Core 4 Jan 07

Q1 $f(x) = (2-5x)^{-2}$

$$(2-5x)^{-2} = 2^{-2}\left(1-\frac{5}{2}x\right)^{-2} = \frac{1}{4}\left(1-\frac{5}{2}x\right)^{-2}$$

$$= \frac{1}{4}\left(1 + (-2)\left(-\frac{5}{2}x\right) + \frac{(-2)(-2-1)}{2!}\left(-\frac{5}{2}x\right)^2 + \frac{(-2)(-2-1)(-2-2)}{3!}\left(-\frac{5}{2}x\right)^3 + \dots\right)$$

$$= \frac{1}{4}\left(1 + 5x + \frac{\cancel{2}\times 3}{\cancel{2}} \times \frac{5^2}{2^2}x^2 + \frac{\cancel{2}\times\cancel{3}\times 4}{\cancel{2}\times\cancel{3}} \times \frac{5^3}{2^3}x^3 + \dots\right)$$

$$= \frac{1}{4}\left(1 + 5x + \frac{75}{4}x^2 + \frac{125}{2}x^3 + \dots\right)$$

$$= \frac{1}{4} + \frac{5}{4}x + \frac{75}{16}x^2 + \frac{125}{8}x^3 + \dots \qquad (5)$$

Q2 (a) $\text{Volume} = \pi\int_a^b y^2\,dx = \pi\int_{-\frac{1}{4}}^{\frac{1}{2}} \frac{1}{9(1+2x)^2}\,dx$

$$= \frac{\pi}{9}\int_{-\frac{1}{4}}^{\frac{1}{2}} (1+2x)^{-2}\,dx = \frac{1}{9}\left[-\frac{1}{2}(1+2x)^{-1}\right]_{-\frac{1}{4}}^{\frac{1}{2}}$$

$$= -\frac{\pi}{18}\left[\left(1+2\times\frac{1}{2}\right)^{-1} - \left(1-2\times\frac{1}{4}\right)^{-1}\right]$$

$\left(1-2\times\frac{1}{4}\right)^{-1} = \left(\frac{1}{2}\right)^{-1} = 2$

$\left(1+2\times\frac{1}{2}\right)^{-1} = 2^{-1} = \frac{1}{2}$

$$= -\frac{\pi}{18}\left(\frac{1}{2} - 2\right)$$

$$= -\frac{\pi}{\cancel{18}_6} \times -\frac{\cancel{3}^1}{2}$$

$$= \frac{\pi}{12}$$

(b) scale factor $SF = \frac{\text{real length}}{\text{model}} = \frac{3}{\frac{3}{4}} = 4$

$\text{Volume} = \text{Volume of model} \times SF^3$

$$= \frac{\pi}{12} \times 4^3 = \frac{16\pi}{3}$$

Jan 07

Q3 $x = 7\cos t - \cos 7t, \quad y = 7\sin t - \sin 7t$

(a) $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$\frac{dx}{dt} = -7\sin t + 7\sin 7t$

$\frac{dy}{dt} = 7\cos t - 7\cos 7t$

$$= \frac{(7\cos t - 7\cos 7t)}{(-7\sin t + 7\sin 7t)}$$

$$= \frac{\cos 7t - \cos t}{-\sin 7t + \sin t}$$

(b) gradient of the normal $(m_n) = -\frac{1}{\text{gradient of tangent } (m_t)}$

$$m_t = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = \frac{\cos 7\times\frac{\pi}{6} - \cos\frac{\pi}{6}}{-\sin\frac{7\pi}{6} + \sin\frac{\pi}{6}} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

$$m_n = -\frac{1}{m_t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Need the coordinates of the point where $t = \frac{\pi}{6}$

$$x\left(\frac{\pi}{6}\right) = 7\cos\frac{\pi}{6} - \cos\frac{7\pi}{6} = 4\sqrt{3}$$

$$y\left(\frac{\pi}{6}\right) = 7\sin\frac{\pi}{6} - \sin\frac{7\pi}{6} = 4$$

The equation: $y - y_1 = m(x - x_1)$

$$y - 4 = \frac{\sqrt{3}}{3}\left(x - 4\sqrt{3}\right)$$

$\frac{\sqrt{3}}{3} \times 4\sqrt{3} = 4$

$$y = \frac{\sqrt{3}}{3}x - 4 + 4$$

$$y = \frac{\sqrt{3}}{3}x$$

Q4 (a) $\frac{2x-1}{(x-1)(2x-3)} \equiv \frac{A}{(x-1)} + \frac{B}{(2x-3)} = \frac{A(2x-3) + B(x-1)}{(x-1)(2x-3)}$

