

C3 JUNE 2013 INT

1. Express

$$\frac{3x+5}{x^2+x-12} - \frac{2}{x-3}$$

as a single fraction in its simplest form.

(4)

$$\frac{3x+5}{(x+4)(x-3)} - \frac{2(x+4)}{(x+4)(x-3)} = \frac{3x+5-2x-8}{(x+4)(x-3)}$$

$$= \frac{x-3}{(x+4)(x-3)} = \frac{1}{x+4}$$

Q1) MUCH EASIER THAN UK PAPER

2.

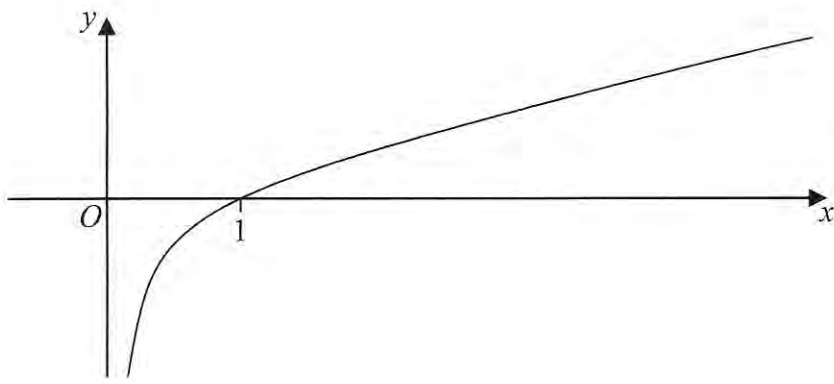


Figure 1

Figure 1 shows a sketch of the curve with equation $y = f(x)$, $x > 0$, where f is an increasing function of x . The curve crosses the x -axis at the point $(1, 0)$ and the line $x = 0$ is an asymptote to the curve.

On separate diagrams, sketch the curve with equation

(a) $y = f(2x)$, $x > 0$

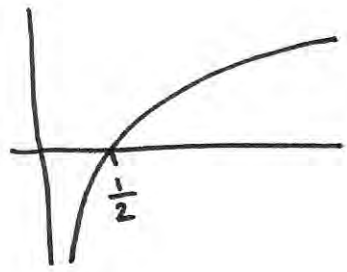
(2)

(b) $y = |f(x)|$, $x > 0$

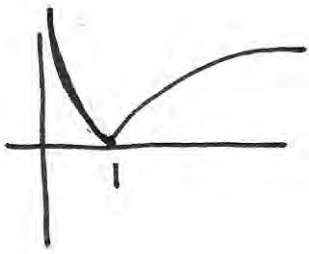
(3)

Indicate clearly on each sketch the coordinates of the point at which the curve crosses or meets the x -axis.

a) $y = f(2x)$
 $\rightarrow 2 \leftarrow$



b) $y = |f(x)|$



Q2. slightly easier than UK. No asymptotes.

$$f(x) = 7\cos x + \sin x$$

Given that $f(x) = R\cos(x - \alpha)$, where $R > 0$ and $0 < \alpha < 90^\circ$,

(a) find the exact value of R and the value of α to one decimal place.

(3)

(b) Hence solve the equation

$$7\cos x + \sin x = 5$$

for $0 \leq x < 360^\circ$, giving your answers to one decimal place.

(5)

(c) State the values of k for which the equation

$$7\cos x + \sin x = k$$

has only one solution in the interval $0 \leq x < 360^\circ$

(2)

$$a) R \cos(x - \alpha) = R \cancel{\cos x} \cos \alpha + R \cancel{\sin x} \sin \alpha$$

$$7\cos x + 1\sin x$$

$$\Rightarrow \frac{R \sin \alpha}{R \cos \alpha} = \frac{1}{7} \Rightarrow \tan \alpha = \frac{1}{7} \Rightarrow 8.1^\circ \quad R^2 = 1^2 + 7^2$$

$$\Rightarrow R^2 = 50$$

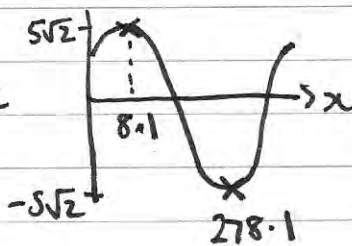
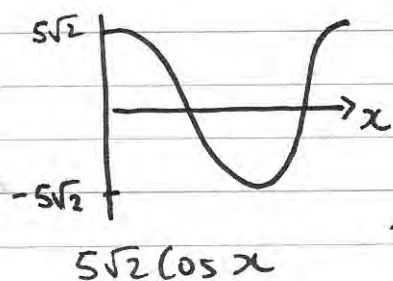
$$\Rightarrow R = 5\sqrt{2}$$

$$b) 5\sqrt{2} \cos(x - 8.1) = 5$$

$$\Rightarrow \cos(x - 8.1) = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \cos(x - 8.1) =$$

$$\Rightarrow x - 8.1 = 45, 315 \quad \therefore x = \underline{53.1}; \underline{323.1}^\circ$$

$$c) 5\sqrt{2} \cos(x - 8.1) = k$$



$$\therefore k = 5\sqrt{2}, -5\sqrt{2}$$

$$x = 8.1, 278.1$$

(no asked)

Q3. MUCH! easier than UK.

