

4727 Further Pure Mathematics 3

1 (i) (a)	$(n =) 3$	B1	1	For correct n
(b)	$(n =) 6$	B1	1	For correct n
(c)	$(n =) 4$	B1	1	For correct n
(ii)	$(n =) 4, 6$	B1		For <i>either</i> 4 or 6
		B1	2	For both 4 and 6 and no extras Ignore all $n \dots 8$ SR B0 B0 if more than 3 values given, even if they include 4 or 6
		5		
2 (i)	$\frac{\sqrt{3}+i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i} = \frac{1}{2} + \frac{1}{2}i\sqrt{3}$	M1		For multiplying top and bottom by complex conjugate
	OR $\frac{\sqrt{3}+i}{\sqrt{3}-i} = \frac{2e^{\frac{1}{6}\pi i}}{2e^{-\frac{1}{6}\pi i}}$			OR for changing top and bottom to polar form
	$= (1)e^{\frac{1}{3}\pi i}$	A1		For $(r =) 1$ (may be implied)
		A1	3	For $(\theta =) \frac{1}{3}\pi$ SR Award maximum A1 A0 if $e^{i\theta}$ form is not seen
(ii)	$\left(e^{\frac{1}{3}\pi i}\right)^6 = e^{2\pi i} = 1 \Rightarrow (n =) 6$	M1		For use of $e^{2\pi i} = 1$, $e^{i\pi} = -1$, $\sin k\pi = 0$ or $\cos k\pi = \pm 1$ (may be implied)
		A1	2	For $(n =) 6$ SR For $(n =) 3$ only, award M1 A0
		5		
3 (i)	$\mathbf{n} = [2, 1, 3] \times [3, 1, 5]$	M1		For using direction vectors and attempt to find vector product
	$= [2, -1, -1]$	A1	2	For correct direction (allow multiples)
(ii)	$d = \frac{ [5, 2, 1] \cdot [2, -1, -1] }{\sqrt{6}}$	B1		For $(\mathbf{AB} =) [5, 2, 1]$ or any vector joining lines
		M1		For attempt at evaluating $\mathbf{AB} \cdot \mathbf{n}$
		M1		For $ \mathbf{n} $ in denominator
	$= \frac{7}{\sqrt{6}} = \frac{7}{6}\sqrt{6} = 2.8577$	A1	4	For correct distance
		6		

4	$m^2 + 4m + 5 (= 0) \Rightarrow m = \frac{-4 \pm \sqrt{16 - 20}}{2}$	M1	For attempt to solve correct auxiliary equation
	$= -2 \pm i$	A1	For correct roots
	CF = $e^{-2x}(C \cos x + D \sin x)$	A1✓	For correct CF (here or later). f.t. from m AEtrig but not forms including e^{ix}
	PI = $p \sin 2x + q \cos 2x$	B1	For stating a trial PI of the correct form
	$y' = 2p \cos 2x - 2q \sin 2x$	M1	For differentiating PI twice and substituting into the DE
	$y'' = -4p \sin 2x - 4q \cos 2x$		
	$\cos 2x(-4q + 8p + 5q)$	A1	For correct equation
	$+ \sin 2x(-4p - 8q + 5p) = 65 \sin 2x$		
	$\left. \begin{matrix} 8p + q = 0 \\ p - 8q = 65 \end{matrix} \right\} \quad p = 1, \quad q = -8$	M1	For equating coefficients of $\cos 2x$ and $\sin 2x$ and attempting to solve for p and/or q
	PI = $\sin 2x - 8 \cos 2x$	A1	For correct p and q
$\Rightarrow y =$	B1✓	For using GS = CF + PI, with 2 arbitrary constants in CF and none in PI	
$e^{-2x}(C \cos x + D \sin x) + \sin 2x - 8 \cos 2x$			
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9			
5 (i)	$y = u - \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{1}{x^2}$	M1	For differentiating substitution
		A1	For correct expression
	$x^3 \left(\frac{du}{dx} + \frac{1}{x^2} \right) = x \left(u - \frac{1}{x} \right) + x + 1$	M1	For substituting y and $\frac{dy}{dx}$ into DE
	$\Rightarrow x^2 \frac{du}{dx} = u$	A1 4	For obtaining correct equation AG
(ii)	METHOD 1		
	$\int \frac{1}{u} du = \int \frac{1}{x^2} dx \Rightarrow \ln ku = -\frac{1}{x}$	M1	For separating variables and attempt at integration
		A1	For correct integration (k not required here)
	$ku = e^{-1/x} \Rightarrow k \left(y + \frac{1}{x} \right) = e^{-1/x}$	M1	For any 2 of $\left. \begin{matrix} k \text{ seen,} \\ \text{exponentiating,} \\ \text{substituting for } u \end{matrix} \right\}$
		M1	
	$\Rightarrow y = Ae^{-1/x} - \frac{1}{x}$	A1 5	For correct solution AEF in form $y = f(x)$
	METHOD 2		
	$\frac{du}{dx} - \frac{1}{x^2} u = 0 \Rightarrow \text{I.F. } e^{\int -1/x^2 dx} = e^{1/x}$	M1	For attempt to find I.F.
