OCR Maths FP3

Past Paper Pack

2007-2014

1 **(i)** By writing z in the form $r\mathrm{e}^{\mathrm{i}\theta}$, show that $zz^* = |z|^2$. [1]

(ii) Given that $zz^* = 9$, describe the locus of z. [2]

2 A line l has equation $\mathbf{r} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k} + t(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$ and a plane Π has equation $8x - 7y + 10z = 7$. Determine whether l lies in Π, is parallel to Π without intersecting it, or intersects Π at one point. [5]

3 Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 8y = \mathrm{e}^{3x}.$$ [6]

4 Elements of the set $\{p, q, r, s, t\}$ are combined according to the operation table shown below.

	p	q	r	s	t
p	t	s	p	r	q
q	s	p	q	t	r
r	p	q	r	s	t
s	r	t	s	q	p
t	q	r	t	p	s

(i) Verify that $q(st) = (qs)t$. [2]

(ii) Assuming that the associative property holds for all elements, prove that the set $\{p, q, r, s, t\}$, with the operation table shown, forms a group G. [4]

(iii) A multiplicative group H is isomorphic to the group G. The identity element of H is e and another element is d. Write down the elements of H in terms of e and d. [2]

5 **(i)** Use de Moivre's theorem to prove that

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$ [4]

(ii) Hence find the largest positive root of the equation

$$64x^6 - 96x^4 + 36x^2 - 3 = 0,$$

giving your answer in trigonometrical form. [4]

6 Lines l_1 and l_2 have equations

$$\frac{x-3}{2} = \frac{y-4}{-1} = \frac{z+1}{1} \quad \text{and} \quad \frac{x-5}{4} = \frac{y-1}{3} = \frac{z-1}{2}$$

respectively.

(i) Find the equation of the plane Π_1 which contains l_1 and is parallel to l_2, giving your answer in the form $\mathbf{r}.\mathbf{n} = p$. [5]

(ii) Find the equation of the plane Π_2 which contains l_2 and is parallel to l_1, giving your answer in the form $\mathbf{r}.\mathbf{n} = p$. [2]

(iii) Find the distance between the planes Π_1 and Π_2. [2]

(iv) State the relationship between the answer to part **(iii)** and the lines l_1 and l_2. [1]

7 **(i)** Show that $(z - e^{i\phi})(z - e^{-i\phi}) \equiv z^2 - (2\cos\phi)z + 1$. [1]

(ii) Write down the seven roots of the equation $z^7 = 1$ in the form $e^{i\theta}$ and show their positions in an Argand diagram. [4]

(iii) Hence express $z^7 - 1$ as the product of one real linear factor and three real quadratic factors. [5]

8 **(i)** Find the general solution of the differential equation

$$\frac{dy}{dx} + y\tan x = \cos^3 x,$$

expressing y in terms of x in your answer. [8]

(ii) Find the particular solution for which $y = 2$ when $x = \pi$. [2]

9 The set S consists of the numbers 3^n, where $n \in \mathbb{Z}$. ($\mathbb{Z}$ denotes the set of integers $\{0, \pm1, \pm2, \ldots\}$.)

(i) Prove that the elements of S, under multiplication, form a commutative group G. (You may assume that **addition** of integers is associative and commutative.) [6]

(ii) Determine whether or not each of the following subsets of S, under multiplication, forms a subgroup of G, justifying your answers.

(a) The numbers 3^{2n}, where $n \in \mathbb{Z}$. [2]

(b) The numbers 3^n, where $n \in \mathbb{Z}$ and $n \geqslant 0$. [2]

(c) The numbers $3^{(\pm n^2)}$, where $n \in \mathbb{Z}$. [2]

1 **(a)** A group G of order 6 has the combination table shown below.

	e	a	b	p	q	r
e	e	a	b	p	q	r
a	a	b	e	r	p	q
b	b	e	a	q	r	p
p	p	q	r	e	a	b
q	q	r	p	b	e	a
r	r	p	q	a	b	e

(i) State, with a reason, whether or not G is commutative. [1]

(ii) State the number of subgroups of G which are of order 2. [1]

(iii) List the elements of the subgroup of G which is of order 3. [1]

(b) A multiplicative group H of order 6 has elements e, c, c^2, c^3, c^4, c^5, where e is the identity. Write down the order of each of the elements c^3, c^4 and c^5. [3]

2 Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 4x. \quad [7]$$

3 Two fixed points, A and B, have position vectors $\mathbf{a}$ and $\mathbf{b}$ relative to the origin O, and a variable point P has position vector $\mathbf{r}$.

(i) Give a geometrical description of the locus of P when $\mathbf{r}$ satisfies the equation $\mathbf{r} = \lambda\mathbf{a}$, where $0 \leqslant \lambda \leqslant 1$. [2]

(ii) Given that P is a point on the line AB, use a property of the vector product to explain why $(\mathbf{r} - \mathbf{a}) \times (\mathbf{r} - \mathbf{b}) = \mathbf{0}$. [2]

(iii) Give a geometrical description of the locus of P when $\mathbf{r}$ satisfies the equation $\mathbf{r} \times (\mathbf{a} - \mathbf{b}) = \mathbf{0}$. [3]

Jan 2008

4 The integrals C and S are defined by

$$C = \int_0^{\frac{1}{2}\pi} e^{2x} \cos 3x \, dx \qquad \text{and} \qquad S = \int_0^{\frac{1}{2}\pi} e^{2x} \sin 3x \, dx.$$

By considering $C + iS$ as a single integral, show that

$$C = -\tfrac{1}{13}\left(2 + 3e^{\pi}\right),$$

and obtain a similar expression for S. [8]

(You may assume that the standard result for $\int e^{kx}\,dx$ remains true when k is a complex constant, so that $\int e^{(a+ib)x}\,dx = \dfrac{1}{a+ib}e^{(a+ib)x}$.)

5 **(i)** Find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = \sin 2x,$$

expressing y in terms of x in your answer. [6]

In a particular case, it is given that $y = \dfrac{2}{\pi}$ when $x = \frac{1}{4}\pi$.

(ii) Find the solution of the differential equation in this case. [2]

(iii) Write down a function to which y approximates when x is large and positive. [1]

6 A tetrahedron $ABCD$ is such that AB is perpendicular to the base BCD. The coordinates of the points A, C and D are $(-1, -7, 2)$, $(5, 0, 3)$ and $(-1, 3, 3)$ respectively, and the equation of the plane BCD is $x + 2y - 2z = -1$.

