about the x -axis to form a solid of

revolution.

(d) Use calculus to calculate the volume of the solid revolution formed, giving your answer in the form $k\pi a^3$, where k is an exact fraction. **E**

Solution:

$$(a) \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-a \sin t}{2a \sin t \cos t} = -\frac{1}{2} \sec t$$

$$(b) P \text{ is } \left(\frac{3}{4}a, \frac{1}{2}a \right), \text{ so } \cos t = \frac{1}{2}$$

$$\Rightarrow M = -\frac{1}{2 \times \frac{1}{2}} = -1$$

$$\therefore \text{ tangent is } y - \frac{1}{2}a = -1 \left(x - \frac{3}{4}a \right)$$

$$\text{or } y = -x + \frac{5}{4}a$$

$$(c) \sin^2 t + \cos^2 t = 1 \Rightarrow \frac{x}{a} + \frac{y^2}{a^2} = 1$$

$$\text{or } y^2 = a^2 - ax$$

$$(d) \text{ volume} = \text{cone} - \pi \int \frac{3}{4}a y^2 dx$$

$$\text{cone} = \frac{1}{3}\pi \left(\frac{1}{2}a \right)^2 \left(\frac{5}{4}a - \frac{3}{4}a \right) = \frac{\pi a^3}{24}$$

$$\begin{aligned} \pi \int \frac{3}{4}a y^2 dx &= \pi \left[a^2 x - \frac{a}{2} x^2 \right] \frac{3}{4}a \\ &= \pi \left[\left(a^3 - \frac{a^3}{2} \right) - \left(\frac{3}{4}a^3 - \frac{9}{32}a^3 \right) \right] = \frac{\pi a^3}{32} \end{aligned}$$

$$\therefore \text{ Volume} = \pi \left(\frac{a^3}{24} - \frac{a^3}{32} \right) = \frac{\pi a^3}{96}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 18

Question:

(a) Using the substitution $u = 1 + 2x$, or otherwise, find

$$\int \frac{4x}{(1 + 2x)^2} dx, x > -\frac{1}{2},$$

(b) Given that $y = \frac{\pi}{4}$ when $x = 0$, solve the differential equation

$$(1 + 2x)^2 \frac{dy}{dx} = \frac{x}{\sin^2 y}$$

E

Solution:

$$(a) I = \int \frac{4x}{(1 + 2x)^2} dx$$

$$u = 1 + 2x$$

$$\Rightarrow \frac{du}{2} = dx \text{ and } 4x = 2(u - 1)$$

$$\therefore I = \int \frac{2(u - 1)}{u^2} \times \frac{du}{2}$$

$$= \int \left(\frac{1}{u} - u^{-2} \right) du$$

$$= \ln |u| + \frac{1}{u} + C$$

$$= \ln |1 + 2x| + \frac{1}{1 + 2x} + C$$

$$(b) (1 + 2x)^2 \frac{dy}{dx} = \frac{x}{\sin^2 y}$$

$$\Rightarrow \int \sin^2 y dy = \int \frac{x}{(1 + 2x)^2} dx$$

$$\Rightarrow \int 4 \sin^2 y dy = \int \frac{4x}{(1 + 2x)^2} dx$$

$$\Rightarrow \int (2 - 2 \cos 2y) dy = I$$

$$\Rightarrow 2y - \sin 2y = \ln |1 + 2x| + \frac{1}{1 + 2x} + C$$

$$x = 0, y = \frac{\pi}{4} \Rightarrow \frac{\pi}{2} - 1 = \ln 1 + 1 + C$$

$$\Rightarrow C = \frac{\pi}{2} - 2$$

$$\therefore 2y - \sin 2y = \ln |1 + 2x| + \frac{1}{1 + 2x} + \frac{\pi}{2} - 2$$

Solutionbank

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Integration

Exercise L, Question 19

Question:

The diagram shows the curve with equation $y = xe^{2x}$, $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

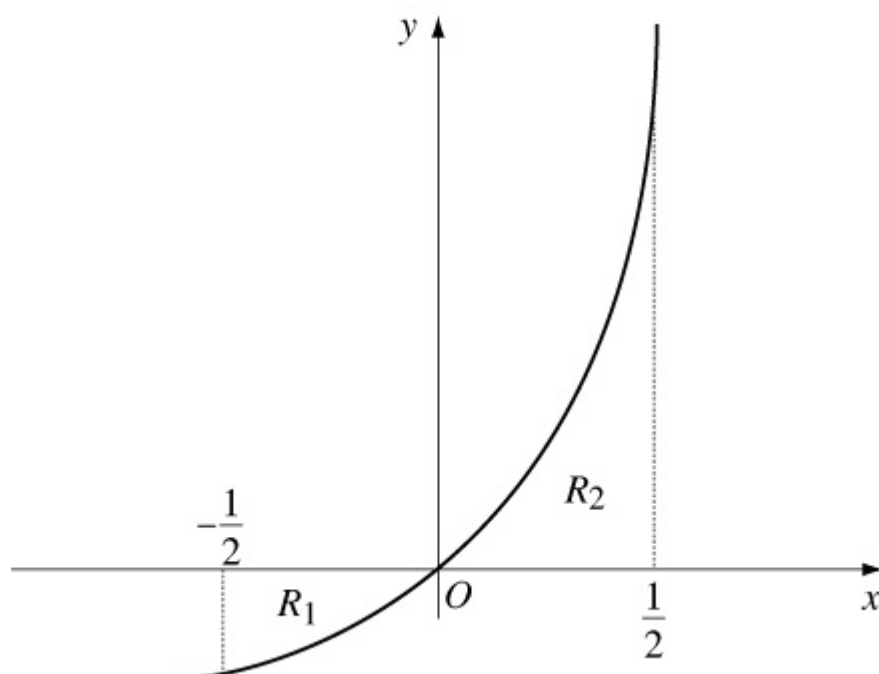
The finite region R_1 bounded by the curve, the x-axis and the line $x = -\frac{1}{2}$ has area A_1 .

