

**ADVANCED GCE
MATHEMATICS (MEI)**

4754/01B

Applications of Advanced Mathematics (C4) Paper B: Comprehension

INSERT

TUESDAY 22 JANUARY 2008

Afternoon
Time: Up to 1 hour

INSTRUCTIONS TO CANDIDATES

- This insert contains the text for use with the questions.

This document consists of **8** printed pages.

The Golden Ratio

Introduction

In mathematics there are many important constants. You have met some of these, such as π , the ratio of the circumference of a circle to its diameter, and e , the base of natural logarithms.

In this article, you will be introduced to another important constant and some of its properties, one of which is linked to sequences. 5

Geometric sequences

The sequence

$$1, x, x^2, x^3, x^4, x^5, \dots \quad (*)$$

is a geometric sequence with first term 1 and common ratio x . You can substitute various values for x and each one will generate a new geometric sequence. 10

For example, $x = 3$ generates the geometric sequence

$$1, 3, 9, 27, 81, 243, \dots$$

and $x = -\frac{1}{2}$ generates the geometric sequence

$$1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \dots \quad 15$$

Notice that neither of these is an arithmetic sequence, prompting the following question.

Are there any values of x for which the geometric sequence generated is also an arithmetic sequence?

In order to answer this question, it is helpful to think about the first three terms of the sequence (*).

Suppose that $1, x, x^2$ are the first three terms of an arithmetic sequence. Then the difference between the first two terms, $x - 1$, is equal to the difference between the next two terms, $x^2 - x$, so that 20

$$x - 1 = x^2 - x.$$

This equation can be written in the form $x^2 - 2x + 1 = 0$. This has a repeated root of $x = 1$; the only value of x for which $1, x, x^2, \dots$ is both an arithmetic and a geometric sequence is $x = 1$.

You will notice that $x = 1$ leads to the rather uninteresting sequence $1, 1, 1, 1, 1, \dots$ 25

Additive sequences

In the Fibonacci sequence, the first two terms are given as 1, 1. After that, each term is the sum of the two previous terms. The Fibonacci sequence is

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

Another sequence with the same property, that each term is the sum of the two previous terms, is the Lucas sequence 30

$$2, 1, 3, 4, 7, 11, 18, 29, \dots$$

In general, you can choose any two numbers for the first two terms, call them u_1 and u_2 , and then use the rule

$$u_{n+1} = u_n + u_{n-1} \text{ for } n \geq 2 \quad 35$$

to generate subsequent terms. For the purposes of this article, any sequence with this property will be called an *additive sequence*.

Now think again about the geometric sequence (*). Are there any values of x for which this sequence is an additive sequence?

To be an additive sequence, x must satisfy each of the following equations. 40

$$\begin{aligned} x^2 &= x + 1 \\ x^3 &= x^2 + x \\ x^4 &= x^3 + x^2 \\ &\vdots \end{aligned}$$

The first equation shows that $x \neq 0$. So each of the other equations reduces to the quadratic equation $x^2 = x + 1$. The two roots of this quadratic equation are $x = \frac{1 + \sqrt{5}}{2}$ and $x = \frac{1 - \sqrt{5}}{2}$. Therefore these are the only two values of x for which the sequence (*) is both a geometric sequence and an additive sequence. 45

The Golden Ratio

The first of these numbers, $\frac{1 + \sqrt{5}}{2}$, is called the Golden Ratio and is denoted by the Greek letter ϕ , pronounced 'phi'. 50

Substituting $x = \phi$ in the sequence (*) gives

$$1, \phi, \phi^2, \phi^3, \phi^4, \phi^5, \dots$$

When written in this form, it is not at all obvious that it is an additive sequence. However, using the fact that $\phi^2 = \phi + 1$, the third term can be written as $\phi + 1$ and the fourth term, ϕ^3 , can be written as $\phi^2 + \phi$ which simplifies to $2\phi + 1$. Continuing in this way, the sequence can be written as 55

$$1, \phi, \phi + 1, 2\phi + 1, 3\phi + 2, 5\phi + 3, \dots$$

confirming the additive property $u_{n+1} = u_n + u_{n-1}$.

The Golden Rectangle

The rectangle ABCD in Fig. 1 has width $\phi = \frac{1 + \sqrt{5}}{2}$ units and height 1 unit.

60

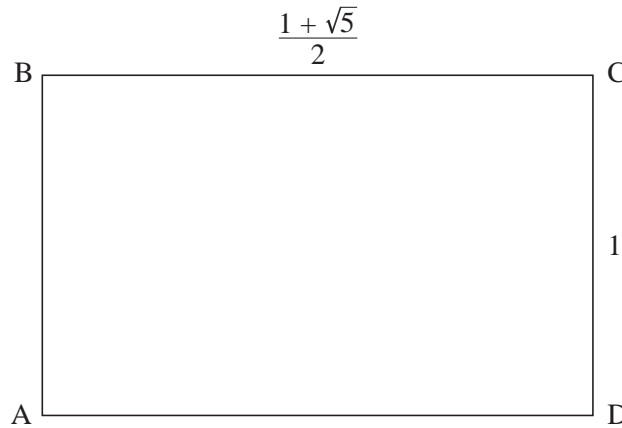


Fig. 1

The ratio of the sides of this rectangle is $\frac{1 + \sqrt{5}}{2} : 1$.

