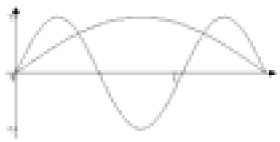
4727 Further Pure Mathematics 3

1 (a) (i) e.g. $ap \neq pa \Rightarrow$ not commutative	B1 **1**	For correct reason and conclusion
(ii) 3	B1 **1**	For correct number
(iii) e, a, b	B1 **1**	For correct elements
(b) c^3 has order 2 c^4 has order 3 c^5 has order 6	B1 B1 B1 **3** **6**	For correct order For correct order For correct order
2 $m^2 - 8m + 16 = 0$ $\Rightarrow m = 4$ $\Rightarrow$ CF $(y =)\ (A + Bx)e^{4x}$ For PI try $y = px + q$ $\Rightarrow -8p + 16(px + q) = 4x$ $\Rightarrow p = \frac{1}{4} \quad q = \frac{1}{8}$ $\Rightarrow$ GS $y = (A + Bx)e^{4x} + \frac{1}{4}x + \frac{1}{8}$	M1 A1 A1√ M1 A1 A1 B1√ **7** **7**	For stating and attempting to solve auxiliary eqn For correct solution For CF of correct form. f.t. from m For using linear expression for PI For correct coefficients For GS = CF + PI. Requires $\boxed{y =}$. f.t. from CF and PI with 2 arbitrary constants in CF and none in PI
3 (i) line segment OA	B1 B1 **2**	For stating line through O *OR* A For correct description **AEF**
(ii) $(\mathbf{r} - \mathbf{a}) \times (\mathbf{r} - \mathbf{b}) = \overrightarrow{AP} \times \overrightarrow{BP}$ $= \lvert AP \rvert \lvert BP \rvert \sin \pi \,.\, \hat{\mathbf{n}} = \mathbf{0}$	B1 B1 **2**	For identifying $\mathbf{r} - \mathbf{a}$ with $\overrightarrow{AP}$ and $\mathbf{r} - \mathbf{b}$ with $\overrightarrow{BP}$ Allow direction errors For using × of 2 parallel vectors = **0** *OR* sin π = 0 *or* sin 0 = 0 in an appropriate vector expression
(iii) line through O parallel to AB	B1 B1 B1 **3** **7**	For stating line For stating through O For stating correct direction **SR** For $\overrightarrow{AB}$ or $\overrightarrow{BA}$ allow B1 B0 B1
4 $(C + \mathrm{i}S =)\ \int_0^{\frac{1}{2}\pi} e^{2x}(\cos 3x + \mathrm{i}\sin 3x)(\mathrm{d}x)$ $\cos 3x + \mathrm{i}\sin 3x = e^{3\mathrm{i}x}$ $\int_0^{\frac{1}{2}\pi} e^{(2+3\mathrm{i})x}(\mathrm{d}x) = \frac{1}{2+3\mathrm{i}}\left[e^{(2+3\mathrm{i})x}\right]_0^{\frac{1}{2}\pi}$ $= \frac{2-3\mathrm{i}}{4+9}\left(e^{(2+3\mathrm{i})\frac{1}{2}\pi} - e^0\right) = \frac{2-3\mathrm{i}}{13}\left(-\mathrm{i}\,e^{\pi} - 1\right)$ $= \left\{\frac{1}{13}\left(-2 - 3e^{\pi} + \mathrm{i}(3 - 2e^{\pi})\right)\right\}$ $C = -\frac{1}{13}\left(2 + 3e^{\pi}\right)$ $S = \frac{1}{13}\left(3 - 2e^{\pi}\right)$	 B1 M1* A1 A1 M1 (dep*) M1 (dep*) A1 A1 **8**	 For using de Moivre, seen or implied For writing as a single integral in exp form For correct integration (ignore limits) For substituting limits correctly (unsimplified) (may be earned at any stage) For multiplying by complex conjugate of 2+3i For equating real and/or imaginary parts For correct expression **AG** For correct expression