The finite region R_2 bounded by the curve, the x-axis and the line $x = \frac{1}{2}$ has area A_2 .

(a) Find the exact values of A_1 and A_2 by integration.

(b) Show that $A_1 : A_2 = (e - 2) : e$.

E



Solution:

(a) $\int xe^{2x} dx$

$$u = x \Rightarrow \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = e^{2x} \Rightarrow v = \frac{1}{2}e^{2x}$$

$$\therefore \int xe^{2x} dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

$$\begin{aligned} A_1 &= - \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} \right] - \frac{1}{2}^0 \\ &= - \left[\left(0 - \frac{1}{4} \right) - \left(-\frac{1}{4}e^{-1} - \frac{1}{4}e^{-1} \right) \right] \\ &= \frac{1}{4} \left(1 - 2e^{-1} \right) \end{aligned}$$

$$\begin{aligned} A_2 &= \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} \right] 0^{\frac{1}{2}} \\ &= \left(\frac{1}{4}e^1 - \frac{1}{4}e^1 \right) - \left(0 - \frac{1}{4} \right) \\ &= \frac{1}{4} \end{aligned}$$

$$(b) \frac{A_1}{A_2} = \frac{\frac{1}{4}(1 - 2e^{-1})}{\frac{1}{4}} = 1 - 2e^{-1} = \frac{e-2}{e}$$

$$\therefore A_1 : A_2 = (e - 2) : e$$

Solutionbank

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Integration

Exercise L, Question 20

Question:

Find $\int x^2 e^{-x} dx$.

Given that $y = 0$ at $x = 0$, solve the differential equation $\frac{dy}{dx} = x^2 e^{3y-x}$. **E**

Solution:

$$I = \int x^2 e^{-x} dx$$

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x$$

$$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$$

$$\therefore I = -x^2 e^{-x} - \int (-e^{-x}) \times 2x dx$$

$$= -x^2 e^{-x} + \int 2x e^{-x} dx$$

$$J = \int 2x e^{-x} dx$$

$$u = 2x \Rightarrow \frac{du}{dx} = 2$$

$$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$$

$$\therefore J = -2x e^{-x} - \int (-e^{-x}) \times 2 dx$$

$$= -2x e^{-x} - 2e^{-x} + k$$

$$\therefore I = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$$

$$\frac{dy}{dx} = x^2 e^{3y-x} = x^2 e^{-x} e^{3y}$$

$$\Rightarrow \int e^{-3y} dy = \int x^2 e^{-x} dx$$

$$\Rightarrow -\frac{1}{3} e^{-3y} = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$$

$$x = 0, y = 0 \Rightarrow -\frac{1}{3} = -2 + C \Rightarrow C = \frac{5}{3}$$

$$\therefore \frac{1}{3} e^{-3y} = x^2 e^{-x} + 2x e^{-x} + 2e^{-x} - \frac{5}{3}$$

Solutionbank

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Integration

Exercise L, Question 21

Question:

The curve with equation $y = e^{3x} + 1$ meets the line $y = 8$ at the point $(h, 8)$.

(a) Find h , giving your answer in terms of natural logarithms.

(b) Show that the area of the finite region enclosed by the curve with equation $y = e^{3x} + 1$, the x -axis, the y -axis and the line $x = h$ is $2 + \frac{1}{3} \ln 7$.

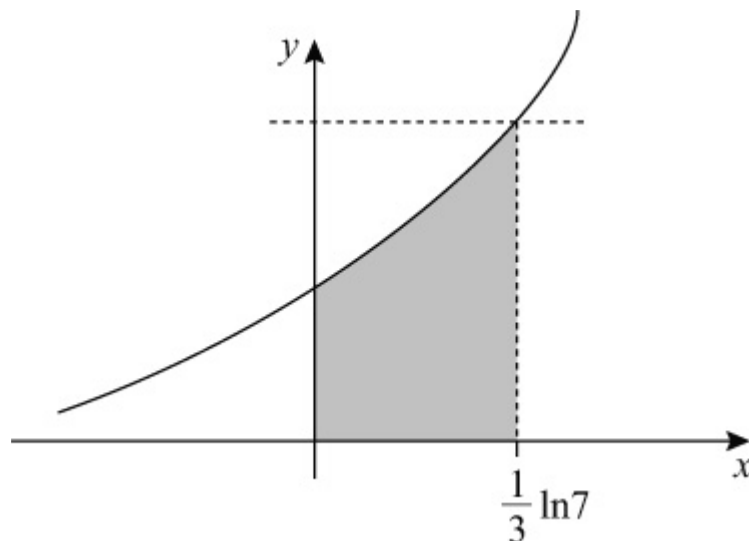
E

Solution:

$$(a) \quad 8 = e^{3x} + 1 \quad \Rightarrow \quad 7 = e^{3x}$$

$$\therefore x = \frac{1}{3} \ln 7, \text{ i.e. } h = \frac{1}{3} \ln 7$$

(b)



$$\begin{aligned} \text{Area} &= \int_0^{\frac{1}{3} \ln 7} y \, dx \\ &= \int_0^{\frac{1}{3} \ln 7} (e^{3x} + 1) \, dx \\ &= \left[\frac{1}{3} e^{3x} + x \right]_0^{\frac{1}{3} \ln 7} \end{aligned}$$

$$\begin{aligned} &= \left(\frac{1}{3}e^{\ln 7} + \frac{1}{3}\ln 7 \right) - \left(\frac{1}{3} + 0 \right) \\ &= \frac{1}{3} \left(7 + \ln 7 \right) - \frac{1}{3} \\ &= \frac{1}{3} \left(6 + \ln 7 \right) \\ &= 2 + \frac{1}{3}\ln 7 \end{aligned}$$

Solutionbank

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Integration

Exercise L, Question 22

Question:

(a) Given that

$$\frac{x^2}{x^2 - 1} \equiv A + \frac{B}{x - 1} + \frac{C}{x + 1},$$

find the values of the constants A , B and C .

