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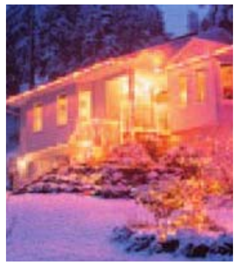


Review Exercise

1 Simplify

a $\dfrac{2x^2 - 7x - 15}{x^2 - 25}$

b $\dfrac{x^3 + 1}{x + 1}$.

2 Express $\dfrac{4x}{x^2 - 2x - 3} + \dfrac{1}{x^2 + x}$ as a single fraction, giving your answer in its simplest form.

3 Express $\dfrac{2x^2 + 3x}{(2x + 3)(x - 2)} - \dfrac{6}{x^2 - x - 2}$ as a single fraction in its simplest form. *E*

4 **a** Given that

$16x^3 - 36x^2 - 12x + 5$
$\equiv (2x + 1)(8x^2 + ax + b)$,

find the value of a and the value of b.

b Hence, or otherwise, simplify

$\dfrac{16x^3 - 36x^2 - 12x + 5}{4x - 1}$

5 $f(x) = 1 - \dfrac{3}{x + 2} + \dfrac{3}{(x + 2)^2}$, $x \neq -2$

a Show that $f(x) = \dfrac{x^2 + x + 1}{(x + 2)^2}$, $x \neq -2$

b Show that $x^2 + x + 1 > 0$ for all values of x.

c Show that $f(x) > 0$ for all values of x, $x \neq -2$. *E*

6 **a** Show that

$\dfrac{4}{(x + 1)^2} - \dfrac{1}{(x + 1)} - \dfrac{1}{2} \equiv \dfrac{5 - 4x - x^2}{2(x + 1)^2}$.

b Hence solve

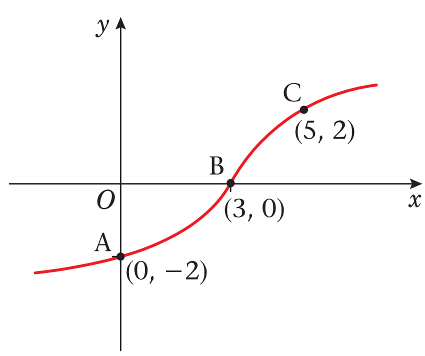
$\dfrac{4}{(x + 1)^2} < \dfrac{1}{(x + 1)} + \dfrac{1}{2}$, $x \neq -1$.

7 $f(x) = \dfrac{x}{x + 3} - \dfrac{x + 24}{2x^2 + 5x - 3}$, $\{x \in \mathbb{R}, x > \tfrac{1}{2}\}$.

a Show that $f(x) = \dfrac{2(x - 4)}{2x - 1}$, $\{x \in \mathbb{R}, x > \tfrac{1}{2}\}$.

b Find $f^{-1}(x)$.

8 The graph of the increasing function f passes through the points A(0, −2), B(3, 0) and C(5, 2), as shown.



a Sketch the graph of f^{-1}, showing the images of A, B and C.

The function g is defined by

$g: x \to \sqrt{x^2 + 2},\ x \in \mathbb{R}$

b Find **i** $fg\sqrt{23}$, **ii** $gf(0)$.

9 The functions f and g are defined by

$f: x \to 3x + 4, \quad x \in \mathbb{R},\ x > 0,$

$g: x \to \dfrac{x}{x - 2}, \quad x \in \mathbb{R},\ x > 2.$

a Find the inverse function $f^{-1}(x)$, stating its domain.

b Find the exact value of $gf\left(\frac{1}{2}\right)$.

c State the range of g.

d Find $g^{-1}(x)$, stating its domain.

10 The function f is defined by

$f : x \to \dfrac{5x + 1}{x^2 + x - 2} - \dfrac{3}{x + 2}, \ x > 1.$

a Show that $f(x) = \dfrac{2}{x - 1}, \quad x > 1.$

b Find $f^{-1}(x)$.

The function g is defined by

$g : x \to x^2 + 5, \quad x \in \mathbb{R}$

c Solve $fg(x) = \frac{1}{4}$. *E*

11 The functions f and g are defined by

$f: x \to (x - 4)^2 - 16, \quad x \in \mathbb{R},\ x > 0,$

$g: x \to \dfrac{8}{1 - x}, \quad x \in \mathbb{R},\ x < 1.$

a Find the range of f.

b Explain why, with the given domain for f, $f^{-1}(x)$ does not exist.

c Show that $fg(x) = \dfrac{64x}{(1 - x)^2}$.

d Find $g^{-1}(x)$, stating its domain.

12 The function f(x) is defined by

$$f(x) = \begin{cases} -2(x + 1) & -2 \leqslant x \leqslant -1 \\ (x + 1)(2 - x) & -1 < x \leqslant 2 \end{cases}$$

a Sketch the graph of f(x).

b Write down the range of f.

c Find the values of x for which $f(x) < 2$.

13 **a** Express $4x^2 - 4x - 3$ in the form $(ax - b)^2 - c$, where a, b and c are positive constants to be found.

