

C3 JAN 11

1. (a) Express $7 \cos x - 24 \sin x$ in the form $R \cos(x + \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.
Give the value of α to 3 decimal places.

(3)

(b) Hence write down the minimum value of $7 \cos x - 24 \sin x$.

(1)

(c) Solve, for $0 \leq x < 2\pi$, the equation

$$7 \cos x - 24 \sin x = 10$$

giving your answers to 2 decimal places.

(5)

$$\begin{aligned} a) \quad R \cos(x + \alpha) &= R \cos x \cos \alpha - R \sin x \sin \alpha \\ R \cos(x + \alpha) &= 7 \cos x - 24 \sin x \end{aligned}$$

$$\frac{R \sin \alpha = 24}{R \cos \alpha = 7} \Rightarrow \tan \alpha = \frac{24}{7} \Rightarrow \alpha = 1.287^\circ$$

$$b) \quad R^2 = 24^2 + 7^2 \Rightarrow R = 25$$

$$25 \cos(x + 1.287) \quad \text{min} = -25$$

$$c) \quad 25 \cos(x + 1.287) = 10$$

$$x + 1.287 = \cos^{-1}\left(\frac{10}{25}\right) = 1.107, 5.176, 7.176$$

$$-1.287 \Rightarrow x = 3.84^\circ, 6.16^\circ$$

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2. (a) Express

$$\frac{4x-1}{2(x-1)} - \frac{3}{2(x-1)(2x-1)}$$

as a single fraction in its simplest form.

(4)

Given that

$$f(x) = \frac{4x-1}{2(x-1)} - \frac{3}{2(x-1)(2x-1)} - 2, \quad x > 1,$$

(b) show that

$$f(x) = \frac{3}{2x-1}$$

(2)

(c) Hence differentiate $f(x)$ and find $f'(2)$.

(3)

$$\begin{aligned} a) \quad \frac{(4x-1)(2x-1) - 3}{2(x-1)(2x-1)} &= \frac{8x^2 - 6x + 1 - 3}{2(x-1)(2x-1)} \\ &= \frac{2(4x+1)(x-1)}{2(x-1)(2x-1)} = \frac{4x+1}{2x-1} \end{aligned}$$

$$b) \quad f(x) = \frac{4x+1}{2x-1} - 2 \times \frac{(2x-1)}{1 \times (2x-1)} = \frac{4x+1 - 4x + 2}{2x-1}$$

$$f(x) = \frac{3}{2x-1} \quad \#$$

$$c) \quad f(x) = 3(2x-1)^{-1} \Rightarrow f'(x) = -3 \times 2(2x-1)^{-2}$$

$$f'(x) = \frac{-6}{(2x-1)^2} \Rightarrow f'(2) = \frac{-6}{3^2} = \frac{-2}{3}$$

3. Find all the solutions of

$$2 \cos 2\theta = 1 - 2 \sin \theta$$

in the interval $0 \leq \theta < 360^\circ$.

(6)

$$2 - 4 \sin^2 \theta = 1 - 2 \sin \theta$$

$$\Rightarrow 4 \sin^2 \theta - 2 \sin \theta - 1 = 0$$

$$\Rightarrow \sin^2 \theta - \frac{1}{2} \sin \theta = \frac{1}{4}$$

$$\Rightarrow \left(\sin \theta - \frac{1}{4}\right)^2 = \frac{1}{4} + \frac{1}{16} = \frac{5}{16}$$

$$\Rightarrow \sin \theta = \frac{1}{4} \pm \frac{\sqrt{5}}{4} = \frac{1 \pm \sqrt{5}}{4}$$

$$\theta = \sin^{-1}\left(\frac{1+\sqrt{5}}{4}\right) = \underline{54^\circ}, \underline{126^\circ}$$

$$\theta = \sin^{-1}\left(\frac{1-\sqrt{5}}{4}\right) = \cancel{-18^\circ}, \underline{198^\circ}, \underline{342^\circ}$$

4. Joan brings a cup of hot tea into a room and places the cup on a table. At time t minutes after Joan places the cup on the table, the temperature, $\theta^\circ\text{C}$, of the tea is modelled by the equation

$$\theta = 20 + Ae^{-kt}$$

where A and k are positive constants.