(i) Find, in either order, the coordinates of B and the length of AB. [5]

(ii) Find the acute angle between the planes ACD and BCD. [6]

7 **(i)** **(a)** Verify, without using a calculator, that $\theta = \frac{1}{8}\pi$ is a solution of the equation $\sin 6\theta = \sin 2\theta$. [1]

(b) By sketching the graphs of $y = \sin 6\theta$ and $y = \sin 2\theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, or otherwise, find the other solution of the equation $\sin 6\theta = \sin 2\theta$ in the interval $0 < \theta < \frac{1}{2}\pi$. [2]

(ii) Use de Moivre's theorem to prove that

$$\sin 6\theta \equiv \sin 2\theta\left(16\cos^4\theta - 16\cos^2\theta + 3\right). \qquad [5]$$

(iii) Hence show that one of the solutions obtained in part **(i)** satisfies $\cos^2\theta = \frac{1}{4}\left(2 - \sqrt{2}\right)$, and justify which solution it is. [3]

Jan 2008

8 Groups A, B, C and D are defined as follows:

A: the set of numbers $\{2, 4, 6, 8\}$ under multiplication modulo 10,

B: the set of numbers $\{1, 5, 7, 11\}$ under multiplication modulo 12,

C: the set of numbers $\{2^0, 2^1, 2^2, 2^3\}$ under multiplication modulo 15,

D: the set of numbers $\left\{\frac{1+2m}{1+2n}, \text{ where } m \text{ and } n \text{ are integers}\right\}$ under multiplication.

(i) Write down the identity element for each of groups A, B, C and D. [2]

(ii) Determine in each case whether the groups

A and B,

B and C,

A and C

are isomorphic or non-isomorphic. Give sufficient reasons for your answers. [5]

(iii) Prove the closure property for group D. [4]

(iv) Elements of the set $\left\{\frac{1+2m}{1+2n}, \text{ where } m \text{ and } n \text{ are integers}\right\}$ are combined under **addition**. State which of the four basic group properties are **not** satisfied. (Justification is not required.) [2]

June 2008

1 **(a)** A cyclic multiplicative group G has order 12. The identity element of G is e and another element is r, with order 12.

(i) Write down, in terms of e and r, the elements of the subgroup of G which is of order 4. [2]

(ii) Explain briefly why there is no proper subgroup of G in which two of the elements are e and r. [1]

(b) A group H has order mnp, where m, n and p are prime. State the possible orders of proper subgroups of H. [2]

2 Find the acute angle between the line with equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{k} + t(\mathbf{i} + 4\mathbf{j} - \mathbf{k})$ and the plane with equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$. [7]

3 **(i)** Use the substitution $z = x + y$ to show that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y+3}{x+y-1} \qquad \text{(A)}$$

may be written in the form $\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{2(z+1)}{z-1}$. [3]

(ii) Hence find the general solution of the differential equation (A). [4]

4 **(i)** By expressing $\cos\theta$ in terms of $\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{e}^{-\mathrm{i}\theta}$, show that

$$\cos^5\theta \equiv \tfrac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta). \qquad [5]$$

(ii) Hence solve the equation $\cos 5\theta + 5\cos 3\theta + 9\cos\theta = 0$ for $0 \leqslant \theta \leqslant \pi$. [4]

5 Two lines have equations

$$\frac{x-k}{2} = \frac{y+1}{-5} = \frac{z-1}{-3} \qquad \text{and} \qquad \frac{x-k}{1} = \frac{y+4}{-4} = \frac{z}{-2},$$

where k is a constant.

(i) Show that, for all values of k, the lines intersect, and find their point of intersection in terms of k. [6]

(ii) For the case $k = 1$, find the equation of the plane in which the lines lie, giving your answer in the form $ax + by + cz = d$. [4]

6 The operation $\circ$ on real numbers is defined by $a \circ b = a|b|$.

(i) Show that $\circ$ is not commutative. [2]

(ii) Prove that $\circ$ is associative. [4]

(iii) Determine whether the set of real numbers, under the operation $\circ$, forms a group. [4]

7 The roots of the equation $z^3 - 1 = 0$ are denoted by 1, ω and ω^2.

(i) Sketch an Argand diagram to show these roots. [1]

(ii) Show that $1 + \omega + \omega^2 = 0$. [2]

(iii) Hence evaluate

(a) $(2 + \omega)(2 + \omega^2)$, [2]

(b) $\dfrac{1}{2+\omega} + \dfrac{1}{2+\omega^2}$. [2]

(iv) Hence find a cubic equation, with integer coefficients, which has roots 2, $\dfrac{1}{2+\omega}$ and $\dfrac{1}{2+\omega^2}$. [4]

8 **(i)** Find the complementary function of the differential equation

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x. \qquad [2]$$

(ii) It is given that $y = p(\ln \sin x)\sin x + qx\cos x$, where p and q are constants, is a particular integral of this differential equation.

(a) Show that $p - 2(p + q)\sin^2 x \equiv 1$. [6]

(b) Deduce the values of p and q. [2]

(iii) Write down the general solution of the differential equation. State the set of values of x, in the interval $0 \leqslant x \leqslant 2\pi$, for which the solution is valid, justifying your answer. [3]

1 In this question G is a group of order n, where $3 \leqslant n < 8$.

(i) In each case, write down the smallest possible value of n:

(a) if G is cyclic, **[1]**

(b) if G has a proper subgroup of order 3, **[1]**

(c) if G has at least two elements of order 2. **[1]**

(ii) Another group has the same order as G, but is not isomorphic to G. Write down the possible value(s) of n. **[2]**

2 **(i)** Express $\dfrac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. **[3]**

(ii) Hence find the smallest positive value of n for which $\left(\dfrac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}\right)^n$ is real and positive. **[2]**

3 Two skew lines have equations

$$\frac{x}{2} = \frac{y+3}{1} = \frac{z-6}{3} \qquad \text{and} \qquad \frac{x-5}{3} = \frac{y+1}{1} = \frac{z-7}{5}.$$

(i) Find the direction of the common perpendicular to the lines. **[2]**

(ii) Find the shortest distance between the lines. **[4]**

4 Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 65\sin 2x. \qquad \textbf{[9]}$$