(b) Given that $x = 2$ at $t = 1$, solve the differential equation

$$\frac{dx}{dt} = 2 - \frac{2}{x^2}, \quad x > 1.$$

You need not simplify your final answer. **E**

Solution:

$$(a) \quad \frac{x^2}{x^2 - 1} \equiv A + \frac{B}{x - 1} + \frac{C}{x + 1}$$

$$\Rightarrow \quad x^2 \equiv A(x - 1)(x + 1) + B(x + 1) + C(x - 1)$$

$$x = 1 \quad \Rightarrow \quad 1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$$

$$x = -1 \quad \Rightarrow \quad 1 = -2C \quad \Rightarrow \quad C = -\frac{1}{2}.$$

$$\text{Coefficients of } x^2: \quad 1 = A \quad \Rightarrow \quad A = 1$$

$$(b) \quad \frac{dx}{dt} = 2 - \frac{2}{x^2}$$

$$\Rightarrow \quad \int \frac{x^2}{x^2 - 1} dx = \int 2 dt$$

$$\Rightarrow \quad \int \left(1 + \frac{\left(\frac{1}{2}\right)}{x - 1} - \frac{\left(\frac{1}{2}\right)}{x + 1} \right) dx = 2t$$

$$\Rightarrow \quad x + \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| = 2t + C$$

$$x = 2, t = 1 \Rightarrow 2 + \frac{1}{2} \ln \frac{1}{3} = 2 + C \Rightarrow C = \frac{1}{2} \ln \frac{1}{3}$$

$$\therefore x + \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| = 2t + \frac{1}{2} \ln \frac{1}{3}$$

Solutionbank

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Integration

Exercise L, Question 23

Question:

The curve C is given by the equations

$$x = 2t, y = t^2,$$

where t is a parameter.

(a) Find an equation of the normal to C at the point P on C where $t = 3$.

The normal meets the y -axis at the point B . The finite region R is bounded by the part of the curve C between the origin O and P , and the lines OB and OP .

(b) Show the region R , together with its boundaries, in a sketch.

The region R is rotated through 2π about the y -axis to form a solid S .

(c) Using integration, and explaining each step in your method, find the volume of S , giving your answer in terms of π .

E

Solution:

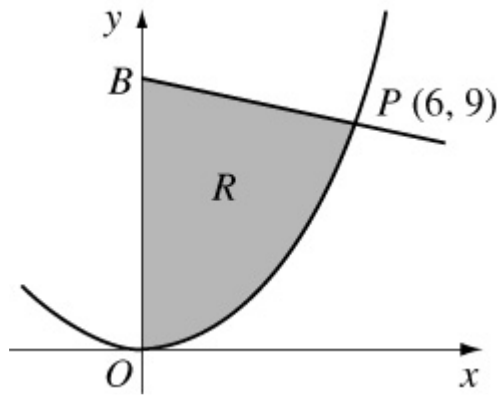
$$(a) \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2t}{2} = t.$$

$$\therefore \text{ at } P(6, 9) \text{ gradient of normal is } -\frac{1}{3}$$

$$\therefore \text{ equation of normal is } y - 9 = -\frac{1}{3} \left(x - 6 \right) \quad \text{or} \quad y = -\frac{1}{3}x + 11$$

$$(b) x = 2t, y = t^2 \quad \Rightarrow \quad y = \frac{x^2}{4}$$

B is $(0, 11)$



$$(c) \text{ volume} = \text{cone} + \pi \int_0^9 x^2 dy$$

$$\text{cone} = \frac{1}{3}\pi \times 6^2 \times 2 = 24\pi$$

$$\pi \int_0^9 x^2 dy = \pi \int_{t=0}^{t=3} 4t^2 \times 2t dt = \pi \int_0^3 8t^3 dt$$

$$= \pi [2t^4]_0^3 = \pi \times 2 \times 81 = 162\pi$$

$$\therefore \text{Volume of } S = 186\pi$$

Solutionbank

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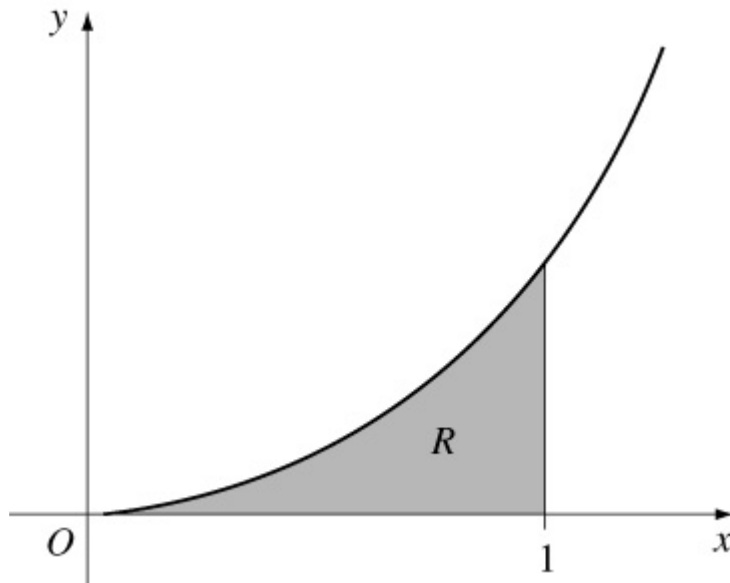
Integration

Exercise L, Question 24

Question:

Shown is part of the curve with equation $y = e^{2x} - e^{-x}$. The shaded region R is bounded by the curve, the x -axis and the line with equation $x = 1$.