Given that the initial temperature of the tea was 90°C ,

(a) find the value of A .

(2)

The tea takes 5 minutes to decrease in temperature from 90°C to 55°C .

(b) Show that $k = \frac{1}{5} \ln 2$.

(3)

(c) Find the rate at which the temperature of the tea is decreasing at the instant when $t = 10$. Give your answer, in $^\circ\text{C}$ per minute, to 3 decimal places.

(3)

$$\text{a) } 90 = 20 + A \Rightarrow \underline{A = 70}$$

$$\text{b) } 55 = 20 + 70e^{-5k} \Rightarrow 35 = 70e^{-5k}$$

$$\Rightarrow e^{-5k} = \frac{1}{2} \Rightarrow -5k = \ln\left(\frac{1}{2}\right) = \ln(2^{-1}) = -\ln 2$$

$$\Rightarrow k = \frac{1}{5} \ln 2 \quad \#$$

$$\text{c) } \frac{d\theta}{dt} = (70 \times -\frac{1}{5} \ln 2) e^{-\frac{1}{5} \ln 2 t} = -14 \ln 2 e^{-\frac{1}{5} \ln 2 t}$$

$$t=10 \quad \frac{d\theta}{dt} = (-14 \ln 2) e^{-2 \ln 2} = -2.426$$

\Rightarrow decreasing at a rate of $2.426^\circ\text{C}/\text{min}$

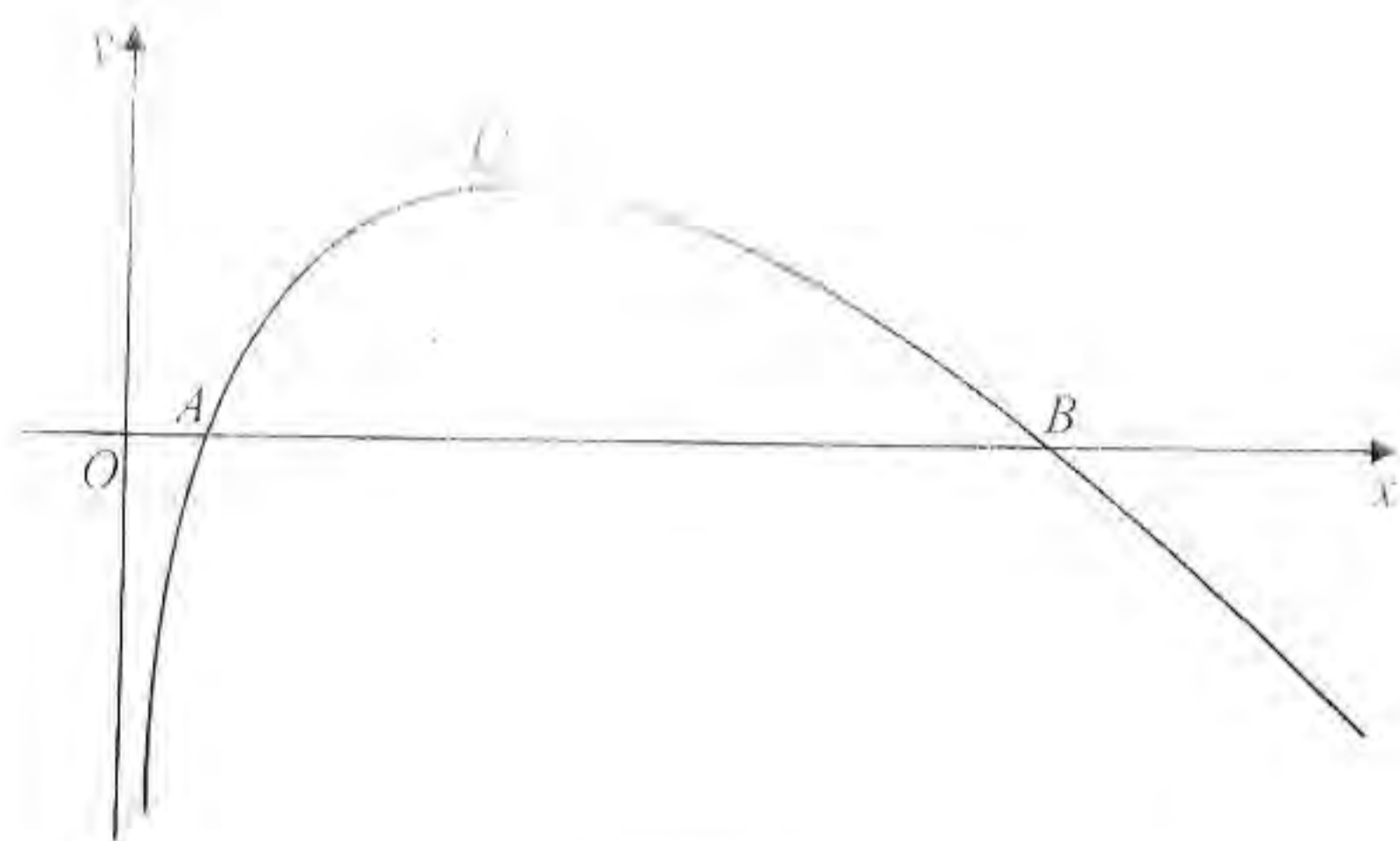


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = f(x)$, where

$$f(x) = (8-x)\ln x, \quad x > 0$$

The curve cuts the x -axis at the points A and B and has a maximum turning point at Q , as shown in Figure 1.