COMPARING THE NUMERATORS: $2x-1 \equiv A(2x-3) + B(x-1)$

let $x = \frac{3}{2}$, then $2\times\frac{3}{2} - 1 = B\left(\frac{3}{2}-1\right)$

$2 = \frac{1}{2}B \implies \underline{B = 4}$

let $x = 1$, then $2\times 1 - 1 = A(2\times 1 - 3)$

$1 = -A \implies \underline{A = -1}$

ANSWER: $\frac{2x-1}{(x-1)(2x-3)} = \frac{-1}{(x-1)} + \frac{4}{(2x-3)}$

(b) $(2x-3)(x-1)\frac{dy}{dx} = (2x-1)y$

Separating the variables:

$$\int\frac{dy}{y} = \int\frac{2x-1}{(2x-3)(x-1)}\,dx$$

Using part (a)

$$\ln y = \int\frac{-1}{(x-1)}\,dx + \int\frac{4}{(2x-3)}\,dx$$

$$\ln y = -\ln|x-1| + 4\times\frac{1}{2}\ln|2x-3| + C$$

$$\ln y = -\ln(x-1) + 2\ln(2x-3) + C$$

When: $x = 2, y = 10 \implies \ln 10 = -\ln 1 + 2\ln 1 + C \qquad \ln 1 = 0$

$$C = \ln 10$$

Hence, $\ln y = -\ln(x-1) + \ln(2x-3)^2 + \ln 10$

$$\ln y = \ln\left[\frac{(2x-3)^2}{x-1} \times 10\right] \implies y = \frac{10(2x-3)^2}{x-1}$$

ANSWER:

Q5 (a) $\sin x + \cos y = 0.5 \qquad (1)$

Differentiating explicitly with respect to x:

$$\cos x - \sin y\frac{dy}{dx} = 0$$

$$\sin y\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin y}$$

(b) $\frac{dy}{dx} = 0 \implies \frac{\cos x}{\sin y} = 0$

i.e. $\cos x = 0$

$$x = \frac{\pm\pi}{2}$$

INTO (1) $\underline{x = \frac{\pi}{2}}$,

$$\sin\frac{\pi}{2} + \cos y = 0.5$$

$$1 + \cos y = 0.5$$

$$\cos y = -0.5$$

$$y = \pm\frac{2\pi}{3}$$

$\sin\left(-\frac{\pi}{2}\right) + \cos y$

$\underline{x = -\frac{\pi}{2}}$ $-1 + \cos y = 0.$

$\cos y = 1.5$

$\cos y \neq 1.5$

No solution

ANSWER: $\left(\frac{\pi}{3}, -\frac{2\pi}{3}\right)$ or $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$

Q6 (a) $y = 2^x$

$\ln y = \ln 2^x$

$\ln y = x \ln 2$ $\quad \Rightarrow 2^x = e^{x\ln 2}$

$y = e^{x\ln 2}$

$$\frac{dy}{dx} = e^{x\ln 2} \times \ln 2$$

$$\frac{dy}{dx} = 2^x \times \ln 2$$

(b) $y = 2^{(x^2)}$

$$\frac{dy}{dx} = 2^{x^2} \ln 2 \times 2x$$

$$= 2x \times 2^{(x^2)} \ln 2$$

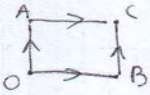
At $(2, 16)$ $\quad = 2 \times 2 \times 2^4 \ln 2$

$$= 64 \ln 2$$

Q7 $\underline{a} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$

(a) $\underline{c} = \underline{a} + \underline{b} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = 3\underline{i} + 3\underline{j} - 3\underline{k}$

(b)



Need to show:
$\vec{OA} = \vec{BC}$, $\vec{OB} = \vec{AC}$
and $\angle AOB = 90°$

$\vec{OA} = \underline{a} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$\vec{BC} = \underline{c} - \underline{b} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ $\quad \Big\} \ \vec{OA} = \vec{BC}$

$\vec{OB} = \underline{b} = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$

$\vec{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$ $\quad \Big\} \ \vec{OB} = \vec{AC}$