Many artists and architects believe that, of all rectangles, ones with this ratio are particularly pleasing to the human eye. For this reason approximations to these rectangles have been used extensively in art and architecture.

Any rectangle with sides in the ratio $\frac{1 + \sqrt{5}}{2} : 1$ is called a Golden Rectangle.

65

Golden Rectangles have many interesting properties, some of which will be described here.

First remove a unit square, ABEF, from a Golden Rectangle, as shown in Fig. 2.

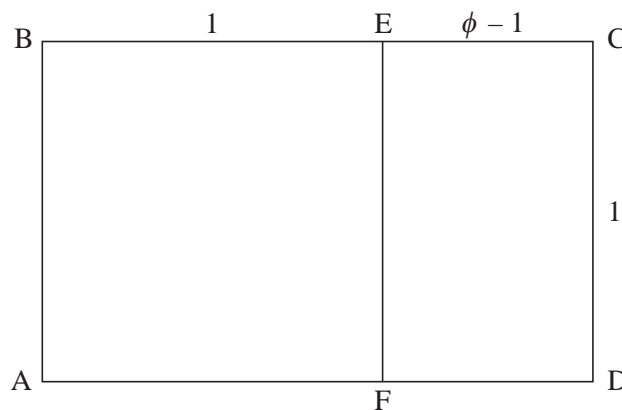


Fig. 2

Then using $\phi^2 = \phi + 1$, the ratio of the sides of rectangle ECDF may be found as follows.

$$\begin{aligned} CD : DF &= 1 : \phi - 1 \\ &= \phi : \phi^2 - \phi \\ &= \phi : 1 \end{aligned}$$

70

Therefore the ratio is the same for rectangle ECDF as for rectangle ABCD. So rectangle ECDF is also a Golden Rectangle.

This shows a property of a Golden Rectangle: any Golden Rectangle can be divided into a square and a smaller Golden Rectangle. 75

So if you remove a square ECGH from the Golden Rectangle ECDF, as shown in Fig. 3, then the ratio of the sides of rectangle FHGD is the same as that for rectangle ABCD.

This implies that the point H must lie on the line BD.

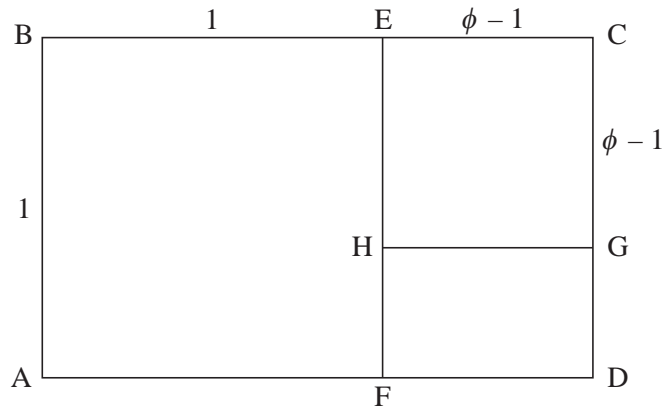


Fig. 3

Continuing in this way, removing a square from the Golden Rectangle just constructed, you can form a sequence of Golden Rectangles as shown in Fig. 4. The lines BD and CF are also shown. 80

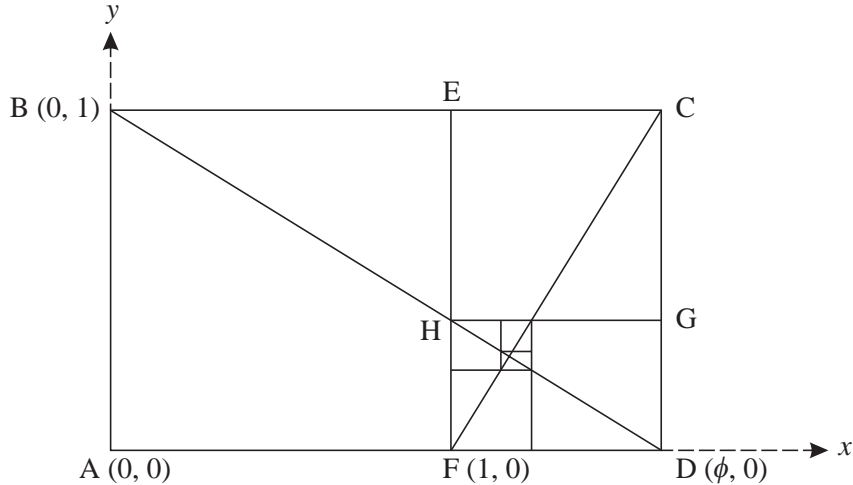


Fig. 4

This process can be continued indefinitely, each rectangle generated being similar to the original rectangle ABCD. Each square has a corner on the line BD and another corner on the line CF.

