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ADVANCED GCE

MATHEMATICS (MEI) 4754B

Applications of Advanced Mathematics (C4) Paper B: Comprehension

INSERT

Friday 14 January 2011
Afternoon

Duration: Up to 1 hour

4754B

INFORMATION FOR CANDIDATES

- This insert contains the text for use with the questions.
- This document consists of **12** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this insert for marking; it should be retained in the centre or destroyed.

The Art Gallery Problem

Introduction

Closed-circuit television (CCTV) is widely used to monitor activities in public areas. Usually a CCTV camera is fixed to a wall, either on the inside or the outside of a building, in such a way that it can rotate to survey a wide region.

Art galleries need to take surveillance very seriously. Many galleries use a combination of guards, who can move between rooms, and fixed cameras.

This article addresses the problem of how to ensure that all points in a gallery can be observed, using the minimum number of guards or cameras. Two typical layouts will be considered: a standard layout consisting of a chain of rectangular rooms with one route through them; and a polygonal, open-plan gallery.

Standard layout

Fig. 1 shows the plan view of an art gallery. It contains six rectangular rooms in a chain.

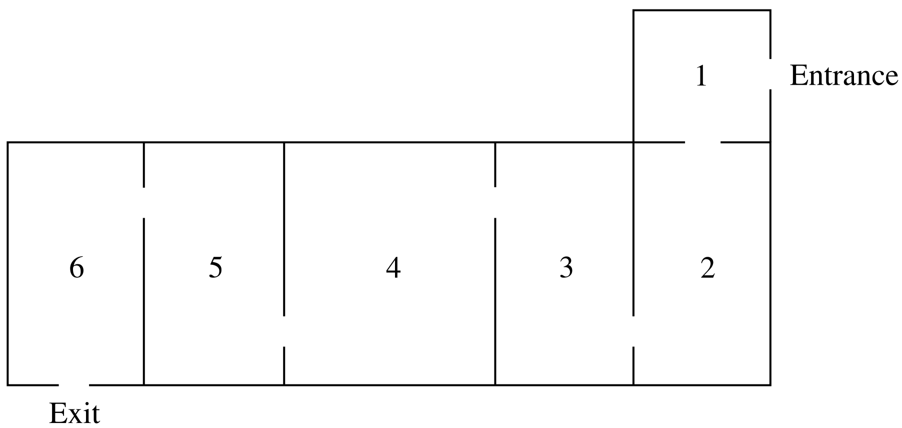

Fig. 1

Imagine you want to employ the minimum number of guards so that every point in the gallery can be observed by at least one of them. How many guards are needed and where would you position them?

It turns out that 3 guards are needed; they should be positioned in the doorways between rooms 1 and 2, rooms 3 and 4 and rooms 5 and 6.

Table 2 shows the number of guards, G, needed for different numbers of rectangular rooms, r, arranged in a chain.

Number of rooms, r	1	2	3	4	5	6	7	8	9	10
Number of guards, G	1	1	2	2	3	3	4	4	5	5

Table 2

One formula which expresses G in terms of r is

$$G = \frac{(2r+1)-(-1)^r}{4}.$$

This can be expressed more concisely using the 'floor function', denoted by $\lfloor x \rfloor$. This is defined as the greatest integer less than or equal to x. For example, $\lfloor 3.9 \rfloor = 3$ and $\lfloor 5 \rfloor = 5$.

Using this notation, the formula which expresses G in terms of r is

$$G = \left\lfloor \frac{r+1}{2} \right\rfloor.$$

It has been assumed that the thickness of the walls does not obscure the view of a guard positioned in a doorway. In reality the guard would need to take a step into a room to ensure that he or she can see into all corners. In this sense, guards have the advantage over fixed cameras as the thickness of the walls between rooms would result in the view of a camera positioned in a doorway being partially blocked.

The remainder of this article is concerned with positioning fixed cameras in galleries which are not divided into separate rooms.

Open-plan gallery

An open-plan gallery has no interior walls.

Figs 3a, 3b, 3c and 3d show the plan views of four different open-plan galleries. Each one is made up of 12 straight exterior walls with no interior walls.

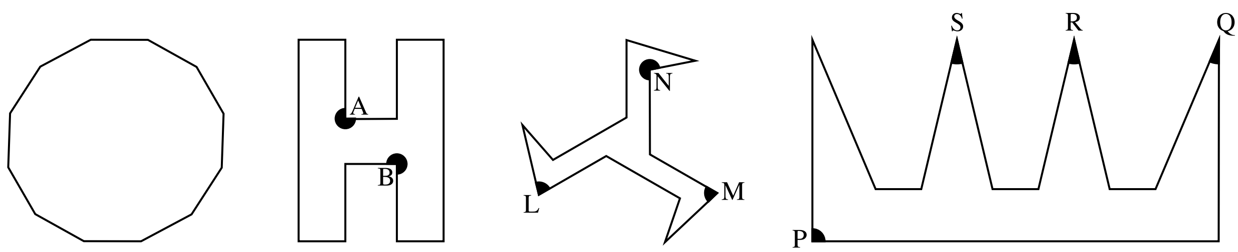

Fig. 3a **Fig. 3b** **Fig. 3c** **Fig. 3d**

Fig. 3a is a gallery in the shape of a convex dodecagon. A single camera mounted at any point on any wall would be able to observe the entire gallery.