(a) Write down the coordinates of A and the coordinates of B .

(2)

(b) Find $f'(x)$.

(3)

(c) Show that the x -coordinate of Q lies between 3.5 and 3.6

(2)

(d) Show that the x -coordinate of Q is the solution of

$$x = \frac{8}{1 + \ln x}$$

(3)

To find an approximation for the x -coordinate of Q , the iteration formula

$$x_{n+1} = \frac{8}{1 + \ln x_n}$$

is used.

(e) Taking $x_0 = 3.55$, find the values of x_1 , x_2 and x_3 .

Give your answers to 3 decimal places.

(3)

5a) $(8-x)\ln x = 0$ when $x=8, x=1$ $A(1,0)$ $B(8,0)$

b) $f'(x) = (8-x)' \ln x + (8-x)(\ln x)'$
 $f'(x) = -\ln x + \frac{8-x}{x}$

c) $f'(3.5) = 0.033 \Rightarrow$ Increasing function at $x=3.5$
 $f'(3.6) = -0.059 \Rightarrow$ decreasing function at $x=3.6$
 \Rightarrow turning point between 3.5 and 3.6.

d) $f'(x) = 0 \Rightarrow \frac{8-x}{x} = \ln x \Rightarrow 8-x = x \ln x$
 $\Rightarrow 8 = x + x \ln x \Rightarrow 8 = x(1 + \ln x) \Rightarrow x = \frac{8}{1 + \ln x}$

e) $x_0 = \underline{3.55}$; $x_1 = \underline{3.529}$; $x_2 = \underline{3.538}$; $x_3 = \underline{3.534}$

6. The function f is defined by

$$f(x) = \frac{3-2x}{x-5}, \quad x \in \mathbb{R}, x \neq 5$$

(a) Find $f^{-1}(x)$.