4. The functions f and g are defined by

$$f: x \mapsto 2|x| + 3, \quad x \in \mathbb{R},$$

$$g: x \mapsto 3 - 4x, \quad x \in \mathbb{R}$$

(a) State the range of f .

(2)

(b) Find $fg(1)$.

(2)

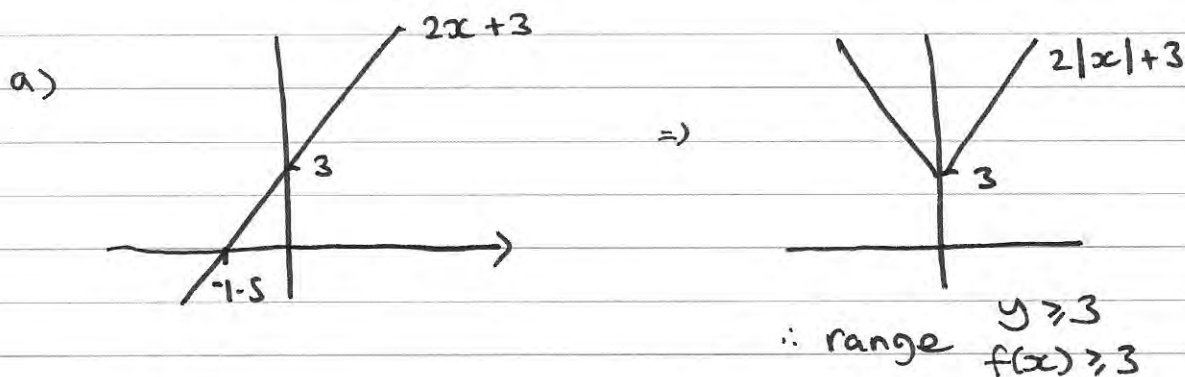
(c) Find g^{-1} , the inverse function of g .

(2)

(d) Solve the equation

$$gg(x) + [g(x)]^2 = 0$$

(5)



b) $g(1) = -1 \quad \therefore fg(1) = f(-1) = 5$

c) $y = 3 - 4x \Rightarrow x = \frac{3 - y}{4} \Rightarrow 4y = 3 - x$
 $\therefore y = g^{-1} = \frac{3 - x}{4}$

d) $gg(x) = 3 - 4(3 - 4x) = 3 - 12 + 16x = 16x - 9$

$$[g(x)]^2 = (3 - 4x)^2 = 9 - 24x + 16x^2$$

$$\Rightarrow 16x - 9 + 9 - 24x + 16x^2 = 0$$

$$\Rightarrow 16x^2 - 8x = 0$$

$$\Rightarrow 8x(2x - 1) = 0$$

$$\therefore x = 0$$

$$x = \frac{1}{2}$$

Q4
Easier
than
uk.

5. (a) Differentiate

$$\frac{\cos 2x}{\sqrt{x}}$$

with respect to x .

(3)

(b) Show that $\frac{d}{dx}(\sec^2 3x)$ can be written in the form

$$\mu(\tan 3x + \tan^3 3x)$$

where μ is a constant.

(3)

(c) Given $x = 2 \sin\left(\frac{y}{3}\right)$, find $\frac{dy}{dx}$ in terms of x , simplifying your answer.

(4)

$$\begin{aligned} \text{a) } u &= \cos 2x & v &= x^{\frac{1}{2}} \\ u' &= -2 \sin 2x & v' &= \frac{1}{2} x^{-\frac{1}{2}} \end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2\sqrt{x} \sin 2x - \frac{\cos 2x}{2\sqrt{x}}}{x}$$

$$\begin{aligned} \text{b) } y &= (\sec 3x)^2 \Rightarrow \frac{dy}{dx} = 2(\sec 3x) \times 3 \sec 3x \tan 3x \\ &= 6 \sec^2 3x \tan 3x \end{aligned}$$

$$\Rightarrow (6 \tan^2 3x + 6) \tan 3x = 6(\tan^3 3x + \tan 3x)$$

$$\text{c) } x = 2 \sin\left(\frac{1}{3}y\right)$$

$$\Rightarrow \frac{dx}{dy} = \frac{2}{3} \cos\left(\frac{1}{3}y\right) \Rightarrow \frac{dy}{dx} = \frac{3}{2 \cos\left(\frac{1}{3}y\right)}$$

$$= \frac{3}{2 \sqrt{\cos^2\left(\frac{1}{3}y\right)}} = \frac{3}{2 \sqrt{1 - \sin^2\left(\frac{1}{3}y\right)}} \therefore \frac{dy}{dx} = \frac{3}{2 \sqrt{1 - \left(\frac{x}{2}\right)^2}}$$

$$\text{alt } \frac{dy}{dx} = \frac{3}{2} \sec\left(\frac{y}{3}\right) = \frac{3}{2} \sec\left[\arcsin\left(\frac{x}{2}\right)\right]$$

(Q5. c) isn't easy, but overall, easier and more standard than UK.