In order to ensure that all points can be observed, it turns out that the galleries shown in Figs 3b, 3c and 3d require 2, 3 and 4 cameras respectively. Possible positions for the cameras are shown. The cameras do not have to be positioned in corners but corners are often convenient locations for them.

An interesting question is whether there could be an open-plan gallery with 12 straight walls that requires more than 4 cameras.

A procedure called *triangulation* helps to answer this question and to determine how many cameras may be required for open-plan galleries with any number of walls. This will be illustrated for a gallery with 11 walls.

Triangulation

Fig. 4 shows a polygon with 11 edges.

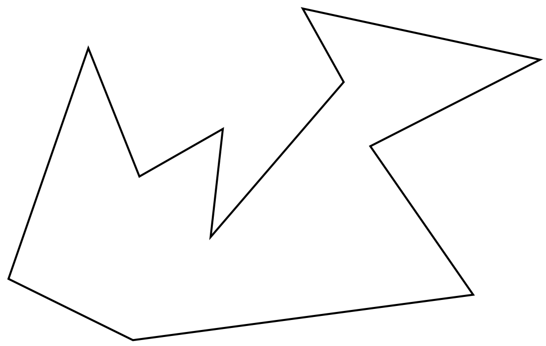

Fig. 4

It can be proved that every polygon can be split into triangles without creating any new vertices; this procedure is called triangulation. Figs 5a and 5b show two ways of triangulating the polygon shown in Fig. 4.

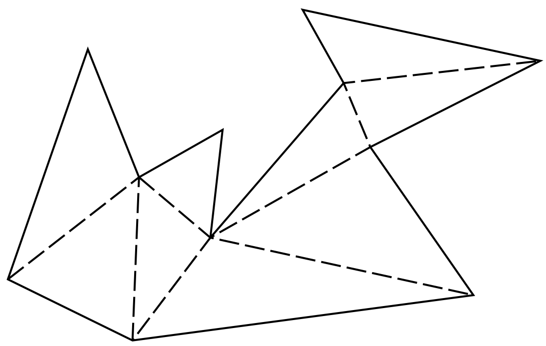

Fig. 5a

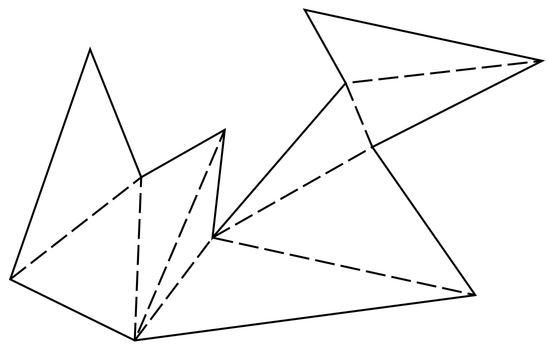

Fig. 5b

There is a two-stage process to decide on possible positions of the cameras in an open-plan gallery. The first stage is to add new internal edges to triangulate the polygon which represents the gallery. Then each vertex is labelled either A, B or C using the procedure shown in Fig. 6.

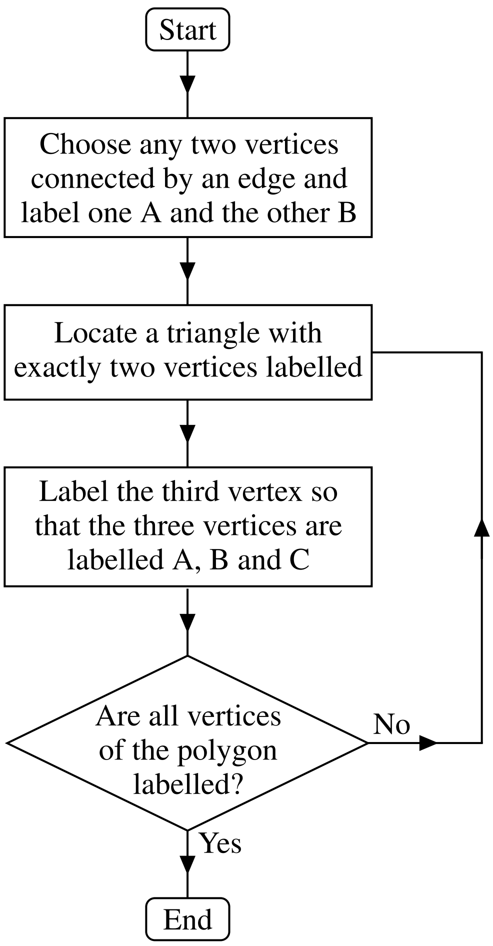

Fig. 6

This procedure is illustrated using the polygon in Fig. 4; for ease of reference the polygon has been reproduced in Fig. 7 with the vertices numbered 1 to 11.