Use calculus to find the area of R , giving your answer in terms of e . **E**



Solution:

$$\begin{aligned}
 \text{Area} &= \int_0^1 (e^{2x} - e^{-x}) \, dx \\
 &= \left[\frac{1}{2}e^{2x} + e^{-x} \right]_0^1 \\
 &= \left(\frac{1}{2}e^2 + e^{-1} \right) - \left(\frac{1}{2} + 1 \right) \\
 &= \frac{1}{2} \left(e^2 + \frac{2}{e} - 3 \right)
 \end{aligned}$$

Solutionbank

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Integration

Exercise L, Question 25

Question:

(a) Given that $2y = x - \sin x \cos x$, show that $\frac{dy}{dx} = \sin^2 x$.

(b) Hence find $\int \sin^2 x \, dx$.

(c) Hence, using integration by parts, find $\int x \sin^2 x \, dx$. **E**

Solution:

(a) $2y = x - \sin x \cos x$

$$\Rightarrow 2 \frac{dy}{dx} = 1 - \left[\cos^2 x + \sin x \left(-\sin x \right) \right] = 1 - \cos^2 x + \sin^2 x$$

$$\therefore \frac{dy}{dx} = \sin^2 x \quad (\text{using } \sin^2 x = 1 - \cos^2 x)$$

(b) $\int \sin^2 x \, dx = y + C_1$

$$= \frac{x}{2} - \frac{1}{2} \sin x \cos x + C_1$$

(c) $\int x \sin^2 x \, dx$

$$u = x \Rightarrow \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \sin^2 x \Rightarrow v = \left(\frac{x}{2} - \frac{1}{2} \sin x \cos x \right)$$

$$\therefore \int x \sin^2 x \, dx = \frac{x^2}{2} - \frac{1}{2} x \sin x \cos x - \int \left(\frac{x}{2} - \frac{1}{2} \sin x \cos x \right) dx$$

$$= \frac{x^2}{2} - \frac{1}{2} x \sin x \cos x - \frac{x^2}{4} + \frac{1}{4} \int \sin 2x \, dx$$

$$= \frac{x^2}{4} - \frac{1}{2} x \sin x \cos x - \frac{1}{8} \cos 2x + C_2$$

Solutionbank

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Integration

Exercise L, Question 26

Question:

The rate, in $\text{cm}^3 \text{s}^{-1}$, at which oil is leaking from an engine sump at any time t seconds is proportional to the volume of oil, $V \text{cm}^3$, in the sump at that instant. At time $t = 0$, $V = A$.

(a) By forming and integrating a differential equation, show that

$$V = Ae^{-kt}$$

where k is a positive constant.

(b) Sketch a graph to show the relation between V and t .

Given further that $V = \frac{1}{2}A$ at $t = T$,

(c) show that $kT = \ln 2$. **E**

Solution:

(a) $\frac{dV}{dt} = -kV$

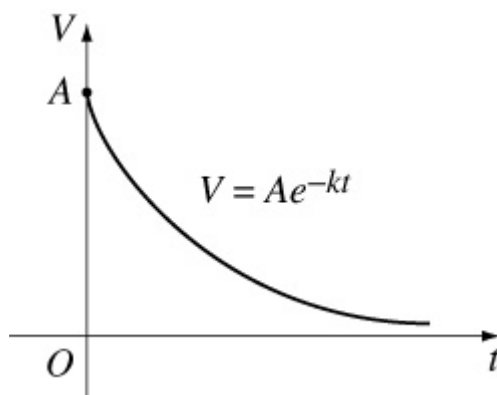
$$\Rightarrow \int \frac{1}{V} dV = \int -k dt$$

$$\Rightarrow \ln |V| = -kt + C$$

$$\Rightarrow V = A_1 e^{-kt}$$

$$t = 0, V = A \Rightarrow V = Ae^{-kt} \quad (A_1 = A)$$

(b)



$$(c) \ t = T, \ V = \frac{1}{2}A \quad \Rightarrow \quad \frac{1}{2}A = Ae^{-kT}$$

$$\Rightarrow \quad -\ln 2 = -kT$$

$$\Rightarrow \quad kT = \ln 2$$

Solutionbank

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Integration

Exercise L, Question 27

Question:

This graph shows part of the curve C with parametric equations

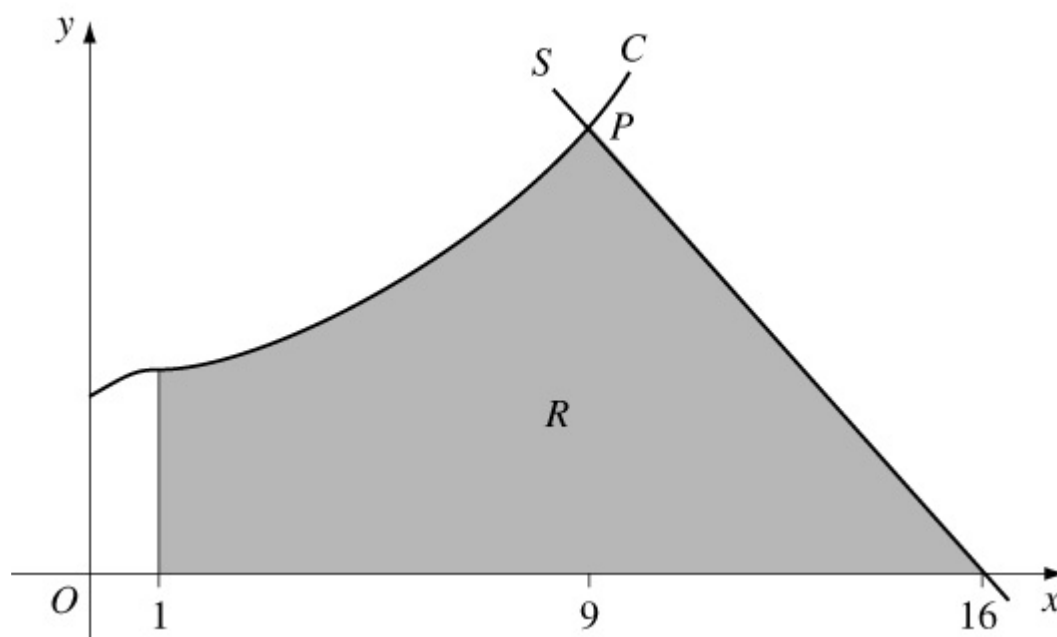
$$x = (t + 1)^2, y = \frac{1}{2}t^3 + 3, t \geq -1.$$