5 The variables x and y are related by the differential equation

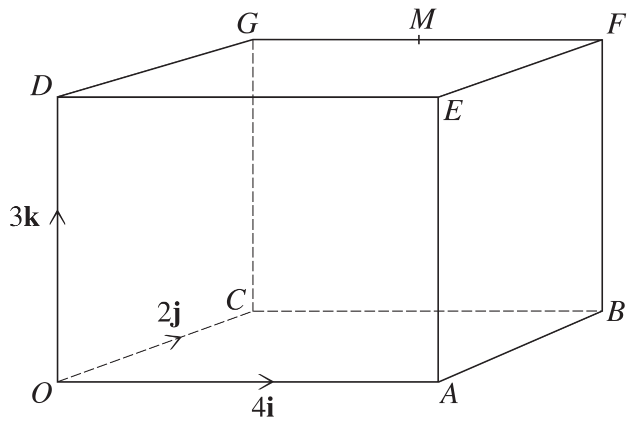
$$x^3\frac{\mathrm{d}y}{\mathrm{d}x} = xy + x + 1. \qquad \text{(A)}$$

(i) Use the substitution $y = u - \dfrac{1}{x}$, where u is a function of x, to show that the differential equation may be written as

$$x^2\frac{\mathrm{d}u}{\mathrm{d}x} = u. \qquad \textbf{[4]}$$

(ii) Hence find the general solution of the differential equation (A), giving your answer in the form $y = \mathrm{f}(x)$. **[5]**

6



The cuboid $OABCDEFG$ shown in the diagram has $\overrightarrow{OA} = 4\mathbf{i}$, $\overrightarrow{OC} = 2\mathbf{j}$, $\overrightarrow{OD} = 3\mathbf{k}$, and M is the mid-point of GF.

(i) Find the equation of the plane $ACGE$, giving your answer in the form $\mathbf{r}.\mathbf{n} = p$. **[4]**

(ii) The plane $OEFC$ has equation $\mathbf{r}.(3\mathbf{i} - 4\mathbf{k}) = 0$. Find the acute angle between the planes $OEFC$ and $ACGE$. **[4]**

(iii) The line AM meets the plane $OEFC$ at the point W. Find the ratio $AW : WM$. **[5]**

7 **(i)** The operation $*$ is defined by $x * y = x + y - a$, where x and y are real numbers and a is a real constant.

(a) Prove that the set of real numbers, together with the operation $*$, forms a group. **[6]**

(b) State, with a reason, whether the group is commutative. **[1]**

(c) Prove that there are no elements of order 2. **[2]**

(ii) The operation $\circ$ is defined by $x \circ y = x + y - 5$, where x and y are **positive** real numbers. By giving a numerical example in each case, show that two of the basic group properties are not necessarily satisfied. **[4]**

8 **(i)** By expressing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^6\theta \equiv -\tfrac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10).$$ **[5]**

(ii) Replace θ by $\left(\tfrac{1}{2}\pi - \theta\right)$ in the identity in part **(i)** to obtain a similar identity for $\cos^6\theta$. **[3]**

(iii) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} \left(\sin^6\theta - \cos^6\theta\right)\,d\theta$. **[4]**

1 Find the cube roots of $\frac{1}{2}\sqrt{3} + \frac{1}{2}\mathrm{i}$, giving your answers in the form $\cos\theta + \mathrm{i}\sin\theta$, where $0 \leqslant \theta < 2\pi$. **[4]**

2 It is given that the set of complex numbers of the form $r\mathrm{e}^{\mathrm{i}\theta}$ for $-\pi < \theta \leqslant \pi$ and $r > 0$, under multiplication, forms a group.

(i) Write down the inverse of $5\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$. **[1]**

(ii) Prove the closure property for the group. **[2]**

(iii) Z denotes the element $\mathrm{e}^{\mathrm{i}\gamma}$, where $\frac{1}{2}\pi < \gamma < \pi$. Express Z^2 in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta < 0$. **[2]**

3 A line l has equation $\dfrac{x-6}{-4} = \dfrac{y+7}{8} = \dfrac{z+10}{7}$ and a plane p has equation $3x - 4y - 2z = 8$.

(i) Find the point of intersection of l and p. **[3]**

(ii) Find the equation of the plane which contains l and is perpendicular to p, giving your answer in the form $ax + by + cz = d$. **[5]**

4 The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{1-x^2}y = (1-x)^{\frac{1}{2}}, \quad \text{where } |x| < 1,$$

can be solved by the integrating factor method.

(i) Use an appropriate result given in the List of Formulae (MF1) to show that the integrating factor can be written as $\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{2}}$. **[2]**

(ii) Hence find the solution of the differential equation for which $y = 2$ when $x = 0$, giving your answer in the form $y = \mathrm{f}(x)$. **[6]**

5 The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = \mathrm{e}^{3x}.$$

(i) Find the complementary function. **[3]**

(ii) Explain briefly why there is no particular integral of either of the forms $y = k\mathrm{e}^{3x}$ or $y = kx\mathrm{e}^{3x}$. **[1]**

(iii) Given that there is a particular integral of the form $y = kx^2\mathrm{e}^{3x}$, find the value of k. **[5]**

6 The plane Π_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix}$.

(i) Express the equation of Π_1 in the form $\mathbf{r.n} = p$. **[4]**

The plane Π_2 has equation $\mathbf{r.}\begin{pmatrix} 7 \\ 17 \\ -3 \end{pmatrix} = 21$.

(ii) Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. **[5]**

7 **(i)** Use de Moivre's theorem to prove that

$$\tan 3\theta \equiv \frac{\tan\theta(3 - \tan^2\theta)}{1 - 3\tan^2\theta}.$$ **[4]**

(ii) **(a)** By putting $\theta = \frac{1}{12}\pi$ in the identity in part **(i)**, show that $\tan\frac{1}{12}\pi$ is a solution of the equation

$$t^3 - 3t^2 - 3t + 1 = 0.$$ **[1]**

(b) Hence show that $\tan\frac{1}{12}\pi = 2 - \sqrt{3}$. **[4]**

(iii) Use the substitution $t = \tan\theta$ to show that

$$\int_0^{2-\sqrt{3}} \frac{t(3 - t^2)}{(1 - 3t^2)(1 + t^2)}\,\mathrm{d}t = a\ln b,$$

where a and b are positive constants to be determined. **[5]**

8 A multiplicative group Q of order 8 has elements $\{e, p, p^2, p^3, a, ap, ap^2, ap^3\}$, where e is the identity. The elements have the properties $p^4 = e$ and $a^2 = p^2 = (ap)^2$.

(i) Prove that $a = pap$ and that $p = apa$. **[2]**

(ii) Find the order of each of the elements p^2, a, ap, ap^2. **[5]**

(iii) Prove that $\{e, a, p^2, ap^2\}$ is a subgroup of Q. **[4]**

(iv) Determine whether Q is a commutative group. **[4]**

1 Determine whether the lines

$$\frac{x-1}{1} = \frac{y+2}{-1} = \frac{z+4}{2} \qquad \text{and} \qquad \frac{x+3}{2} = \frac{y-1}{3} = \frac{z-5}{4}$$

intersect or are skew. **[5]**

2 H denotes the set of numbers of the form $a + b\sqrt{5}$, where a and b are rational. The numbers are combined under multiplication.