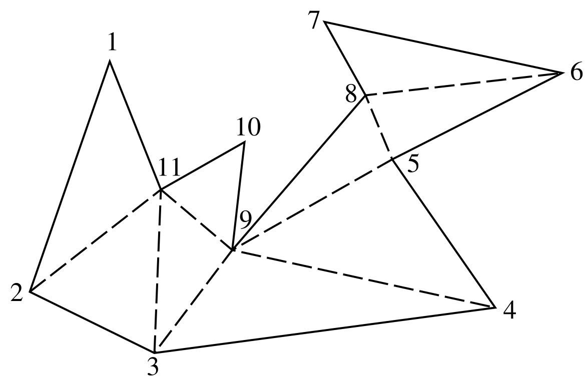

Fig. 7

Fig. 8 shows the result after choosing vertices 1 and 2 and labelling them A and B respectively.

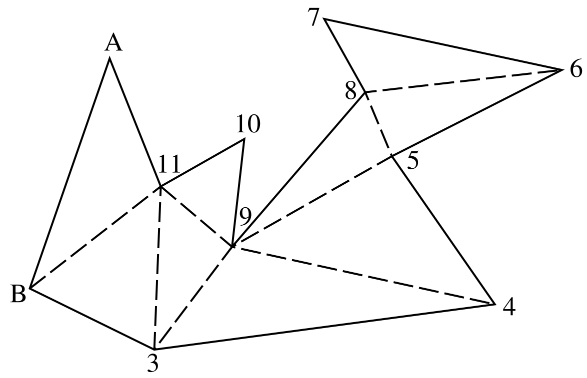

Fig. 8

Vertex 11 is now assigned the label C so that this triangle (shaded in Fig. 9) has vertices labelled A, B and C.

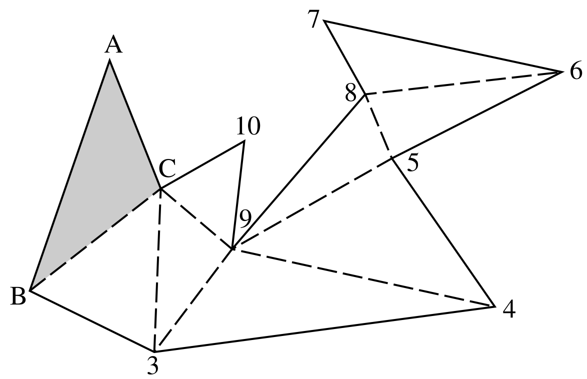

Fig. 9

The next vertex to be labelled is vertex 3; this is assigned the label A. The remaining vertices are labelled in the following order.

Vertex 9 is labelled B;
Vertex 10 is labelled A;
Vertex 4 is labelled C;
Vertex 5 is labelled A;
Vertex 8 is labelled C;
Vertex 6 is labelled B;
Vertex 7 is labelled A.

The resulting labelling is shown in Fig. 10.

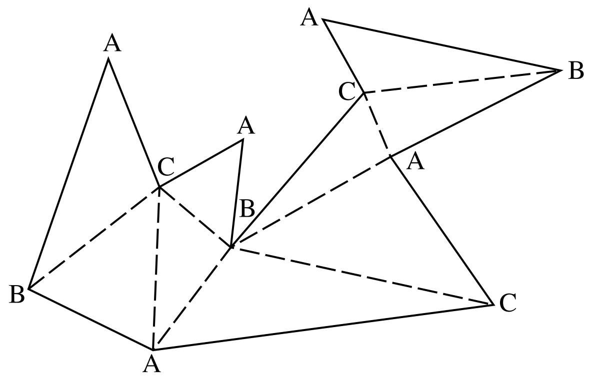

Fig. 10

The 11 vertices have all been assigned a label A, B or C; the numbers of vertices with each label is given in Table 11.

Label	Number of vertices assigned this label
A	5
B	3
C	3

Table 11

Since every triangle contains a vertex labelled A, positioning a camera at each vertex A will ensure that the whole gallery can be observed. This would require 5 cameras.

Alternatively, positioning cameras at the 3 vertices labelled B would ensure that the whole gallery can be observed using only 3 cameras.