P is the point on the curve where $t = 2$. The line S is the normal to C at P .

(a) Find an equation of S .

The shaded region R is bounded by C , S , the x -axis and the line with equation $x = 1$.

(b) Using integration and showing all your working, find the area of R . **E**



Solution:

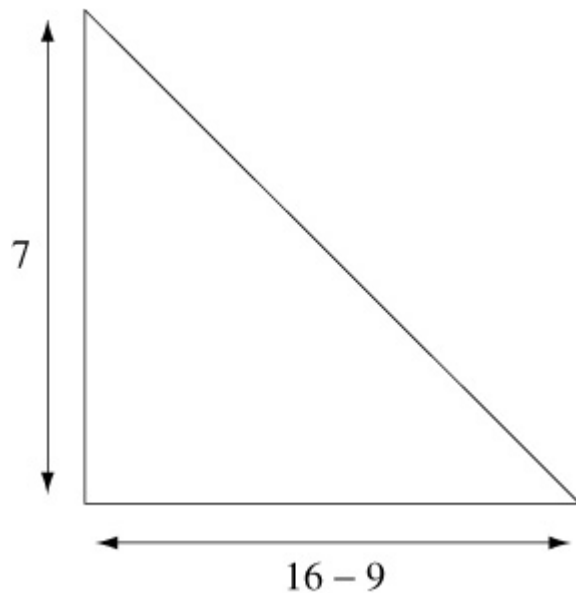
$$(a) \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\frac{3}{2}t^2}{2(t+1)} = \frac{3t^2}{4(t+1)}$$

$$\text{At } P(9, 7) \text{ gradient of normal is } -\frac{4 \times 3}{3 \times 2^2} = -1$$

$$\therefore \text{ equation of line } S \text{ is } y - 7 = -1(x - 9)$$

i.e. $y = -x + 16$ or $y + x = 16$

(b) Area = $\int_{x=1}^{x=9} y \, dx$ + area of triangle shown below



$$\begin{aligned}
 \int_{x=1}^{x=9} y \, dx &= \int_{t=0}^{t=2} \left(\frac{1}{2}t^3 + 3 \right) \cdot 2 \left(t + 1 \right) dt \\
 &= \int_0^2 (t^4 + t^3 + 6t + 6) dt \\
 &= \left[\frac{1}{5}t^5 + \frac{1}{4}t^4 + \frac{6t^2}{2} + 6t \right]_0^2 \\
 &= \left(\frac{32}{5} + \frac{16}{4} + 3 \times 4 + 6 \times 2 \right) - \left(0 \right) \\
 &= 34.4 \\
 \therefore \text{Area} &= 34.4 + \frac{1}{2} \times 7^2 = 58.9
 \end{aligned}$$

Solutionbank

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Integration

Exercise L, Question 28

Question:

Shown is part of the curve C with parametric equations

$$x = t^2, y = \sin 2t, t \geq 0.$$

The point A is an intersection of C with the x -axis.

(a) Find, in terms of π , the x -coordinate of A .

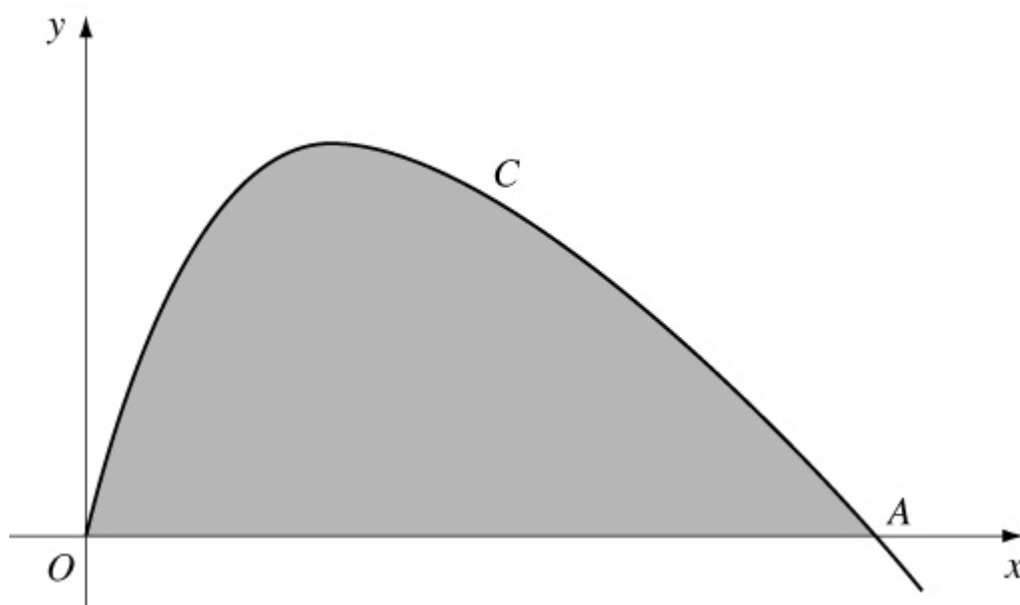
(b) Find $\frac{dy}{dx}$ in terms of t , $t > 0$.