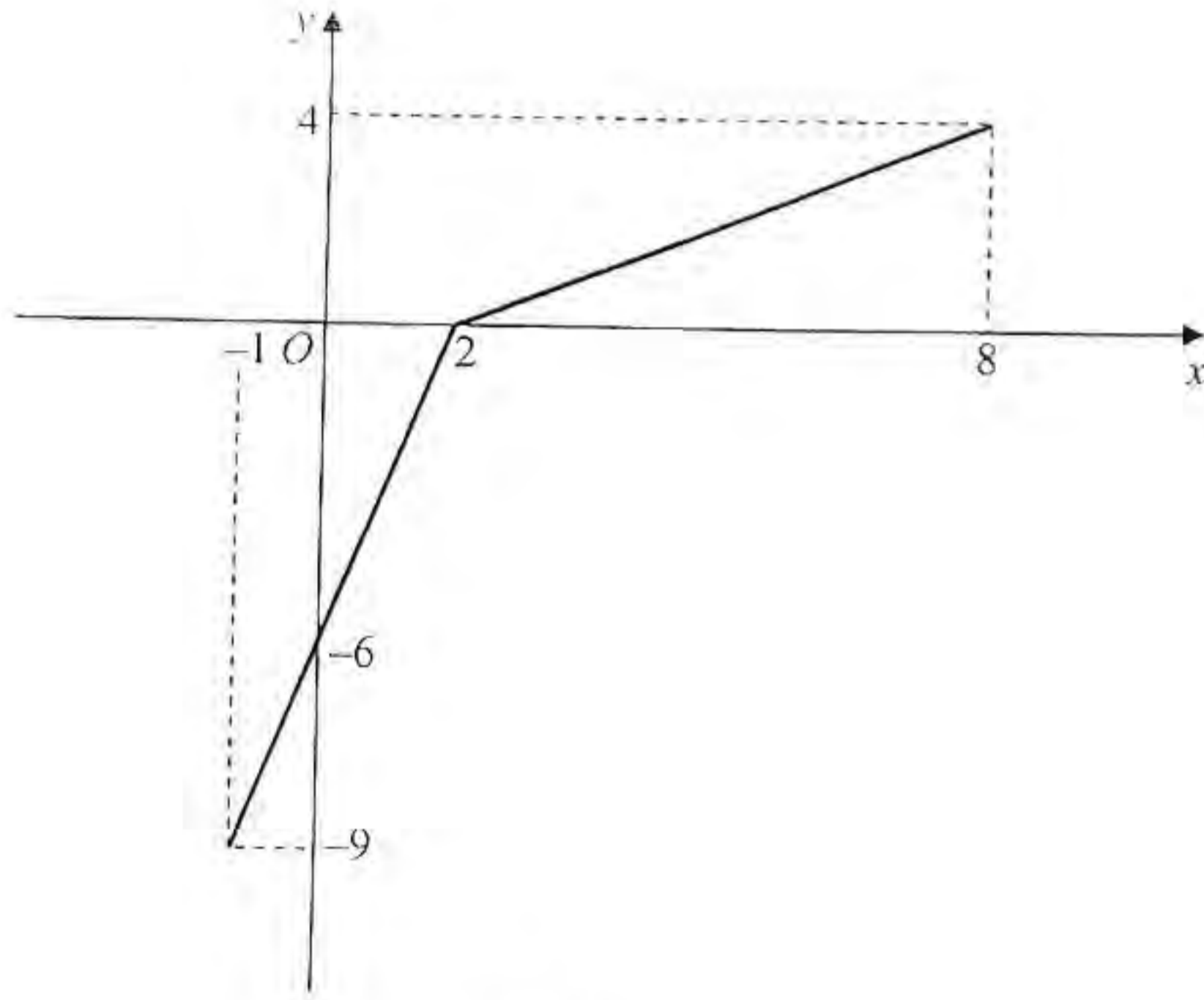


Figure 2

The function g has domain $-1 \leq x \leq 8$, and is linear from $(-1, -9)$ to $(2, 0)$ and from $(2, 0)$ to $(8, 4)$. Figure 2 shows a sketch of the graph of $y = g(x)$.

(b) Write down the range of g .

(1)

(c) Find $gg(2)$.

(2)

(d) Find $fg(8)$.

(2)

(e) On separate diagrams, sketch the graph with equation

(i) $y = |g(x)|$,

(ii) $y = g^{-1}(x)$.

Show on each sketch the coordinates of each point at which the graph meets or cuts the axes.

(4)

(f) State the domain of the inverse function g^{-1} .