The function f is defined by

$f: x \to 4x^2 - 4x - 3, \quad \left\{x \in \mathbb{R},\ x \geqslant \frac{1}{2}\right\}.$

b Sketch the graph of f.

c Sketch the graph of f^{-1}.

d Find $f^{-1}(x)$, stating its domain.

14 The functions f and g are defined by

$f: x \to \dfrac{x + 2}{x}, \quad x \in \mathbb{R},\ x \neq 0.$

$g: x \to \ln(2x - 5), \quad x \in \mathbb{R},\ x > 2\frac{1}{2}.$

a Sketch the graph of f.

b Show that $f^2(x) = \dfrac{3x + 2}{x + 2}$.

[$f^2(x)$ means $ff(x)$]

c Find the exact value of $gf\left(\frac{1}{4}\right)$.

d Find $g^{-1}(x)$, stating its domain.

15 Solve the following equations, giving your answers to 3 significant figures.

a $3e^{(2x+5)} = 4$

b $3^x = 5^{1-x}$

c $2\ln(2x - 1) = 1 + \ln 7$

16 Find the exact solutions to the equations

a $\ln x + \ln 3 = \ln 6$

b $e^x + 3e^{-x} = 4$ *E*

17 The function f is defined by

$f: x \to 3 - \ln(x + 2), \quad x \in \mathbb{R},\ x > -2.$

The graph of $y = f(x)$ crosses the x-axis at the point A and crosses the y-axis at the point B.

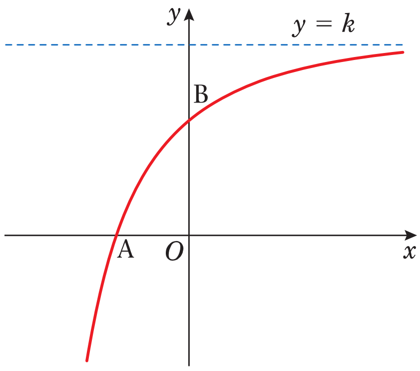
a Find the exact coordinates of A and B.

b Sketch the graph of $y = f(x)$, $x > -2$.

18 The graph of the function

$f(x) = 144 - 36e^{-2x}, \quad x \in \mathbb{R}$

has an asymptote $y = k$, and crosses the x and y axes at A and B respectively, as shown overleaf.



a Write down the value of k and the y-coordinate of B.

b Express the x-coordinate of A in terms of ln2.

19 [Part **d** requires the differentiation of e^{ax}, see Ch 8]

The functions f and g are defined by

$f : x \to 2x + \ln 2, \quad x \in \mathbb{R}$

$g : x \to e^{2x}, \quad x \in \mathbb{R}.$

a Prove that the composite function gf is

$gf : x \to 4e^{4x}, \quad x \in \mathbb{R}.$

b Sketch the curve with equation $y = gf(x)$, and show the coordinates of the point where the curve crosses the y-axis.

c Write down the range of gf.

d Find the value of x for which $\frac{d}{dx}[gf(x)] = 3$, giving your answer to 3 significant figures. *E*

20 **a** Show that $e^x - e^{-x} = 4$ can be rewritten in the form $e^{2x} - 4e^x - 1 = 0$

b Hence find the exact value of the real solution of $e^x - e^{-x} = 4$.

c For this value of x, find the exact value of $e^x + e^{-x}$.

21 At time $t = 0$, a lake is stocked with k fish. The number, n, of fish in the lake at time t days can be represented by the equation

$n = 3000 + 1450e^{0.04t}.$

a State the value of k.

b Calculate the increase in the population of fish 3 weeks after stocking the lake.

c Find how many days pass, from the day the lake was stocked, before the number of fish increases to over 7000.

22 A heated metal ball S is dropped into a liquid. As S cools its temperature, T °C, t minutes after it enters the liquid is given by

$T = 400e^{-0.05t} + 25, \quad t \geqslant 0.$

a Find the temperature of S as it enters the liquid.

b Find how long S is in the liquid before its temperature drops to 300 °C. Give your answer to 3 significant figures.

c Find the rate, in °C per minute to 3 significant figures, at which the temperature of S is decreasing at the instant $t = 50$.

d With reference to the equation given above, explain why the temperature of S can never drop to 20 °C *E*

23 A breeding programme for a particular animal is being monitored. Initially there were k breeding pairs in the survey. A suggested model for the number of breeding pairs, n, after t years is

$$n = \frac{400}{1 + 9e^{-\frac{1}{9}t}}.$$

a Find the value of k.

b Show that the above equation can be written in the form $t = 9 \ln \left(\frac{9n}{400 - n}\right)$

c Hence, or otherwise, calculate the number of years, according to the model, after which the number of breeding pairs will first exceed 100.

The model predicts that the number of breeding pairs cannot exceed the value A.

d Find the value of A.

24 $f(x) = x^3 - \frac{1}{x} - 2, \quad x \neq 0.$

a Show that the equation $f(x) = 0$ has a root between 1 and 2.