(c) Show that an equation of the tangent to C at A is $4x + 2\pi y = \pi^2$.

The shaded region is bounded by C and the x -axis.

(d) Use calculus to find, in terms of π , the area of the shaded region.

E



Solution:

(a) At A , $y = 0 \Rightarrow \sin 2t = 0 \Rightarrow 2t = 0 \text{ or } \pi \Rightarrow t = \frac{\pi}{2}$

$$\therefore A \text{ is } \left(\left(\frac{\pi}{2} \right)^2, 0 \right) \text{ or } \left(\frac{\pi^2}{4}, 0 \right)$$

$$(b) \frac{dy}{dx} = \frac{2 \cos 2t}{2t} = \frac{\cos 2t}{t}$$

$$(c) \text{ Gradient of tangent at } A \text{ is } \frac{\cos \pi}{\left(\frac{\pi}{2}\right)} = - \frac{1}{\left(\frac{\pi}{2}\right)} = - \frac{2}{\pi}$$

$$\therefore \text{ equation of tangent is } y - 0 = - \frac{2}{\pi} \left(x - \frac{\pi^2}{4} \right)$$

$$\Rightarrow \pi y = -2x + \frac{2\pi^2}{4}$$

$$\text{or } 2\pi y + 4x = \pi^2$$

$$(d) \text{ Area} = \int y dx = \int_{t=0}^{t=\frac{\pi}{2}} \frac{\pi}{2} \sin 2t \times 2t dt$$

$$u = t \Rightarrow \frac{du}{dt} = 1$$

$$\frac{dv}{dt} = 2 \sin 2t \Rightarrow v = -\cos 2t$$

$$\therefore \text{ Area} = \left[-t \cos 2t \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(-\cos 2t \right) dt$$

$$= \left(+ \frac{\pi}{2} \right) - \left(0 \right) + \left[\frac{1}{2} \sin 2t \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 29

Question:

Showing your method clearly in each case, find

(a) $\int \sin^2 x \cos x \, dx$,

(b) $\int x \ln x \, dx$.

Using the substitution $t^2 = x + 1$, where $x > -1$, $t > 0$,

(c) Find $\int \frac{x}{\sqrt{x+1}} \, dx$.

(d) Hence evaluate $\int_0^3 \frac{x}{\sqrt{x+1}} \, dx$. **E**

Solution:

(a) Let $y = \sin^3 x \Rightarrow \frac{dy}{dx} = 3 \sin^2 x \cos x$

$$\therefore \int \sin^2 x \cos x \, dx = \frac{1}{3} \sin^3 x + C$$

(b) $\int x \ln x \, dx$

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x \Rightarrow v = \frac{1}{2}x^2$$

$$\therefore \int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \times \frac{1}{x} \, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{x^2}{4} + C$$

(c) $t^2 = x + 1 \Rightarrow 2t \, dt = dx$

$$\therefore I = \int \frac{x}{\sqrt{x+1}} \, dx$$

$$= \int \frac{t^2 - 1}{t} \times 2t \, dt$$

$$\begin{aligned}
 &= \int (2t^2 - 2) dt \\
 &= \frac{2}{3}t^3 - 2t + C \\
 &= \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + C \\
 &= \frac{2}{3}\sqrt{x+1} \left(x - 2 \right) + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(d) } \int_0^3 \frac{x}{\sqrt{x+1}} dx &= \left[\frac{2}{3} \left(x - 2 \right) \sqrt{x+1} \right]_0^3 \\
 &= \left(\frac{2}{3} \times 2 \right) - \left(-\frac{4}{3} \right) = \frac{8}{3}
 \end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 30

Question:

(a) Using the substitution $u = 1 + 2x^2$, find $\int x (1 + 2x^2)^5 dx$.

(b) Given that $y = \frac{\pi}{8}$ at $x = 0$, solve the differential equation

$$\frac{dy}{dx} = x (1 + 2x^2)^5 \cos^2 2y. \quad \text{E}$$

Solution:

$$(a) \quad u = 1 + 2x^2 \quad \Rightarrow \quad du = 4x dx \quad \Rightarrow \quad x dx = \frac{du}{4}$$

$$\text{So } \int x (1 + 2x^2)^5 dx = \int \frac{u^5}{4} du = \frac{u^6}{24} + C_1 = \frac{(1 + 2x^2)^6}{24} + C_1$$

$$(b) \quad \frac{dy}{dx} = x (1 + 2x^2)^5 \cos^2 2y$$

$$\Rightarrow \int \sec^2 2y dy = \int x (1 + 2x^2)^5 dx$$

$$\Rightarrow \frac{1}{2} \tan 2y = \frac{(1 + 2x^2)^6}{24} + C_2$$

$$y = \frac{\pi}{8}, x = 0 \quad \Rightarrow \quad \frac{1}{2} = \frac{1}{24} + C_2 \quad \Rightarrow \quad C_2 = \frac{11}{24}$$

$$\therefore \tan 2y = \frac{(1 + 2x^2)^6}{12} + \frac{11}{12}$$

Solutionbank

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Integration

Exercise L, Question 31

Question:

Find $\int x^2 \ln 2x \, dx$.