	$\Rightarrow \frac{d}{dx} (u e^{1/x}) = 0$	A1	For correct result
	$u e^{1/x} = k \Rightarrow y + \frac{1}{x} = k e^{-1/x}$	M1	From $\boxed{u \times \text{I.F.} =}$, for k seen for substituting for u } in either order
		M1	
	$\Rightarrow y = k e^{-1/x} - \frac{1}{x}$	A1	For correct solution AEF in form $y = f(x)$
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6 (i)	METHOD 1		
	Use 2 of [−4, 2, 0], [0, 0, 3], [−4, 2, 3], [4, −2, 3] or multiples	M1	For finding vector product of 2 appropriate vectors in plane <i>ACGE</i>
	$\mathbf{n} = k [1, 2, 0]$	A1	For correct $\mathbf{n}$
	Use $A[4, 0, 0], C[0, 2, 0], G[0, 2, 3]$ OR $E[4, 0, 3]$	M1	For substituting a point in the plane
	$\mathbf{r} \cdot [1, 2, 0] = 4$	A1	4 For correct equation. AEF in this form
	METHOD 2		
	$\mathbf{r} = [4, 0, 0] + \lambda [-4, 2, 0] + \mu [0, 0, 3]$	M1	For writing plane in 2-parameter form
	$\Rightarrow x = 4 - 4\lambda, y = 2\lambda, z = 3\mu$	A1	For 3 correct equations
	$x + 2y = 4$	M1	For eliminating λ (and μ)
(ii)	$\Rightarrow \mathbf{r} \cdot [1, 2, 0] = 4$	A1	For correct equation. AEF in this form
	$\theta = \cos^{-1} \frac{ [3, 0, -4] \cdot [1, 2, 0] }{\sqrt{3^2 + 0^2 + 4^2} \sqrt{1^2 + 2^2 + 0^2}}$	B1√	For using correct vectors (allow multiples). f.t. from $\mathbf{n}$
		M1	For using scalar product
		M1	For multiplying both moduli in denominator
	$\theta = \cos^{-1} \frac{3}{5\sqrt{5}} = 74.4^\circ$ (74.435...°, 1.299...)	A1	4 For correct angle
(iii)	AM: $(\mathbf{r} =) [4, 0, 0] + t[-2, 2, 3]$ (or $[2, 2, 3] + t[-2, 2, 3]$)	M1	For obtaining parametric expression for AM
		A1	For correct expression seen or implied
	$3(4 - 2t) - 4(3t) = 0$ (or $3(2 - 2t) - 4(3 + 3t) = 0$)	M1	For finding intersection of AM with <i>ACGE</i>
	$t = \frac{2}{3}$ (or $t = -\frac{1}{3}$) OR $\mathbf{w} = [\frac{8}{3}, \frac{4}{3}, 2]$	A1	For correct t OR position vector
	$AW : WM = 2 : 1$	A1	5 For correct ratio
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7 (i)	$x + y - a \in R$	B1	For stating closure is satisfied
	(a) $(x * y) * z = (x + y - a) * z = x + y + z - 2a$	M1	For using 3 distinct elements bracketed both ways
	$x * (y * z) = x * (y + z - a) = x + y + z - 2a$	A1	For obtaining the same result twice for associativity
	$x + e - a = x \Rightarrow e = a$	B1	SR 3 distinct elements bracketed once, expanded, and symmetry noted scores M1 A1
	$x + x^{-1} - a = a \Rightarrow x^{-1} = 2a - x$	M1	For stating identity = a
		M1	For attempting to obtain inverse of x
		A1	6 For obtaining inverse = $2a - x$
			OR for showing that inverses exist, where $x + x^{-1} = 2a$