(1)

$$6) \quad x = \frac{3-2y}{y-5} \Rightarrow xy - 5x = 3 - 2y \Rightarrow xy + 2y = 5x + 3$$

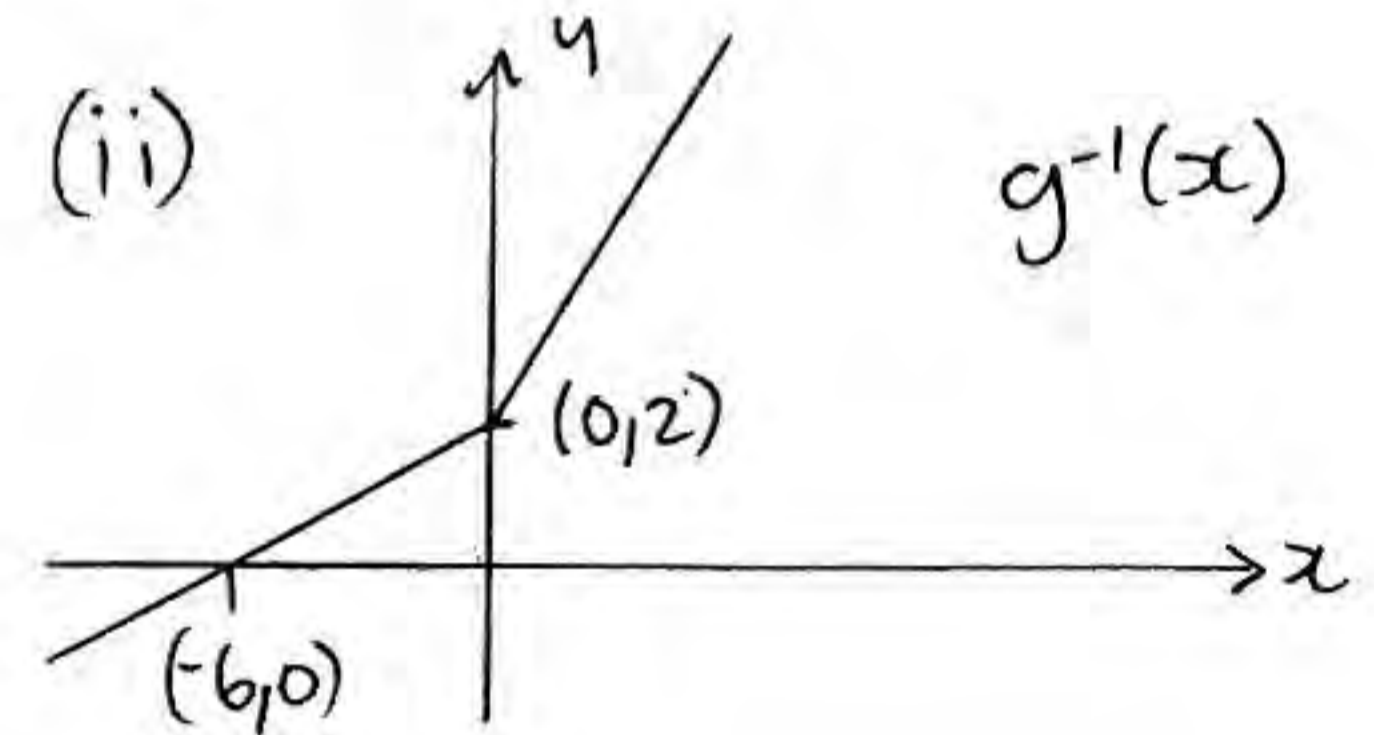
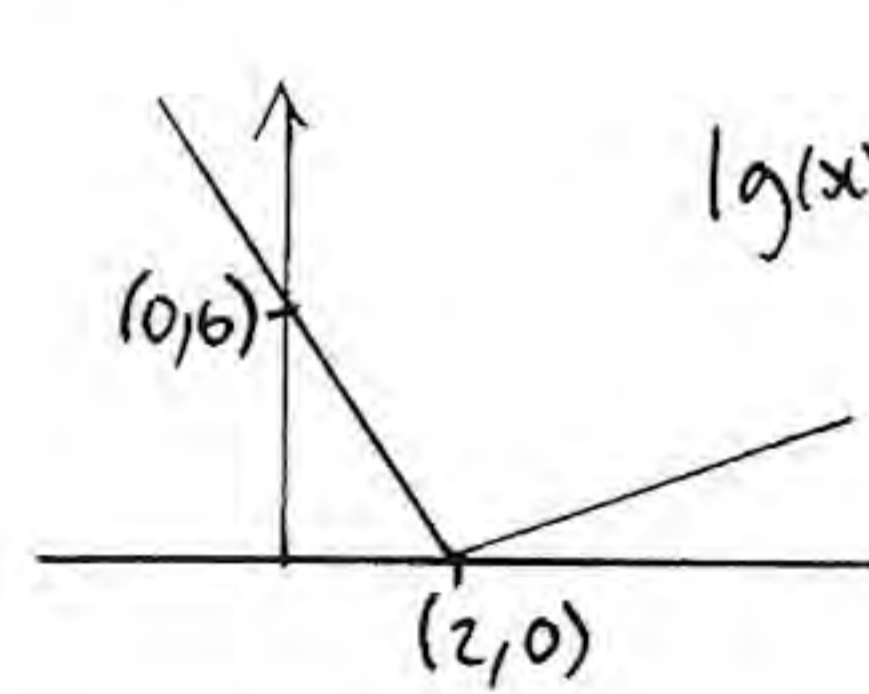
$$\Rightarrow (x+2)y = 5x+3 \Rightarrow y = f^{-1}(x) = \frac{5x+3}{x+2}$$