(i) Show that the product of any two members of H is a member of H. **[2]**

It is now given that, for a and b not both zero, H forms a group under multiplication.

(ii) State the identity element of the group. **[1]**

(iii) Find the inverse of $a + b\sqrt{5}$. **[2]**

(iv) With reference to your answer to part **(iii)**, state a property of the number 5 which ensures that every number in the group has an inverse. **[1]**

3 Use the integrating factor method to find the solution of the differential equation

$$\frac{dy}{dx} + 2y = e^{-3x}$$

for which $y = 1$ when $x = 0$. Express your answer in the form $y = f(x)$. **[6]**

4 **(i)** Write down, in cartesian form, the roots of the equation $z^4 = 16$. **[2]**

(ii) Hence solve the equation $w^4 = 16(1-w)^4$, giving your answers in cartesian form. **[5]**

5 A regular tetrahedron has vertices at the points

$$A\left(0, 0, \tfrac{2}{3}\sqrt{6}\right), \quad B\left(\tfrac{2}{3}\sqrt{3}, 0, 0\right), \quad C\left(-\tfrac{1}{3}\sqrt{3}, 1, 0\right), \quad D\left(-\tfrac{1}{3}\sqrt{3}, -1, 0\right).$$

(i) Obtain the equation of the face ABC in the form

$$x + \sqrt{3}y + \left(\tfrac{1}{2}\sqrt{2}\right)z = \tfrac{2}{3}\sqrt{3}.$$ **[5]**

(Answers which only verify the given equation will not receive full credit.)

(ii) Give a geometrical reason why the equation of the face ABD can be expressed as

$$x - \sqrt{3}y + \left(\tfrac{1}{2}\sqrt{2}\right)z = \tfrac{2}{3}\sqrt{3}.$$ **[2]**

(iii) Hence find the cosine of the angle between two faces of the tetrahedron. **[4]**

6 The variables x and y satisfy the differential equation

$$\frac{d^2y}{dx^2} + 16y = 8\cos 4x.$$

(i) Find the complementary function of the differential equation. **[2]**

(ii) Given that there is a particular integral of the form $y = px\sin 4x$, where p is a constant, find the general solution of the equation. **[6]**

(iii) Find the solution of the equation for which $y = 2$ and $\frac{dy}{dx} = 0$ when $x = 0$. **[4]**

7 **(i)** Solve the equation $\cos 6\theta = 0$, for $0 < \theta < \pi$. **[3]**

(ii) By using de Moivre's theorem, show that

$$\cos 6\theta \equiv (2\cos^2\theta - 1)(16\cos^4\theta - 16\cos^2\theta + 1).$$ **[5]**

(iii) Hence find the exact value of

$$\cos\left(\tfrac{1}{12}\pi\right)\cos\left(\tfrac{5}{12}\pi\right)\cos\left(\tfrac{7}{12}\pi\right)\cos\left(\tfrac{11}{12}\pi\right),$$

justifying your answer. **[5]**

8 The function f is defined by $\mathrm{f} : x \mapsto \frac{1}{2-2x}$ for $x \in \mathbb{R}$, $x \neq 0$, $x \neq \frac{1}{2}$, $x \neq 1$. The function g is defined by $\mathrm{g}(x) = \mathrm{ff}(x)$.

(i) Show that $\mathrm{g}(x) = \frac{1-x}{1-2x}$ and that $\mathrm{gg}(x) = x$. **[4]**

It is given that f and g are elements of a group K under the operation of composition of functions. The element e is the identity, where $\mathrm{e} : x \mapsto x$ for $x \in \mathbb{R}$, $x \neq 0$, $x \neq \frac{1}{2}$, $x \neq 1$.

(ii) State the orders of the elements f and g. **[2]**

(iii) The inverse of the element f is denoted by h. Find $\mathrm{h}(x)$. **[2]**

(iv) Construct the operation table for the elements e, f, g, h of the group K. **[4]**

1 The line l_1 passes through the points (0, 0, 10) and (7, 0, 0) and the line l_2 passes through the points (4, 6, 0) and (3, 3, 1). Find the shortest distance between l_1 and l_2. **[7]**

2 A multiplicative group with identity e contains distinct elements a and r, with the properties $r^6 = e$ and $ar = r^5a$.

(i) Prove that $rar = a$. **[2]**

(ii) Prove, by induction or otherwise, that $r^n ar^n = a$ for all positive integers n. **[4]**

3 In this question, w denotes the complex number $\cos \frac{2}{5}\pi + \mathrm{i}\sin \frac{2}{5}\pi$.

(i) Express w^2, w^3 and w^* in polar form, with arguments in the interval $0 \leqslant \theta < 2\pi$. **[4]**

(ii) The points in an Argand diagram which represent the numbers

$$1, \quad 1 + w, \quad 1 + w + w^2, \quad 1 + w + w^2 + w^3, \quad 1 + w + w^2 + w^3 + w^4$$

are denoted by A, B, C, D, E respectively. Sketch the Argand diagram to show these points and join them in the order stated. (Your diagram need not be exactly to scale, but it should show the important features.) **[4]**

(iii) Write down a polynomial equation of degree 5 which is satisfied by w. **[1]**

4 (i) Use the substitution $y = xz$ to find the general solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x\cos\left(\frac{y}{x}\right),$$

giving your answer in a form without logarithms. (You may quote an appropriate result given in the List of Formulae (MF1).) **[6]**

(ii) Find the solution of the differential equation for which $y = \pi$ when $x = 4$. **[2]**

5 Convergent infinite series C and S are defined by

$$C = 1 + \tfrac{1}{2}\cos\theta + \tfrac{1}{4}\cos 2\theta + \tfrac{1}{8}\cos 3\theta + \ \dots\ ,$$
$$S = \tfrac{1}{2}\sin\theta + \tfrac{1}{4}\sin 2\theta + \tfrac{1}{8}\sin 3\theta + \ \dots\ .$$

(i) Show that $C + \mathrm{i}S = \dfrac{2}{2 - \mathrm{e}^{\mathrm{i}\theta}}$. **[4]**

(ii) Hence show that $C = \dfrac{4 - 2\cos\theta}{5 - 4\cos\theta}$, and find a similar expression for S. **[4]**

6 **(i)** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 17y = 17x + 36. \qquad [7]$$

(ii) Show that, when x is large and positive, the solution approximates to a linear function, and state its equation. **[2]**

7 A line l has equation $\mathbf{r} = \begin{pmatrix} -7 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$. A plane Π passes through the points (1, 3, 5) and (5, 2, 5), and is parallel to l.