6. (i) Use an appropriate double angle formula to show that

$$\operatorname{cosec} 2x = \lambda \operatorname{cosec} x \sec x,$$

and state the value of the constant λ .

(3)

(ii) Solve, for $0 \leq \theta < 2\pi$, the equation

$$3\sec^2\theta + 3\sec\theta = 2\tan^2\theta$$

You must show all your working. Give your answers in terms of π .

(6)

$$\text{i) } \frac{1}{\sin 2x} = \frac{1}{2\sin x \cos x} = \frac{1}{2} \operatorname{cosec} x \sec x$$

$\lambda = \frac{1}{2}$

$$\text{ii) } 3\sec^2\theta + 3\sec\theta = 2(\sec^2\theta - 1)$$

$$\Rightarrow 3\sec^2\theta + 3\sec\theta = 2\sec^2\theta - 2$$

$$\Rightarrow \sec^2\theta + 3\sec\theta + 2 = 0$$

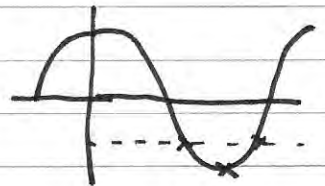
$$\Rightarrow (\sec\theta + 2)(\sec\theta + 1) = 0$$

$$\sec\theta = -2 \quad \sec\theta = -1$$

$$\cos\theta = \frac{-1}{2} \quad \cos\theta = -1$$

$$\therefore \theta = 120^\circ, 240^\circ, 180^\circ$$

$$= \frac{2\pi}{3}, \frac{4\pi}{3}, \pi$$



Q6. v. easy, standard, trig qo.

overall trig was MUCH easier on the paper ↓

7.

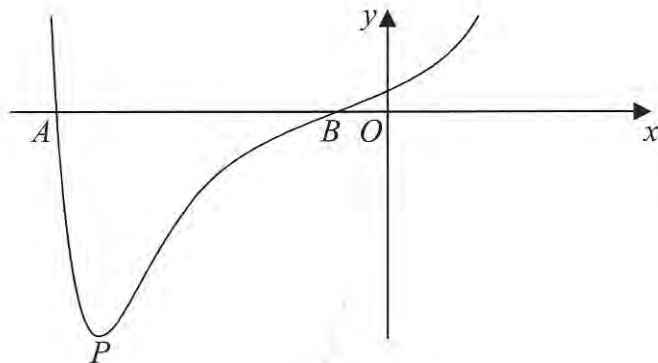


Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = f(x)$ where

$$f(x) = (x^2 + 3x + 1)e^{x^2}$$

The curve cuts the x -axis at points A and B as shown in Figure 2.

(a) Calculate the x coordinate of A and the x coordinate of B , giving your answers to 3 decimal places. (2)

(b) Find $f'(x)$. (3)

The curve has a minimum turning point at the point P as shown in Figure 2.

(c) Show that the x coordinate of P is the solution of

$$x = -\frac{3(2x^2 + 1)}{2(x^2 + 2)} \quad (3)$$

(d) Use the iteration formula

$$x_{n+1} = -\frac{3(2x_n^2 + 1)}{2(x_n^2 + 2)}, \quad \text{with } x_0 = -2.4,$$

to calculate the values of x_1 , x_2 and x_3 , giving your answers to 3 decimal places. (3)

The x coordinate of P is α .

(e) By choosing a suitable interval, prove that $\alpha = -2.43$ to 2 decimal places. (2)

$$a) \quad x^2 + 3x + 1 = 0 \quad \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{4}{4} = 0$$

$$\Rightarrow \left(x + \frac{3}{2}\right)^2 = \frac{5}{4} \quad \Rightarrow \quad x + \frac{3}{2} = \pm \frac{\sqrt{5}}{2} \quad \therefore \quad x = \frac{-3 \pm \sqrt{5}}{2}$$

$$x_A = \frac{-3 - \sqrt{5}}{2} = \underline{\underline{-2.618}} \quad x_B = \frac{-3 + \sqrt{5}}{2} = \underline{\underline{-0.382}}$$