(b)	$x + y - a = y + x - a \Rightarrow$ commutative	B1	1 For stating commutativity is satisfied, with justification
(c)	x order 2 $\Rightarrow x * x = e \Rightarrow 2x - a = e$	M1	For obtaining equation for an element of order 2
	$\Rightarrow 2x - a = a \Rightarrow x = a = e$	A1	2 For solving and showing that the only solution is the identity (which has order 1)
	OR $x = x^{-1} \Rightarrow x = 2a - x \Rightarrow x = a = e$ $\Rightarrow$ no elements of order 2		OR For proving that there are no self-inverse elements (other than the identity)

(ii)	e.g. $2+1-5=-2 \notin \mathbb{R}^+$ $\Rightarrow$ not closed	M1	For attempting to disprove closure
		A1	For stating closure is not necessarily satisfied ($0 < x+y$, 5 required)
	e.g. $2 \times 5 - 11 = -1 \notin \mathbb{R}^+$ $\Rightarrow$ no inverse	M1	For attempting to find an element with no inverse
		A1 4	For stating inverse is not necessarily satisfied ($x \dots 10$ required)
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8 (i)	$\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$	B1	z may be used for $e^{i\theta}$ throughout For expression for $\sin \theta$ seen or implied
		M1	For expanding $(e^{i\theta} - e^{-i\theta})^6$
	$\sin^6 \theta =$		At least 4 terms and 3 binomial coefficients required.
	$-\frac{1}{64}(e^{6i\theta} - 6e^{4i\theta} + 15e^{2i\theta} - 20 + 15e^{-2i\theta} - 6e^{-4i\theta} + e^{-6i\theta})$		For correct expansion. Allow $\frac{\pm(i)}{64}(\dots)$
	$= -\frac{1}{64}(2 \cos 6\theta - 12 \cos 4\theta + 30 \cos 2\theta - 20)$	A1	
	$\sin^6 \theta = -\frac{1}{32}(\cos 6\theta - 6 \cos 4\theta + 15 \cos 2\theta - 10)$	M1	For grouping terms and using multiple angles
		A1 5	For answer obtained correctly AG
(ii)	$\cos^6 \theta = \text{OR } \sin^6\left(\frac{1}{2}\pi - \theta\right) =$	M1	For substituting $\left(\frac{1}{2}\pi - \theta\right)$ for θ throughout
	$-\frac{1}{32}(\cos(3\pi - 6\theta) - 6 \cos(2\pi - 4\theta) + 15 \cos(\pi - 2\theta) - 10)$	A1	For correct unsimplified expression
	$\cos^6 \theta = \frac{1}{32}(\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10)$	A1 3	For correct expression with $\cos n\theta$ terms AEF
(iii)	$\int_0^{\frac{1}{4}\pi} \frac{1}{32}(-2 \cos 6\theta - 30 \cos 2\theta) d\theta$	B1✓	For correct integral. f.t. from $\sin^6 \theta - \cos^6 \theta$
	$= -\frac{1}{16} \left[\frac{1}{6} \sin 6\theta + \frac{15}{2} \sin 2\theta \right]_0^{\frac{1}{4}\pi}$	M1	For integrating $\cos n\theta$, $\sin n\theta$ or $e^{in\theta}$
		A1✓	For correct integration. f.t. from integrand
	$= -\frac{11}{24}$	A1 4	For correct answer WWW
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