(i) Find an equation of Π, giving your answer in the form $\mathbf{r}.\mathbf{n} = p$. **[4]**

(ii) Find the distance between l and Π. **[4]**

(iii) Find an equation of the line which is the reflection of l in Π, giving your answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. **[4]**

8 A set of matrices M is defined by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad C = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix},$$

where ω and ω^2 are the complex cube roots of 1. It is given that M is a group under matrix multiplication.

(i) Write down the elements of a subgroup of order 2. **[1]**

(ii) Explain why there is no element X of the group, other than A, which satisfies the equation $X^5 = A$. **[2]**

(iii) By finding BE and EB, verify the closure property for the pair of elements B and E. **[4]**

(iv) Find the inverses of B and E. **[3]**

(v) Determine whether the group M is isomorphic to the group N which is defined as the set of numbers {1, 2, 4, 8, 7, 5} under multiplication modulo 9. Justify your answer clearly. **[3]**

1 **(i)** Find the general solution of the differential equation

$$\frac{dy}{dx} + xy = xe^{\frac{1}{2}x^2},$$

giving your answer in the form $y = f(x)$. **[4]**

(ii) Find the particular solution for which $y = 1$ when $x = 0$. **[2]**

2 Two intersecting lines, lying in a plane p, have equations

$$\frac{x-1}{2} = \frac{y-3}{1} = \frac{z-4}{-3} \quad \text{and} \quad \frac{x-1}{-1} = \frac{y-3}{2} = \frac{z-4}{4}.$$

(i) Obtain the equation of p in the form $2x - y + z = 3$. **[3]**

(ii) Plane q has equation $2x - y + z = 21$. Find the distance between p and q. **[3]**

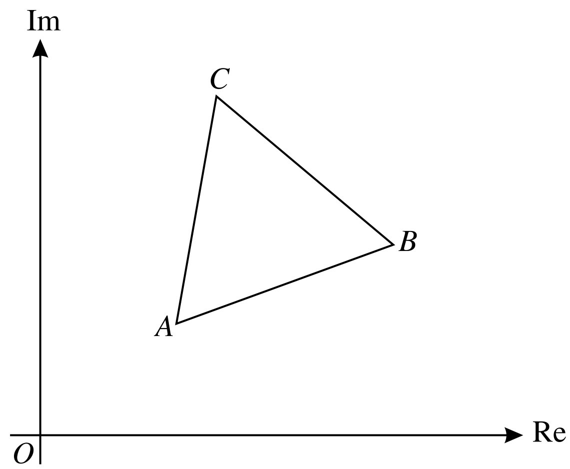
3 **(i)** Express $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$ and show that

$$\sin^4\theta \equiv \tfrac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3). \quad \textbf{[4]}$$

(ii) Hence find the exact value of $\int_0^{\frac{1}{6}\pi} \sin^4\theta \, d\theta$. **[4]**

4 The cube roots of 1 are denoted by 1, ω and ω^2, where the imaginary part of ω is positive.

(i) Show that $1 + \omega + \omega^2 = 0$. **[2]**



In the diagram, ABC is an equilateral triangle, labelled anticlockwise. The points A, B and C represent the complex numbers z_1, z_2 and z_3 respectively.

(ii) State the geometrical effect of multiplication by ω and hence explain why $z_1 - z_3 = \omega(z_3 - z_2)$. **[4]**

(iii) Hence show that $z_1 + \omega z_2 + \omega^2 z_3 = 0$. **[2]**

5 **(i)** Find the general solution of the differential equation

$$3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = -2x + 13.$$ **[7]**

(ii) Find the particular solution for which $y = -\frac{7}{2}$ and $\frac{dy}{dx} = 0$ when $x = 0$. **[5]**

(iii) Write down the function to which y approximates when x is large and positive. **[1]**

6 Q is a multiplicative group of order 12.

(i) Two elements of Q are a and r. It is given that r has order 6 and that $a^2 = r^3$. Find the orders of the elements a, a^2, a^3 and r^2. **[4]**

The table below shows the number of elements of Q with each possible order.

Order of element	1	2	3	4	6
Number of elements	1	1	2	6	2

G and H are the non-cyclic groups of order 4 and 6 respectively.

(ii) Construct two tables, similar to the one above, to show the number of elements with each possible order for the groups G and H. Hence explain why there are no non-cyclic proper subgroups of Q. **[5]**

7 Three planes Π_1, Π_2 and Π_3 have equations

$$\mathbf{r}.(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 5, \qquad \mathbf{r}.(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 6, \qquad \mathbf{r}.(\mathbf{i} + 5\mathbf{j} - 12\mathbf{k}) = 12,$$

respectively. Planes Π_1 and Π_2 intersect in a line l; planes Π_2 and Π_3 intersect in a line m.

(i) Show that l and m are in the same direction. **[5]**

(ii) Write down what you can deduce about the line of intersection of planes Π_1 and Π_3. **[1]**

(iii) By considering the cartesian equations of Π_1, Π_2 and Π_3, or otherwise, determine whether or not the three planes have a common line of intersection. **[4]**

[Question 8 is printed overleaf.]

8 The operation $*$ is defined on the elements (x, y), where $x, y \in \mathbb{R}$, by

$$(a, b) * (c, d) = (ac, ad + b).$$

It is given that the identity element is $(1, 0)$.

(i) Prove that $*$ is associative. **[3]**

(ii) Find all the elements which commute with $(1, 1)$. **[3]**

(iii) It is given that the particular element (m, n) has an inverse denoted by (p, q), where

$$(m, n) * (p, q) = (p, q) * (m, n) = (1, 0).$$

Find (p, q) in terms of m and n. **[2]**

(iv) Find all self-inverse elements. **[3]**

(v) Give a reason why the elements (x, y), under the operation $*$, do not form a group. **[1]**

1 A line l has equation $\frac{x-1}{5} = \frac{y-6}{6} = \frac{z+3}{-7}$ and a plane p has equation $x + 2y - z = 40$.

(i) Find the acute angle between l and p. **[4]**

(ii) Find the perpendicular distance from the point $(1, 6, -3)$ to p. **[2]**

2 It is given that $z = e^{i\theta}$, where $0 < \theta < 2\pi$, and $w = \frac{1+z}{1-z}$.

(i) Prove that $w = i\cot\frac{1}{2}\theta$. **[3]**

(ii) Sketch separate Argand diagrams to show the locus of z and the locus of w. You should show the direction in which each locus is described when θ increases in the interval $0 < \theta < 2\pi$. **[3]**

3 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} + 4y = 5\cos 3x.$$

(i) Find the complementary function. **[2]**

(ii) Hence, or otherwise, find the general solution. **[7]**

(iii) Find the approximate range of values of y when x is large and positive. **[2]**

4 A group G, of order 8, is generated by the elements a, b, c. G has the properties

$$a^2 = b^2 = c^2 = e, \quad ab = ba, \quad bc = cb, \quad ca = ac,$$

where e is the identity.