If $\cos \angle AOB = \dfrac{\underline{a} \cdot \underline{b}}{|\underline{a}| \times |\underline{b}|} = 0$ ie $\underline{a} \cdot \underline{b} = 0$

then $\angle AOB = 90°$

$$\underline{a} \cdot \underline{b} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$$

$$= 2 \times 1 + 2 \times 1 + 1 \times (-4)$$

$$= 0 \quad \Rightarrow \quad \angle AOB = 90°$$

Therefore OACB is a rectangle

Q7. (c) diagonals of a rectangle bisect each other so need the midpoint, D, of $\vec{OC} = \underline{c}$

$\underline{d} = \vec{OD} = \frac{1}{2}\underline{c} = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix} \qquad D\left(\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}\right)$

(d)

Angle α is an angle between $\vec{OC}$ and $\vec{BA}$

$\vec{BA} = \underline{a} - \underline{b} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

$|\vec{BA}| = \sqrt{1^2 + 1^2 + 5^2} = \sqrt{27} = 3\sqrt{3}$

$\vec{OC} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$

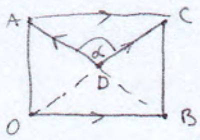
$|\vec{OC}| = \sqrt{3^2 + 3^2 + (-3)^2} = \sqrt{27} = 3\sqrt{3}$

$$\cos \alpha = \frac{\vec{BA} \cdot \vec{OC}}{|\vec{BA}| \times |\vec{OC}|} = \frac{\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}}{3\sqrt{3} \times 3\sqrt{3}} = \frac{1 \times 3 + 1 \times 3 + 5 \times (-3)}{27}$$

$$= -\frac{1}{3}$$

$\alpha = \arccos\left(-\frac{1}{3}\right)$
$= 109.5°$

Q7 Alternatively



$\vec{DC} = \underline{c} - \underline{d} = \frac{1}{2}\underline{c} = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix}$

$$\cos \alpha = \frac{\vec{DC} \cdot \vec{DA}}{|\vec{DC}| \times |\vec{DA}|}$$

$\vec{DA} = \underline{a} - \underline{d} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{5}{2} \end{pmatrix}$

$|\vec{DC}| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{27}{4}}$

$|\vec{DA}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{\frac{27}{4}}$

$$\cos \alpha = \frac{\begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{5}{2} \end{pmatrix}}{\sqrt{\frac{27}{4}} \times \sqrt{\frac{27}{4}}} = \frac{\frac{3}{2} \times \frac{1}{2} + \frac{3}{2} \times \frac{1}{2} - \frac{3}{2} \times \frac{5}{2}}{\frac{27}{4}} = \cdots$$

$\alpha = \arccos\left(-\frac{1}{3}\right)$

Q8 $I = \int_0^5 e^{\sqrt{3x+1}}\,dx$

(a)

x	0	1	2	3	4	5
y	e^1	e^2	$e^{\sqrt{7}}$	$e^{\sqrt{10}}$	$e^{\sqrt{13}}$	e^4

(b) $I \approx \frac{h}{2}\left[y_0 + y_n + 2(y_1 + \dots + y_{n-1})\right]$

$h=1$

$= \frac{1}{2}\left[e^1 + e^4 + 2\left(e^2 + e^{\sqrt{7}} + e^{\sqrt{10}} + e^{\sqrt{13}}\right)\right]$

$= 110.6$ (4sf)

(c) Let $t = \sqrt{3x+1}$ then $t^2 = 3x+1$

differentiating implicitly with respect to x:

$2t\frac{dt}{dx} = 3$

So $\frac{2}{3}t\,dt = dx$

$I = \int_0^5 e^{\sqrt{3x+1}}\,dx$

Substituting in for t:

Limits change:

$t(0) = \sqrt{1} = 1$

$t(5) = \sqrt{3\times5+1} = \sqrt{16} = 4$

$I = \int_{t(0)}^{t(5)} e^t \times \frac{2}{3}t\,dt$

$= \frac{2}{3}\int_1^4 te^t\,dt$

(d) $\int te^t\,dt$

Integrating by parts:

$u = t$ $\quad \frac{dv}{dt} = e^t$

$\frac{du}{dt} = 1$ $\quad v = e^t$

Q8 (c) $\int te^t\,dt = te^t - \int e^t\,dt$

$= te^t - e^t$

So $I = \frac{2}{3}\left[te^t - e^t\right]_1^4 = \frac{2}{3}\left[(4e^4 - e^4) - (e^1 - e^1)\right]$

$= \frac{2}{\cancel{3}} \times \cancel{3}e^4$

$= 2e^4$

$= 109.2$ (4sf)