b) $-9 \leq g(x) \leq 4 \quad g(x) \in \mathbb{R}$.

c) $gg(2) = g(0) = -6$

d) $fg(8) = f(4) = \frac{3-8}{4-5} = \frac{-5}{-1} = 5$

e)



f) domain $g^{-1} = \text{range } g \Rightarrow -9 \leq x \leq 4 \quad x \in \mathbb{R}$

7. The curve C has equation

$$y = \frac{3 + \sin 2x}{2 + \cos 2x}$$

(a) Show that

$$\frac{dy}{dx} = \frac{6 \sin 2x + 4 \cos 2x + 2}{(2 + \cos 2x)^2}$$

(4)

(b) Find an equation of the tangent to C at the point on C where $x = \frac{\pi}{2}$.

Write your answer in the form $y = ax + b$, where a and b are exact constants.

(4)

$$a) \frac{dy}{dx} = \frac{(3 + \sin 2x)'(2 + \cos 2x) - (3 + \sin 2x)(2 + \cos 2x)'}{(2 + \cos 2x)^2}$$

$$= \frac{2 \cos 2x (2 + \cos 2x) - (3 + \sin 2x)(-2 \sin 2x)}{(2 + \cos 2x)^2}$$

$$= \frac{4 \cos 2x + 2 \cos^2 2x + 6 \sin 2x + 2 \sin^2 2x}{(2 + \cos 2x)^2}$$

$$= \frac{6 \sin 2x + 4 \cos 2x + 2(\sin^2 2x + \cos^2 2x)}{(2 + \cos 2x)^2} \quad \overset{=1}{}$$

$$= \frac{6 \sin 2x + 4 \cos 2x + 2}{(2 + \cos 2x)^2} \quad \#$$

$$b) x = \frac{\pi}{2}; y = 3; M_t = -2$$

$$y - 3 = -2(x - \frac{\pi}{2}) \Rightarrow y = -2x + \pi + 3$$

Exam blank

8. (a) Given that

$$\frac{d}{dx}(\cos x) = -\sin x$$

show that $\frac{d}{dx}(\sec x) = \sec x \tan x$.

(3)

Given that

$$x = \sec 2y$$

(b) find $\frac{dx}{dy}$ in terms of y .

(2)

(c) Hence find $\frac{dy}{dx}$ in terms of x .

(4)

$$a) \frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{d}{dx}(\cos x)^{-1}$$

$$= -(\cos x)^{-2} \times -\sin x = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$$

$$= \sec x \tan x \quad \#$$

$$b) x = \sec 2y$$

$$\frac{dx}{dy} = 2 \sec 2y \tan 2y$$

$$c) \frac{dy}{dx} = \frac{1}{2 \sec 2y \tan 2y}$$

$$\sec 2y = x$$

$$\tan^2 2y + 1 = \sec^2 2y$$

$$\frac{dy}{dx} = \frac{1}{2x\sqrt{x^2-1}}$$

$$\tan^2 2y = x^2 - 1$$

$$\tan 2y = \sqrt{x^2 - 1}$$