Positioning cameras at the 3 vertices labelled C would also be sufficient.

Since $\frac{11}{3} < 4$, in any 11-sided polygon at least one of A, B or C must appear as a label at most 3 times.

A generalisation of this argument demonstrates that an open-plan gallery with n walls can be covered with $\left\lfloor \frac{n}{3} \right\rfloor$ cameras or fewer. Table 12 shows this information.

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$\left\lfloor \frac{n}{3} \right\rfloor$	1	1	1	2	2	2	3	3	3	4	4	4	5

Table 12

It is always possible to design an open-plan gallery with n walls that requires $\left\lfloor \frac{n}{3} \right\rfloor$ cameras.

Triangulation in practice

For a given open-plan gallery, different triangulations may suggest different numbers of cameras.

Figs 13a and 13b show two different triangulations on a particular hexagon. Fig. 13a shows that the whole gallery can be observed using 2 cameras (as in Table 12). However, Fig. 13b shows that only one camera is necessary.

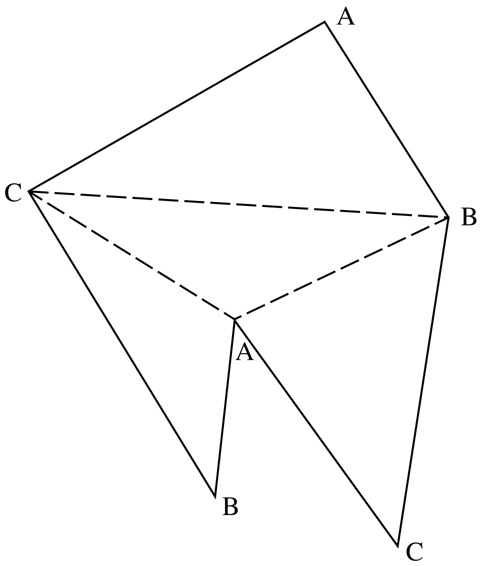

Fig. 13a

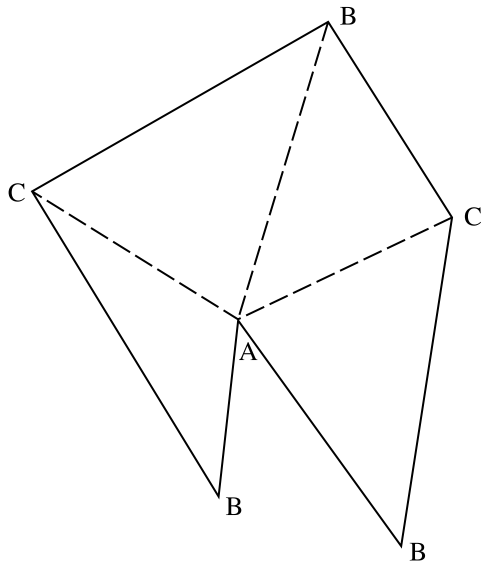

Fig. 13b

Surveillance of the outside of a building

Fig. 14 shows a pentagonal building with corners numbered, in order, from 1 to 5. To observe all of the outside of this building, 3 cameras could be positioned at the odd-numbered corners as shown.

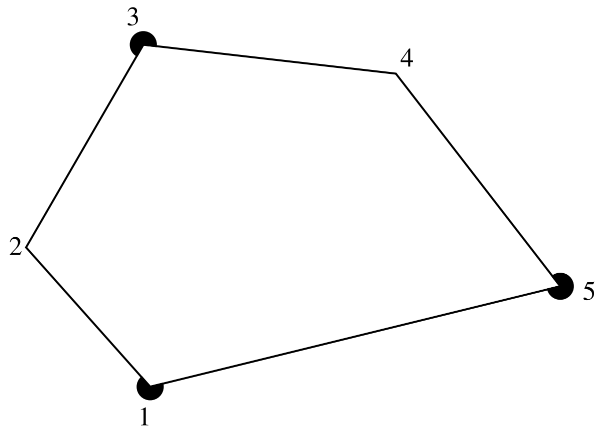

Fig. 14

This method of positioning cameras at odd-numbered corners can be extended to a polygonal building with any number of walls, showing that $\left\lfloor \frac{n+1}{2} \right\rfloor$ cameras are sufficient to observe all the outside of any n-sided polygonal building.

If the cameras did not need to be mounted on the walls, but could be positioned further away from the building, then fewer cameras would usually suffice.

Conclusion

Triangulation provides an elegant proof when analysing the minimum number of cameras needed in open-plan galleries. With more complex layouts in two and three dimensions, such elegant solutions have not been discovered although some necessary and some sufficient conditions have been found. In general, optimal solutions are found by applying computer algorithms to mathematical models of galleries.