(i) Using these properties and basic group properties as necessary, prove that $abc = cba$. **[2]**

The operation table for G is shown below.

	e	a	b	c	bc	ca	ab	abc
e	e	a	b	c	bc	ca	ab	abc
a	a	e	ab	ca	abc	c	b	bc
b	b	ab	e	bc	c	abc	a	ca
c	c	ca	bc	e	b	a	abc	ab
bc	bc	abc	c	b	e	ab	ca	a
ca	ca	c	abc	a	ab	e	bc	b
ab	ab	b	a	abc	ca	bc	e	c
abc	abc	bc	ca	ab	a	b	c	e

(ii) List all the subgroups of order 2. **[2]**

(iii) List five subgroups of order 4. **[3]**

(iv) Determine whether all the subgroups of G which are of order 4 are isomorphic. **[2]**

5 The substitution $y = u^k$, where k is an integer, is to be used to solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = x^2y^2 \qquad \text{(A)}$$

by changing it into an equation (B) in the variables u and x.

(i) Show that equation (B) may be written in the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{3}{kx}u = \frac{1}{k}xu^{k+1}.$$ **[4]**

(ii) Write down the value of k for which the integrating factor method may be used to solve equation (B). **[1]**

(iii) Using this value of k, solve equation (B) and hence find the general solution of equation (A), giving your answer in the form $y = \mathrm{f}(x)$. **[4]**

6 **(a)** The set of polynomials $\{ax + b\}$, where $a, b \in \mathbb{R}$, is denoted by P. Assuming that the associativity property holds, prove that P, under addition, is a group. **[4]**

(b) The set of polynomials $\{ax + b\}$, where $a, b \in \{0, 1, 2\}$, is denoted by Q. It is given that Q, under addition modulo 3, is a group, denoted by $(Q, +(\text{mod}3))$.

(i) State the order of the group. **[1]**

(ii) Write down the inverse of the element $2x + 1$. **[1]**

(iii) $\mathrm{q}(x) = ax + b$ is any element of Q other than the identity. Find the order of $\mathrm{q}(x)$ and hence determine whether $(Q, +(\text{mod}3))$ is a cyclic group. **[4]**

7 (In this question, the notation ΔABC denotes the area of the triangle ABC.)

The points P, Q and R have position vectors $p\mathbf{i}$, $q\mathbf{j}$ and $r\mathbf{k}$ respectively, relative to the origin O, where p, q and r are positive. The points O, P, Q and R are joined to form a tetrahedron.

(i) Draw a sketch of the tetrahedron and write down the values of ΔOPQ, ΔOQR and ΔORP. **[3]**

(ii) Use the definition of the vector product to show that $\frac{1}{2}\left|\overrightarrow{RP} \times \overrightarrow{RQ}\right| = \Delta PQR$. **[1]**

(iii) Show that $(\Delta OPQ)^2 + (\Delta OQR)^2 + (\Delta ORP)^2 = (\Delta PQR)^2$. **[6]**

8 **(i)** Use de Moivre's theorem to express $\cos 4\theta$ as a polynomial in $\cos\theta$. **[4]**

(ii) Hence prove that $\cos 4\theta \cos 2\theta \equiv 16\cos^6\theta - 24\cos^4\theta + 10\cos^2\theta - 1$. **[1]**

(iii) Use part **(ii)** to show that the only roots of the equation $\cos 4\theta \cos 2\theta = 1$ are $\theta = n\pi$, where n is an integer. **[3]**

(iv) Show that $\cos 4\theta \cos 2\theta = -1$ only when $\cos\theta = 0$. **[3]**

1 The variables x and y are related by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^2 + y^2}{xy}. \qquad \text{(A)}$$

(i) Use the substitution $y = ux$, where u is a function of x, to obtain the differential equation

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{u}.$$ **[3]**

(ii) Hence find the general solution of the differential equation (A), giving your answer in the form $y^2 = \mathrm{f}(x)$. **[4]**

2 **(i)** Show that $\left(z^n - \mathrm{e}^{\mathrm{i}\theta}\right)\left(z^n - \mathrm{e}^{-\mathrm{i}\theta}\right) \equiv z^{2n} - (2\cos\theta)\,z^n + 1$. **[1]**

(ii) Express $z^4 - z^2 + 1$ as the product of four factors of the form $\left(z - \mathrm{e}^{\mathrm{i}\alpha}\right)$, where $0 \leqslant \alpha < 2\pi$. **[6]**

3 A multiplicative group contains the distinct elements e, x and y, where e is the identity.

(i) Prove that $x^{-1}y^{-1} = (yx)^{-1}$. **[2]**

(ii) Given that $x^n y^n = (xy)^n$ for some integer $n \geqslant 2$, prove that $x^{n-1}y^{n-1} = (yx)^{n-1}$. **[3]**

(iii) If $x^{n-1}y^{n-1} = (yx)^{n-1}$, does it follow that $x^n y^n = (xy)^n$? Give a reason for your answer. **[2]**

4 The line l has equations $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+1}{2}$ and the point A is $(7, 3, 7)$. M is the point where the perpendicular from A meets l.

(i) Find, in either order, the coordinates of M and the perpendicular distance from A to l. **[7]**

(ii) Find the coordinates of the point B on AM such that $\overrightarrow{AB} = 3\overrightarrow{BM}$. **[3]**

5 The variables x and y satisfy the differential equation

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 5\mathrm{e}^{-2x}.$$

(i) Find the complementary function of the differential equation. **[2]**

(ii) Given that there is a particular integral of the form $y = px\mathrm{e}^{-2x}$, find the constant p. **[4]**

(iii) Find the solution of the equation for which $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 0$. **[5]**

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6 The plane Π has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}$ and the line l has equation $\mathbf{r} = \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$.