$$b) \quad u = x^2 + 3x + 1 \quad v = e^{x^2}$$

$$u' = 2x + 3$$

$$v' = 2xe^{x^2}$$

$$\therefore f'(x) = (2x+3)e^{x^2} + 2x(x^2+3x+1)e^{x^2}$$

$$c) \quad \text{TP when } f'(x) = 0$$

$$(2x+3 + 2x^3 + 6x^2 + 2x)e^{x^2} = 0$$

$$\Rightarrow 2x^3 + 6x^2 + 4x + 3 = 0$$

$$\Rightarrow 2x^3 + 4x = -6x^2 - 3$$

$$\Rightarrow x \times 2(x^2 + 2) = -3(2x^2 + 1)$$

$$\therefore x = -\frac{3(2x^2 + 1)}{2(x^2 + 2)} \quad \#$$

$$d) \quad x_0 = -2.4$$

$$x_1 = -2.420$$

$$x_2 = -2.427$$

$$x_3 = -2.430$$

$$e) \quad f'(-2.425) = 22. > 0$$

$$f'(-2.435) = -15 < 0$$

\therefore by sign change rule $\alpha = -2.43$ (2dp)

Q7 - similar difficulty

8.

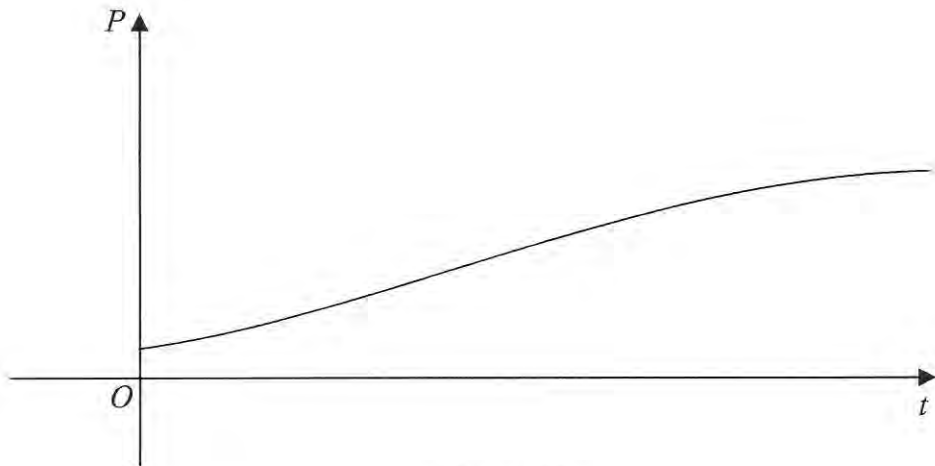


Figure 3

The population of a town is being studied. The population P , at time t years from the start of the study, is assumed to be

$$P = \frac{8000}{1 + 7e^{-kt}}, \quad t \geq 0,$$

where k is a positive constant.

The graph of P against t is shown in Figure 3.

Use the given equation to

(a) find the population at the start of the study, (2)

(b) find a value for the expected upper limit of the population. (1)

Given also that the population reaches 2500 at 3 years from the start of the study,

(c) calculate the value of k to 3 decimal places. (5)

Using this value for k ,

(d) find the population at 10 years from the start of the study, giving your answer to 3 significant figures. (2)

(e) Find, using $\frac{dP}{dt}$, the rate at which the population is growing at 10 years from the start of the study. (3)

$$a) t=0 \Rightarrow P = \frac{8000}{8} = \underline{1000}$$

$$b) \text{ as } t \rightarrow \infty \quad 7e^{-ut} \rightarrow 0 \quad \therefore P \rightarrow \frac{8000}{1} = \underline{8000}$$

$$c) \quad \cancel{2500} = \frac{8000}{1+7e^{-3u}} \Rightarrow 25 + 175e^{-3u} = 80$$

$$\Rightarrow 175e^{-3u} = 55 \Rightarrow e^{-3u} = \frac{11}{35} \Rightarrow -3u = \ln\left(\frac{11}{35}\right)$$

$$\therefore u = \underline{\underline{-\frac{1}{3} \ln\left(\frac{11}{35}\right) = 0.386}}$$

$$d) \quad \frac{8000}{1+7e^{-0.386 \times 10}} = \underline{6970}$$

$$e) \quad P = 8000(1+7e^{-ut})^{-1}$$

$$\Rightarrow \frac{dP}{dt} = -8000(1+7e^{-ut})^{-2} \times (-7ue^{-ut})$$

$$\frac{dP}{dt} = \frac{56000ue^{-ut}}{(1+7e^{-ut})^2}$$

$$\text{at } t=10 \quad \frac{dP}{dt} = 346.2$$

Γ e) is awkward but otherwise a reasonable question ↓

This paper was MUCH easier than uκ !