An approximation for this root is found using the iteration formula

$$x_{n+1} = \left(2 + \frac{1}{x_n}\right)^{\frac{1}{3}}, \text{ with } x_0 = 1.5.$$

b By calculating the values of x_1, x_2, x_3 and x_4 find an approximation to this root, giving your answer to 3 decimal places.

c By considering the change of sign of $f(x)$ in a suitable interval, verify that your answer to part **b** is correct to 3 decimal places. *E*

25 **a** By sketching the graphs of $y = -x$ and $y = \ln x$, $x > 0$, on the same axes, show that the solution to the equation $x + \ln x = 0$ lies between 0 and 1.

b Show that $x + \ln x = 0$ may be written in the form $x = \frac{(2x - \ln x)}{3}$.

c Use the iterative formula

$$x_{n+1} = \frac{(2x_n - \ln x_n)}{3}, \quad x_0 = 1,$$

to find the solution of $x + \ln x = 0$ correct to 5 decimal places.

26 Show that the equation $e^{2x} - 8x = 0$ has a root k between $x = 1$ and $x = 2$.

The iterative formula

$$x_n = \tfrac{1}{2}\ln 8x, \quad x_0 = 1.2,$$

is used to find to find an approximation for k.

a Calculate the values of x_1, x_2 and x_3, giving your answers to 3 decimal places.

b Show that, to 3 decimal places, $k = 1.077$.

c Deduce the value, to 2 decimal places, of one of the roots of $e^x = 4x$.

27 The curve C has equation $y = x^5 - 1$. The tangent to C at the point $P(-1, -2)$ meets the curve again at the point Q, whose x-coordinate is k.

a Show that k is a root of the equation $x^5 - 5x - 4 = 0$.

b Show that $x^5 - 5x - 4 = 0$ can be rearranged in the form $x = \sqrt[4]{5 + \frac{4}{x}}$.

The iterative formula

$$x_{n+1} = \sqrt[4]{5 + \frac{4}{x_n}}, \quad x_0 = 1.5,$$

is used to find to find an approximation for k.

c Write down the values of x_1, x_2, x_3 and x_4, giving your answers to 5 significant figures.

d Show that $k = 1.6506$ correct to 5 significant figures.

28 $f(x) = 2x^3 - x - 4$.

a Show that the equation $f(x) = 0$ can be written as

$$x = \sqrt{\left(\frac{2}{x} + \frac{1}{2}\right)}.$$

The equation $2x^3 - x - 4 = 0$ has a root between 1.35 and 1.4.

b Use the iteration formula

$$x_{n+1} = \sqrt{\left(\frac{2}{x_n} + \frac{1}{2}\right)},$$

with $x_0 = 1.35$, to find, to 2 decimal places, the value of x_1, x_2 and x_3.

The only real root of $f(x) = 0$ is α.

c By choosing a suitable interval, prove that $\alpha = 1.392$, to 3 decimal places. *E*

29 The function f is defined by

$$f : x \to -5 + 4e^{2x}, \quad x \in \mathbb{R}, \quad x > 0.$$

a Show that the inverse function of f is defined by

$$f^{-1}: x \to \frac{1}{2}\ln\left(\frac{x + 5}{4}\right),$$

and write down the domain of f^{-1}.

b Write down the range of f^{-1}.

The graphs of $y = \frac{1}{2}x$ and $y = f^{-1}(x)$, drawn on the same axes, meet at the point with the x-coordinate k.

The iterative formula

$$x_{n+1} = \ln\left(\frac{x_n + 5}{4}\right), \quad x_0 = 0.3,$$

is used to find to find an approximation for k.

c Calculate the values of x_1 and x_2, giving your answers to 4 decimal places.

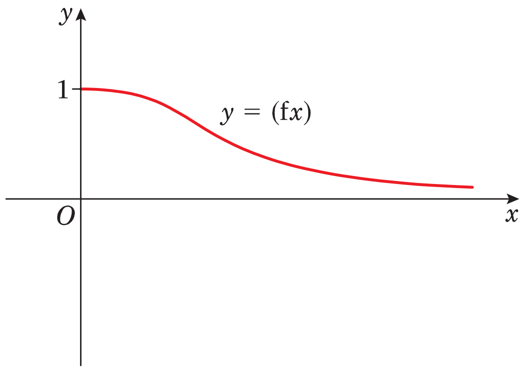
d Continue the iterative process until there are two values which are the same to 4 decimal places.

e Prove that this value does give k, correct to 4 decimal places.

30 The graph of the function f, defined by

$$f: x \to \frac{1}{1+x^2}, \quad x \in \mathbb{R}, \quad x \geqslant 0,$$

is shown.



a Copy the sketch and add to it the graph of $y = f^{-1}(x)$, showing the coordinates of the point where it meets the x-axis.

The two curves meet in the point A, with x-coordinate k.

b Explain why k is a solution of the equation $x = \frac{1}{1+x^2}$.

The iterative formula

$$x_{n+1} = \frac{1}{1+x_n^2}, \quad x_0 = 0.7$$

is used to find an approximation for k.

c Calculate the values of x_1, x_2, x_3 and x_4, giving your answers to 4 decimal places.

d Show that $k = 0.682$, correct to 3 decimal places.