(i) Express the equation of Π in the form $\mathbf{r}.\mathbf{n} = p$. **[4]**

(ii) Find the point of intersection of l and Π. **[2]**

(iii) The equation of Π may be expressed in the form $\mathbf{r} = \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \mu\mathbf{c}$, where $\mathbf{c}$ is perpendicular to $\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$. Find $\mathbf{c}$. **[3]**

7 The set M consists of the six matrices $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$, where $n \in \{0, 1, 2, 3, 4, 5\}$. It is given that M forms a group $(M, \times)$ under matrix multiplication, with numerical addition and multiplication both being carried out modulo 6.

(i) Determine whether $(M, \times)$ is a commutative group, justifying your answer. **[2]**

(ii) Write down the identity element of the group and find the inverse of $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. **[3]**

(iii) State the order of $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ and give a reason why $(M, \times)$ has no subgroup of order 4. **[2]**

(iv) The multiplicative group G has order 6. All the elements of G, apart from the identity, have order 2 or 3. Determine whether G is isomorphic to $(M, \times)$, justifying your answer. **[2]**

8 **(i)** Use de Moivre's theorem to prove that

$$\tan 5\theta \equiv \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$ **[4]**

(ii) Solve the equation $\tan 5\theta = 1$, for $0 \leqslant \theta < \pi$. **[3]**

(iii) Show that the roots of the equation

$$t^4 - 4t^3 - 14t^2 - 4t + 1 = 0$$

may be expressed in the form $\tan\alpha$, stating the exact values of α, where $0 \leqslant \alpha < \pi$. **[5]**

1 The plane p has equation $\mathbf{r}.(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 4$ and the line l_1 has equation $\mathbf{r} = 2\mathbf{j} - \mathbf{k} + t(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. The line l_2 is parallel to p and perpendicular to l_1, and passes through the point with position vector $\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. Find the equation of l_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. **[4]**

2 **(i)** Solve the equation $z^4 = 2(1 + \mathrm{i}\sqrt{3})$, giving the roots exactly in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. **[5]**

(ii) Sketch an Argand diagram to show the lines from the origin to the point representing $2(1 + \mathrm{i}\sqrt{3})$ and from the origin to the points which represent the roots of the equation in part **(i)**. **[3]**

3 Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = 2x$$

for which $y = 2$ when $x = \frac{1}{6}\pi$. Give your answer in the form $y = \mathrm{f}(x)$. **[9]**

4 The elements a, b, c, d are combined according to the operation table below, to form a group G of order 4.

	a	b	c	d
a	b	a	d	c
b	a	b	c	d
c	d	c	a	b
d	c	d	b	a

Group G is isomorphic **either** to the multiplicative group $H = \{e, r, r^2, r^3\}$ **or** to the multiplicative group $K = \{e, p, q, pq\}$. It is given that $r^4 = e$ in group H and that $p^2 = q^2 = e$ in group K, where e denotes the identity in each group.

(i) Write down the operation tables for H and K. **[4]**

(ii) State the identity element of G. **[1]**

(iii) Demonstrate the isomorphism between G and either H or K by listing how the elements of G correspond to the elements of the other group. If the correspondence can be shown in more than one way, list the alternative correspondence(s). **[4]**

5 **(i)** By expressing $\sin\theta$ and $\cos\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, prove that

$$\sin^3\theta\cos^2\theta \equiv -\tfrac{1}{16}(\sin 5\theta - \sin 3\theta - 2\sin\theta).$$ **[6]**

(ii) Hence show that all the roots of the equation

$$\sin 5\theta = \sin 3\theta + 2\sin\theta$$

are of the form $\theta = \dfrac{n\pi}{k}$, where n is any integer and k is to be determined. **[3]**

6 The variables x and y satisfy the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 12e^{2x}.$$

(i) Find the general solution of the differential equation. **[6]**

(ii) It is given that the curve which represents a particular solution of the differential equation has gradient 6 when $x = 0$, and approximates to $y = e^{2x}$ when x is large and positive. Find the equation of the curve. **[4]**

7 With respect to the origin O, the position vectors of the points U, V and W are $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively. The mid-points of the sides VW, WU and UV of the triangle UVW are M, N and P respectively.

(i) Show that $\overrightarrow{UM} = \tfrac{1}{2}(\mathbf{v} + \mathbf{w} - 2\mathbf{u})$. **[2]**

(ii) Verify that the point G with position vector $\tfrac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$ lies on UM, and deduce that the lines UM, VN and WP intersect at G. **[5]**

(iii) Write down, in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, an equation of the line through G which is perpendicular to the plane UVW. (It is not necessary to simplify the expression for $\mathbf{b}$.) **[2]**

(iv) It is now given that $\mathbf{u} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix}0\\0\\1\end{pmatrix}$. Find the perpendicular distance from O to the plane UVW. **[3]**

8 The set M of matrices $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, where a, b, c and d are real and $ad - bc = 1$, forms a group $(M, \times)$ under matrix multiplication. R denotes the set of all matrices $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$.

(i) Prove that $(R, \times)$ is a subgroup of $(M, \times)$. **[6]**

(ii) By considering geometrical transformations in the x-y plane, find a subgroup of $(R, \times)$ of order 6. Give the elements of this subgroup in exact numerical form. **[5]**

1 Two planes have equations

$$x + 2y + 5z = 12 \text{ and } 2x - y + 3z = 5.$$

(i) Find the acute angle between the planes. **[3]**

(ii) Find a vector equation of the line of intersection of the planes. **[4]**

2 The elements of a group G are the complex numbers $a + b\mathrm{i}$ where $a, b \in \{0, 1, 2, 3, 4\}$. These elements are combined under the operation of addition modulo 5.

(i) State the identity element and the order of G. **[2]**

(ii) Write down the inverse of $2 + 4\mathrm{i}$. **[1]**

(iii) Show that every non-zero element of G has order 5. **[3]**

3 Solve the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = x^4\mathrm{e}^{2x}$ for y in terms of x, given that $y = 0$ when $x = 1$. **[8]**

4 The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$$

respectively.

(i) Find the shortest distance between the lines. **[5]**

(ii) Find a cartesian equation of the plane which contains l_1 and which is parallel to l_2. **[2]**

5 **(i)** Solve the equation $z^5 = 1$, giving your answers in polar form. **[2]**

(ii) Hence, by considering the equation $(z + 1)^5 = z^5$, show that the roots of

$$5z^4 + 10z^3 + 10z^2 + 5z + 1 = 0$$

can be expressed in the form $\frac{1}{\mathrm{e}^{\mathrm{i}\theta} - 1}$, stating the values of θ. **[5]**

6 The differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = \sin kx$ is to be solved, where k is a constant.

(i) In the case $k = 2$, by using a particular integral of the form $ax\cos 2x + bx\sin 2x$, find the general solution. **[7]**

(ii) Describe briefly the behaviour of y when $x \to \infty$. **[2]**

(iii) In the case $k \neq 2$, explain whether y would exhibit the same behaviour as in part **(ii)** when $x \to \infty$. **[2]**

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7 Let $S = \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{2\mathrm{i}\theta} + \mathrm{e}^{3\mathrm{i}\theta} + \ldots + \mathrm{e}^{10\mathrm{i}\theta}$.