E

Solution:

$$I = \int x^2 \ln 2x \, dx$$

$$u = \ln 2x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x^2 \quad \Rightarrow \quad v = \frac{x^3}{3}$$

$$\therefore I = \frac{x^3}{3} \ln 2x - \int \frac{x^3}{3} \times \frac{1}{x} \, dx$$

$$= \frac{x^3}{3} \ln 2x - \int \frac{x^2}{3} \, dx$$

$$= \frac{x^3}{3} \ln 2x - \frac{x^3}{9} + C$$

Solutionbank

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Exercise L, Question 32

Question:

Obtain the solution of

$$x \left(x + 2 \right) \frac{dy}{dx} = y, y > 0, x > 0,$$

for which $y = 2$ at $x = 2$, giving your answer in the form $y^2 = f(x)$.

E

Solution:

$$x \left(x + 2 \right) \frac{dy}{dx} = y$$

$$\Rightarrow \int \frac{1}{y} dy = \int \frac{1}{x(x+2)} dx$$

$$\frac{1}{x(x+2)} \equiv \frac{A}{x} + \frac{B}{x+2}$$

$$\Rightarrow 1 \equiv A(x+2) + Bx$$

$$x = 0 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$x = -2 \Rightarrow 1 = -2B \Rightarrow B = -\frac{1}{2}$$

$$\text{So } \ln y = \int \left(\frac{\left(\frac{1}{2}\right)}{x} - \frac{\left(\frac{1}{2}\right)}{x+2} \right) dx$$

$$= \frac{1}{2} \ln |x| - \frac{1}{2} \ln |x+2| + C$$

$$\therefore y = \sqrt{\frac{kx}{x+2}} \quad \left(C = \frac{1}{2} \ln k \right)$$

$$x = 2, y = 2 \Rightarrow 2 = \sqrt{\frac{2k}{4}} \Rightarrow 4 \times 2 = k$$

$$\therefore y = \sqrt{\frac{8x}{x+2}} \quad \text{or} \quad y^2 = \frac{8x}{x+2}$$

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Integration

Exercise L, Question 33

Question:

(a) Use integration by parts to show that

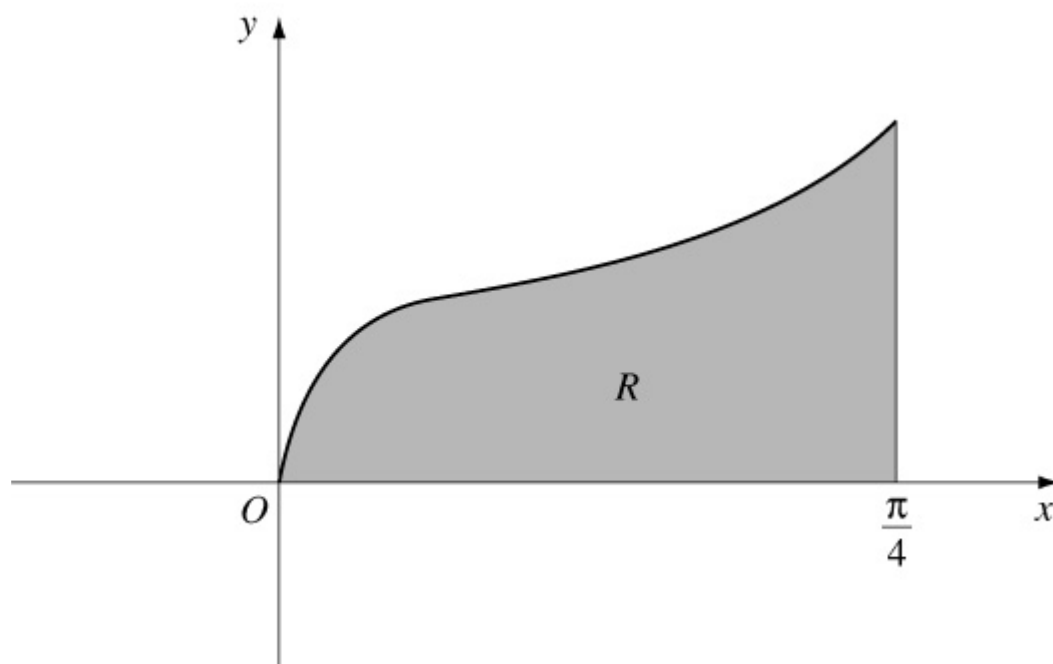
$$\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx = \frac{1}{4}\pi - \frac{1}{2} \ln 2.$$

The finite region R , bounded by the curve with equation $y = x^{\frac{1}{2}} \sec x$, the line $x = \frac{\pi}{4}$ and the x -axis is shown. The region R is rotated through 2π radians about the x -axis.

(b) Find the volume of the solid of revolution generated.

(c) Find the gradient of the curve with equation $y = x^{\frac{1}{2}} \sec x$ at the point where $x = \frac{\pi}{4}$.

E



Solution:

(a) $I = \int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$

$$u = x \Rightarrow \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$$

$$\begin{aligned} \therefore I &= [x \tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, dx \\ &= \left(\frac{\pi}{4} \right) - \left(0 \right) - [\ln |\sec x|]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{4} - \left[\left(\ln \sqrt{2} \right) - \left(\ln 1 \right) \right] \\ &= \frac{\pi}{4} - \frac{1}{2} \ln 2 \end{aligned}$$

$$(b) V = \pi \int_0^{\frac{\pi}{4}} y^2 \, dx = \pi \int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$$

$$\text{Using (a)} \quad V = \frac{\pi^2}{4} - \frac{\pi}{2} \ln 2 = 1.38 \text{ (3 s.f.)}$$

$$(c) \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \sec x + x^{\frac{1}{2}} \sec x \tan x$$

$$\text{At } x = \frac{\pi}{4} \quad \frac{dy}{dx} = \frac{1}{2} \times \frac{2}{\sqrt{\pi}} \times \sqrt{2} + \frac{\sqrt{\pi}}{2} \times \sqrt{2} \times 1 = \sqrt{\frac{2}{\pi}} + \sqrt{\frac{\pi}{2}} = 2.05 \text{ (3 s.f.)}$$

Solutionbank

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Integration

Exercise L, Question 34

Question:

Part of the design of a stained glass window is shown. The two loops enclose an area of blue glass. The remaining area within the rectangle $ABCD$ is red glass. The loops are described by the curve with parametric equations

$$x = 3 \cos t, y = 9 \sin 2t, 0 \leq t < 2\pi.$$

(a) Find the cartesian equation of the curve in the form $y^2 = f(x)$.