5 **(i)** IF $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$ *OR* $x\frac{dy}{dx} + y = x\sin 2x$	M1	For correct process for finding integrating factor *OR* for multiplying equation through by x
$\Rightarrow \frac{d}{dx}(xy) = x\sin 2x$	A1	For writing DE in this form (may be implied)
$\Rightarrow xy = \int x\sin 2x(dx)$	M1	For integration by parts the correct way round
$xy = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x(dx)$	A1	For 1st term correct
$xy = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\ (+c)$	M1	For their 1st term and attempt at integration of $\frac{\cos}{\sin}kx$
$\Rightarrow y = -\frac{1}{2}\cos 2x + \frac{1}{4x}\sin 2x + \frac{c}{x}$	A1 **6**	For correct expression for y
(ii) $\left(\frac{1}{4}\pi, \frac{2}{\pi}\right) \Rightarrow \frac{2}{\pi} = \frac{1}{\pi} + \frac{4c}{\pi} \Rightarrow c = \frac{1}{4}$	M1	For substituting $\left(\frac{1}{4}\pi, \frac{2}{\pi}\right)$ in solution
$\Rightarrow y = -\frac{1}{2}\cos 2x + \frac{1}{4x}\sin 2x + \frac{1}{4x}$	A1 **2**	For correct solution. Requires $\boxed{y=}$.
(iii) $(y \approx) -\frac{1}{2}\cos 2x$	B1√ **1** $\boxed{9}$	For correct function **AEF** f.t. from **(ii)**
6 **(i)**		***Either coordinates or vectors may be used*** Methods 1 and 2 may be combined, for a maximum of 5 marks
METHOD 1 State $B = (-1, -7, 2) + t(1, 2, -2)$	M1	For using vector normal to plane
On plane $\Rightarrow (-1+t) + 2(-7+2t) - 2(2-2t) = -1$	M1 M1	For substituting parametric form into plane For solving a linear equation in t
$\Rightarrow t = 2 \Rightarrow B = (1, -3, -2)$	A1	For correct coordinates
$AB = \sqrt{2^2 + 4^2 + 4^2}$ *OR* $2\sqrt{1^2 + 2^2 + 2^2} = 6$	A1 **5**	For correct length of AB
METHOD 2 $AB = \left\lvert \frac{-1 - 14 - 4 + 1}{\sqrt{1^2 + 2^2 + 2^2}} \right\rvert = 6$	M1	For using a correct distance formula
OR $AB = \mathbf{AC} \cdot \hat{\mathbf{AB}} = \frac{[6, 7, 1] \cdot [1, 2, -2]}{\sqrt{1^2 + 2^2 + 2^2}} = 6$	A1	For correct length of AB
$B = (-1, -7, 2) \pm 6\frac{(1, 2, -2)}{\sqrt{1^2 + 2^2 + 2^2}}$	M1	For using $B = A$ + length of $AB \times$ unit normal
$B = (-1, -7, 2) \pm (2, 4, -4)$	B1	For checking whether + or – is needed (substitute into plane equation)
$B = (1, -3, -2)$	A1	For correct coordinates (allow even if B0)
(ii) Find vector product of any two of $\pm[6, 7, 1]$, $\pm[6, -3, 0]$, $\pm(0, 10, 1)$	M1	For finding vector product of two relevant vectors
Obtain $k[1, 2, -20]$	A1	For correct vector **n**
$\theta = \cos^{-1}\frac{\lvert[1, 2, -2] \cdot [1, 2, -20]\rvert}{\sqrt{1^2 + 2^2 + 2^2}\sqrt{1^2 + 2^2 + 20^2}}$	M1* M1 (dep*)	For using scalar product of two normal vectors For stating both moduli in denominator
$\theta = \cos^{-1}\frac{45}{\sqrt{9}\sqrt{405}} = 41.8°\ (41.810...°, 0.72972...)$	A1√ A1 **6** $\boxed{11}$	For correct scalar product. f.t. from **n** For correct angle