(i) (a) Show that, for $\theta \neq 2n\pi$, where n is an integer,

$$S = \frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}\theta}\left(\mathrm{e}^{10\mathrm{i}\theta} - 1\right)}{2\mathrm{i}\sin\left(\frac{1}{2}\theta\right)}.$$ **[4]**

(b) State the value of S for $\theta = 2n\pi$, where n is an integer. **[1]**

(ii) Hence show that, for $\theta \neq 2n\pi$, where n is an integer,

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = \frac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} - \frac{1}{2}.$$ **[3]**

(iii) Hence show that $\theta = \frac{1}{11}\pi$ is a root of $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = 0$ and find another root in the interval $0 < \theta < \frac{1}{4}\pi$. **[4]**

8 A multiplicative group H has the elements $\{e, a, a^2, a^3, w, aw, a^2w, a^3w\}$ where e is the identity, elements a and w have orders 4 and 2 respectively and $wa = a^3w$.

(i) Show that $wa^2 = a^2w$ and also that $wa^3 = aw$. **[6]**

(ii) Hence show that each of aw, a^2w and a^3w has order 2. **[4]**

(iii) Find two non-cyclic subgroups of H of order 4, and show that they are not cyclic. **[4]**

1 The plane Π passes through the points with coordinates (1, 6, 2), (5, 2, 1) and (1, 0, –2).

(i) Find a vector equation of Π in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. **[2]**

(ii) Find a cartesian equation of Π. **[4]**

2 G consists of the set $\{1, 3, 5, 7\}$ with the operation of multiplication modulo 8.

(i) Write down the operation table and, assuming associativity, show that G is a group. **[5]**

(ii) State the order of each element. **[1]**

(iii) Find all the proper subgroups of G. **[1]**

The group H consists of the set $\{1, 3, 7, 9\}$ with the operation of multiplication modulo 10.

(iv) Explaining your reasoning, determine whether H is isomorphic to G. **[2]**

3 The differential equation

$$3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y^3 = \frac{\cos x}{x}$$

is to be solved for $x > 0$. Use the substitution $u = y^3$ to find the general solution for y in terms of x. **[8]**

4 The complex numbers 0, 3 and $3\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$ are represented in an Argand diagram by the points O, A and B respectively.

(i) Sketch the triangle OAB and show that it is equilateral. **[3]**

(ii) Hence express $3 - 3\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$ in polar form. **[2]**

(iii) Hence find $(3 - 3\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}})^5$, giving your answer in the form $a + b\sqrt{3}\,\mathrm{i}$ where a and b are rational numbers. **[3]**

5 Find the solution of the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = \mathrm{e}^{-x}$ for which $y = \frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$. **[11]**

6 The plane Π has equation $x + 2y - 2z = 5$. The line l has equation $\frac{x-1}{2} = \frac{y+1}{5} = \frac{z-2}{1}$.

(i) Find the coordinates of the point of intersection of l with the plane Π. **[3]**

(ii) Calculate the acute angle between l and Π. **[3]**

(iii) Find the coordinates of the two points on the line l such that the distance of each point from the plane Π is 2. **[5]**

7 A commutative group G has order 18. The elements a, b and c have orders 2, 3 and 9 respectively.

(i) Prove that ab has order 6. **[4]**

(ii) Show that G is cyclic. **[3]**

8 **(i)** Use de Moivre's theorem to show that $\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **[5]**

(ii) Hence find the roots of $16x^4 - 20x^2 + 5 = 0$ in the form $\cos\alpha$ where $0 \leqslant \alpha \leqslant \pi$. **[4]**

(iii) Hence find the exact value of $\cos\frac{1}{10}\pi$. **[3]**

1 **(i)** Find a vector equation of the line of intersection of the planes $2x + y - z = 4$ and $3x + 5y + 2z = 13$. **[4]**

(ii) Find the exact distance of the point $(2, 5, -2)$ from the plane $2x + y - z = 4$. **[2]**

2 Use the substitution $u = y^2$ to find the general solution of the differential equation

$$\frac{dy}{dx} - 2y = \frac{e^x}{y}$$

for y in terms of x. **[8]**

3 **(i)** Solve the equation $z^6 = 1$, giving your answers in the form $re^{i\theta}$, and sketch an Argand diagram showing the positions of the roots. **[4]**

(ii) Show that $(1 + i)^6 = -8i$. **[3]**

(iii) Hence, or otherwise, solve the equation $z^6 + 8i = 0$, giving your answers in the form $re^{i\theta}$. **[3]**

4 The group G consists of the set $\{1, 3, 7, 9, 11, 13, 17, 19\}$ combined under multiplication modulo 20.

(i) Find the inverse of each element. **[3]**

(ii) Show that G is not cyclic. **[3]**

(iii) Find two isomorphic subgroups of order 4 and state an isomorphism between them. **[5]**

5 Solve the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-x}$$

subject to the conditions $y = \frac{dy}{dx} = 0$ when $x = 0$. **[10]**

6 The line l has equations $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-7}{5}$. The plane Π has equation $4x - y - z = 8$.

(i) Show that l is parallel to Π but does not lie in Π. **[3]**

(ii) The point A $(1, -2, 7)$ is on l. Write down a vector equation of the line through A which is perpendicular to Π. Hence find the position vector of the point on Π which is closest to A. **[4]**

(iii) Hence write down a vector equation of the line in Π which is parallel to l and closest to it. **[1]**

7 **(i)** By expressing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\sin^5\theta \equiv \tfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$ **[4]**

(ii) Hence solve the equation

$$\sin 5\theta + 4\sin\theta = 5\sin 3\theta$$

for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. **[4]**

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8 G consists of the set of matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where a and b are real and $a^2 + b^2 \neq 0$, combined under the operation of matrix multiplication.

(i) Prove that G is a group. You may assume that matrix multiplication is associative. **[6]**

(ii) Determine whether G is commutative. **[2]**

(iii) Find the order of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. **[3]**

END OF QUESTION PAPER