(b) Show that the shaded area enclosed by the curve and the x -axis, is given by

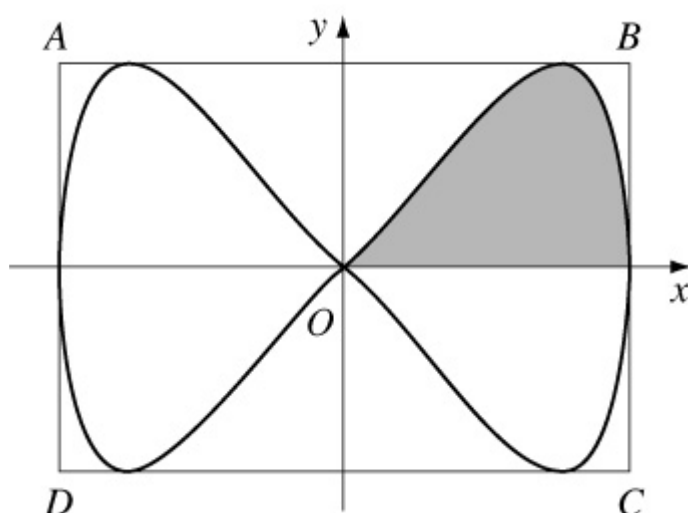
$$\int_0^{\frac{\pi}{2}} \frac{\pi}{2} A \sin 2t \sin t \, dt, \text{ stating the value of the constant } A.$$

(c) Find the value of this integral.

The sides of the rectangle $ABCD$ are the tangents to the curve that are parallel to the coordinate axes. Given that 1 unit on each axis represents 1 cm,

(d) find the total area of the red glass.

E



Solution:

(a) $x = 3 \cos t$

$$y = 9 \sin 2t \Rightarrow y = 18 \cos t \sin t$$

$$\Rightarrow y = 6x \sin t$$

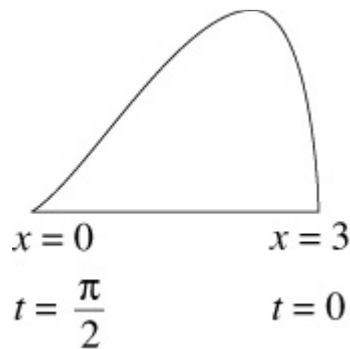
$$\therefore \cos t = \frac{x}{3}, \sin t = \frac{y}{6x}$$

$$\cos^2 t + \sin^2 t = 1 \Rightarrow \frac{x^2}{9} + \frac{y^2}{36x^2} = 1$$

$$\text{i.e. } 4x^4 + y^2 = 36x^2$$

$$\text{or } y^2 = 4x^2 (9 - x^2)$$

(b)



$$\text{Area} = \int y \, dx$$

$$= \int_{t=\frac{\pi}{2}}^{t=0} 9 \sin 2t \times \left(-3 \sin t \right) dt$$

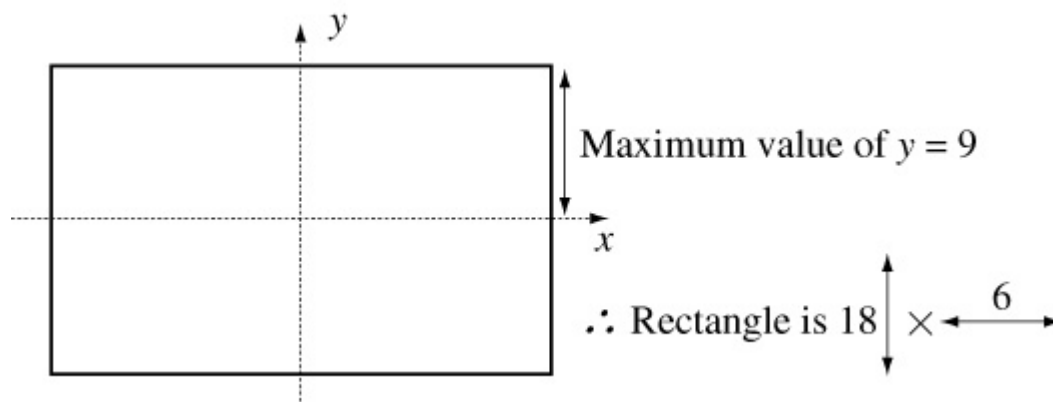
$$= 27 \int_0^{\frac{\pi}{2}} \sin 2t \sin t \, dt$$

$$(c) 27 \int_0^{\frac{\pi}{2}} \sin 2t \sin t \, dt = 54 \int_0^{\frac{\pi}{2}} \sin^2 t \cos t \, dt$$

$$= \left[\frac{54 \sin^3 t}{3} \right]_0^{\frac{\pi}{2}}$$

$$= (18 \times 1) - (0)$$

$$= 18$$

(d) Area of blue glass is $18 \times 4 = 72$ 

$$\text{Area of rectangle} = 108$$

$$\therefore \text{Area of red glass} = 108 - 72 = 36 \text{ cm}^2$$