Taking A as the origin, AD as the positive x-axis and AB as the positive y-axis, as in Fig. 4, it can be shown that the equation of the line BD is

$$\phi y + x = \phi$$

and the equation of the line CF is

$$(\phi - 1)y = x - 1,$$

where $\phi = \frac{1 + \sqrt{5}}{2}$.

It follows that lines BD and CF are perpendicular and Q, the point of intersection of these two lines, has coordinates $\left(\frac{\phi + 1}{2\phi - 1}, \frac{\phi - 1}{2\phi - 1}\right)$.

90

Spirals

In Fig. 5, circular arcs have been drawn in each square. Arc AE has centre F, arc EG has centre H, and so on.

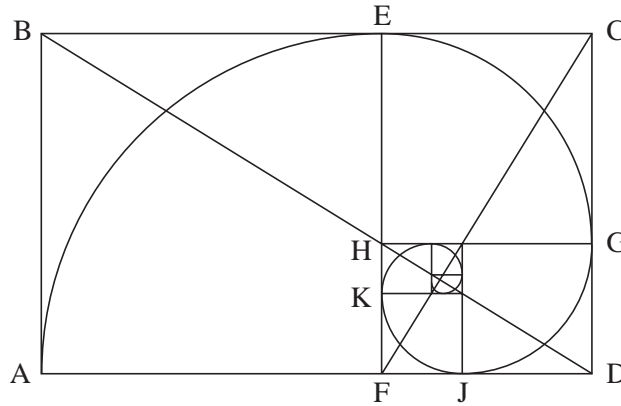


Fig. 5

In this spiral, the radius changes abruptly as it moves into each new square. For example, at the point E the radius changes from 1 unit in square ABEF to $\frac{-1 + \sqrt{5}}{2}$ units in square ECGH.

95

It is, however, possible to construct a spiral through points A, E, G, J, K and so on (see Fig. 6) in a different way so that these 'jumps' are smoothed out.

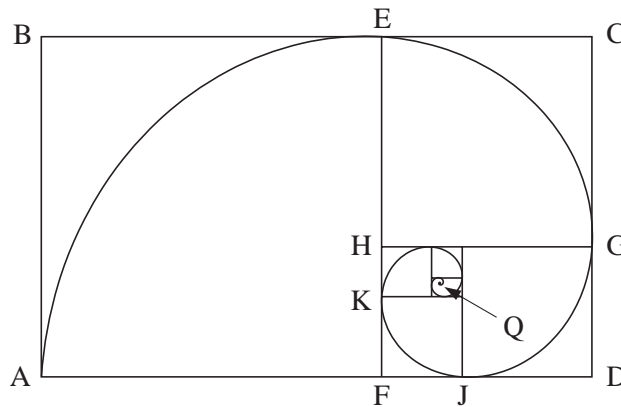


Fig. 6

As a point P moves along the resulting spiral from A towards Q, the distance QP decreases uniformly. During each full turn, the distance QP decreases by a constant factor of

$$\phi^4 = \left(\frac{1 + \sqrt{5}}{2}\right)^4 = \frac{7 + 3\sqrt{5}}{2} \approx 6.9.$$

100

For example, if you draw a straight line from Q, such as the one shown in Fig. 7, and measure the distances QP_1 and QP_2 you will find that $QP_1 \approx 6.9 \times QP_2$.

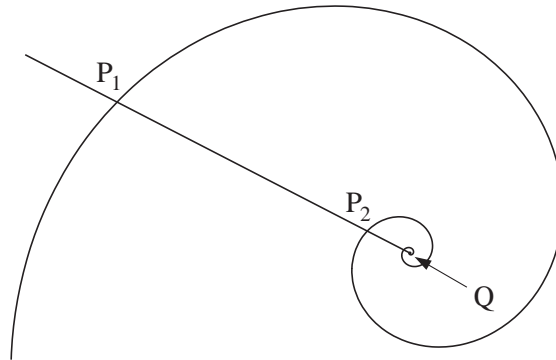


Fig. 7

This spiral is called a logarithmic spiral. It has the property that, although any two segments of the spiral differ in size, they have the same shape. If you zoomed in near the centre of the spiral and then enlarged this view, the shape would fit exactly onto the original spiral.

105



© iStockphoto.com / Horst Gossmann

Fig. 8

Fig. 8 is a photograph of the shell of a nautilus, a marine creature. The shape of the shell forms a logarithmic spiral. This is just one of many examples where the Golden Ratio may be found in the natural world.

Acknowledgements:

Fig. 8 © iStockphoto.com / Horst Gossmann

Permission to reproduce items where third-party owned material protected by copyright is included has been sought and cleared where possible. Every reasonable effort has been made by the publisher (OCR) to trace copyright holders, but if any items requiring clearance have unwittingly been included, the publisher will be pleased to make amends at the earliest possible opportunity.

OCR is part of the Cambridge Assessment Group. Cambridge Assessment is the brand name of University of Cambridge Local Examinations Syndicate (UCLES), which is itself a department of the University of Cambridge.