7 (i) (a) $\sin\frac{6}{8}\pi = \frac{1}{\sqrt{2}}$, $\sin\frac{2}{8}\pi = \frac{1}{\sqrt{2}}$	B1 **1**	For verifying $\theta = \frac{1}{8}\pi$
(b) 	M1	For sketching $y = \sin 6\theta$ and $y = \sin 2\theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ *OR* any other correct method for solving $\sin 6\theta = \sin 2\theta$ for $\theta \neq k\frac{\pi}{2}$ *OR* appropriate use of symmetry *OR* attempt to verify a reasonable guess for θ
$\theta = \frac{3}{8}\pi$	A1 **2**	For correct θ
(ii) $\text{Im}\,(c + \text{i}\,s)^6 = 6c^5 s - 20c^3 s^3 + 6cs^5$	M1 A1	For expanding $(c + \text{i}\,s)^6$; at least 3 terms and 3 binomial coefficients needed For 3 correct terms
$\sin 6\theta = \sin\theta\left(6c^5 - 20c^3(1-c^2) + 6c(1-c^2)^2\right)$	M1	For using $s^2 = 1 - c^2$
$\sin 6\theta = \sin\theta\left(32c^5 - 32c^3 + 6c\right)$	A1	For any correct intermediate stage
$\sin 6\theta = 2\sin\theta\cos\theta\left(16c^4 - 16c^2 + 3\right)$	A1	For obtaining this expression correctly
$\sin 6\theta = \sin 2\theta\left(16\cos^4\theta - 16\cos^2\theta + 3\right)$	**5**	**AG**
(iii) $16c^4 - 16c^2 + 3 = 1$	M1	For stating this equation **AEF**
$\Rightarrow c^2 = \frac{2 \pm \sqrt{2}}{4}$	A1	For obtaining both values of c^2
− sign requires larger $\theta = \frac{3}{8}\pi$	A1 **3**	For stating and justifying $\theta = \frac{3}{8}\pi$ Calculator OK if figures seen
	11	

8 **(i)** Group A: $e = 6$ Group B: $e = 1$ Group C: $e = 2^0$ *OR* 1 Group D: $e = 1$	B1 B1 **2**	For any two correct identities For two other correct identities **AEF** for D, but not " $m = n$ "

(ii) *EITHER* / *OR*

A	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

orders of elements 1, 2, 4, 4 *OR* cyclic group

B	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

orders of elements 1, 2, 2, 2 *OR* non-cyclic group *OR* Klein group

C	2^0	2^1	2^2	2^3
2^0	2^0	2^1	2^2	2^3
2^1	2^1	2^2	2^3	2^0
2^2	2^2	2^3	2^0	2^1
2^3	2^3	2^0	2^1	2^2

orders of elements 1, 2, 4, 4 *OR* cyclic group

	B1* B1*	For showing group table *OR* sufficient details of orders of elements *OR* stating cyclic / non-cyclic / Klein group (as appropriate) for one of groups A, B, C for another of groups A, B, C
$A \not\cong B$	B1 (dep*)	For stating non-isomorphic } with sufficient detail
$B \not\cong C$	B1 (dep*)	For stating non-isomorphic } relating to the first 2 marks
$A \cong C$	B1 (dep*) **5**	For stating isomorphic }
(iii) $\dfrac{1+2m}{1+2n} \times \dfrac{1+2p}{1+2q} = \dfrac{1+2m+2p+4mp}{1+2n+2q+4nq}$	M1* M1 (dep*)	For considering product of 2 distinct elements of this form For multiplying out
$= \dfrac{1+2(m+p+2mp)}{1+2(n+q+2nq)} \equiv \dfrac{1+2r}{1+2s}$	A1 A1 **4**	For simplifying to form shown For identifying as correct form, so closed **SR** $\dfrac{\text{odd}}{\text{odd}} \times \dfrac{\text{odd}}{\text{odd}} = \dfrac{\text{odd}}{\text{odd}}$ earns full credit **SR** If clearly attempting to prove commutativity, allow at most M1
(iv) Closure not satisfied Identity and inverse not satisfied	B1 B1 **2** **13**	For stating closure For stating identity and inverse **SR** If associativity is stated as not satisfied, then award at most B1 B0 *OR* B0 B1