

Mathematics

Advanced GCE

Unit **4726**: Further Pure Mathematics 2

Mark Scheme for January 2011

OCR (Oxford Cambridge and RSA) is a leading UK awarding body, providing a wide range of qualifications to meet the needs of pupils of all ages and abilities. OCR qualifications include AS/A Levels, Diplomas, GCSEs, OCR Nationals, Functional Skills, Key Skills, Entry Level qualifications, NVQs and vocational qualifications in areas such as IT, business, languages, teaching/training, administration and secretarial skills.

It is also responsible for developing new specifications to meet national requirements and the needs of students and teachers. OCR is a not-for-profit organisation; any surplus made is invested back into the establishment to help towards the development of qualifications and support which keep pace with the changing needs of today's society.

This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by Examiners. It does not indicate the details of the discussions which took place at an Examiners' meeting before marking commenced.

All Examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the Report on the Examination.

OCR will not enter into any discussion or correspondence in connection with this mark scheme.

© OCR 2011

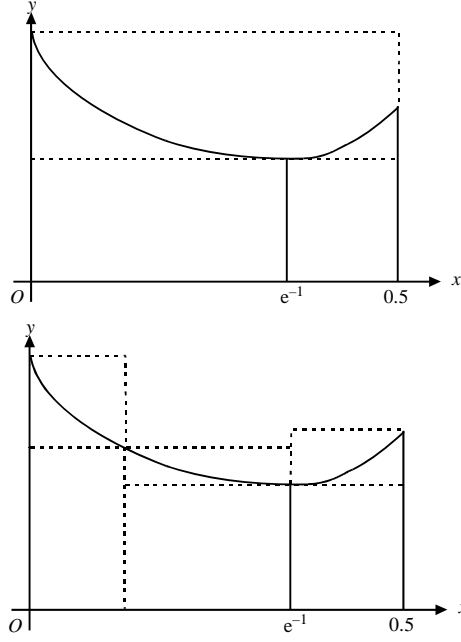
Any enquiries about publications should be addressed to:

OCR Publications
PO Box 5050
Annesley
NOTTINGHAM
NG15 0DL

Telephone: 0870 770 6622
Facsimile: 01223 552610
E-mail: publications@ocr.org.uk

1	$t = \tan \frac{1}{2}x \Rightarrow dt = \frac{1}{2} \sec^2 \frac{1}{2}x dx = \frac{1}{2}(1+t^2) dx$ $\int \frac{1}{1+\sin x + \cos x} dx = \int \frac{1}{1+\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt$ $= \int \frac{1}{1+t} dt = \ln 1+t (+c)$ $= \ln k \left 1 + \tan \frac{1}{2}x \right (+c)$	B1 M1 A1 M1 A1	For correct result AEF (may be implied) For substituting throughout for x For correct unsimplified t integral For integrating (even incorrectly) to $a \ln f(t) $. Allow $ $ or $()$ For correct x expression k may be numerical, c is not required
5			
2 (i)	$f(x) = \tanh^{-1} x, f'(x) = \frac{1}{1-x^2}, f''(x) = \frac{2x}{(1-x^2)^2}$ $f'''(x) =$ $\frac{2(1-x^2)^2 - 2x \cdot 2(1-x^2) \cdot -2x}{(1-x^2)^4} \text{ OR } \frac{2x \cdot 4x}{(1-x^2)^3} + \frac{2}{(1-x^2)^2}$ $= \frac{2(1-x^2)^2 + 8x^2(1-x^2)}{(1-x^2)^4} \text{ OR } \frac{8x^2}{(1-x^2)^3} + \frac{2(1-x^2)}{(1-x^2)^3}$ $= \frac{2(1+3x^2)}{(1-x^2)^3}$	M1 A1 M1 A1 A1	For quoting $f'(x) = \frac{1}{1 \pm x^2}$ and attempting to differentiate $f'(x)$ For $f''(x)$ correct WWW For using quotient <i>OR</i> product rule on $f''(x)$ For correct unsimplified $f'''(x)$ For simplified $f'''(x)$ WWW AG
(ii)	$f(0) = 0, f'(0) = 1, f''(0) = 0$ $f'''(0) = 2 \Rightarrow \tanh^{-1} x = x + \frac{1}{3}x^3$	B1√ M1 A1	For all values correct (may be implied) f.t. from (i) For evaluating $f'''(0)$ and using Maclaurin expansion For correct series
8			
3 (i)(a)	Asymptote $y = 0$	B1	1 For correct equation (allow x -axis)
(b)	METHOD 1 $y = \frac{5ax}{x^2 + a^2} \Rightarrow yx^2 - 5ax + a^2y = 0$ $b^2 \geq 4ac \Rightarrow 25a^2 \geq 4a^2y^2 \Rightarrow -\frac{5}{2} \leq y \leq \frac{5}{2}$	M1 M1 A1 A1	For expressing as a quadratic in x For using $b^2 - 4ac \leq 0$ For $25a^2 - 4a^2y^2$ seen or implied For correct range
	METHOD 2 $y = \frac{5ax}{x^2 + a^2} \Rightarrow \frac{dy}{dx} = \frac{-5a(x^2 - a^2)}{(x^2 + a^2)^2}$ $\frac{dy}{dx} = 0 \Rightarrow x = \pm a \Rightarrow y = \pm \frac{5}{2}$ Asymptote, sketch etc $\Rightarrow -\frac{5}{2} \leq y \leq \frac{5}{2}$	M1* A1 M1 A1 (*dep)	For differentiating y by quotient <i>OR</i> product rule For correct values of x For finding y values and giving argument for range For correct range
(ii)(a)	$y = 0$	B1	1 For correct equation (allow x -axis)
(b)	Maximum $\sqrt{\frac{5}{2}}$, minimum $-\sqrt{\frac{5}{2}}$	B1√ B1√	2 For correct maximum f.t. from (i)(b) For correct minimum f.t. from (i)(b) Allow decimals
(c)	$x \geq 0$	B1	1 For correct set of values (allow in words)
9			

4 (i)	$8\sinh^4 x \equiv \frac{8}{16}(e^x - e^{-x})^4$ $\equiv \frac{8}{16}(e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x})$ $\equiv \frac{1}{2}(e^{4x} + e^{-4x}) - \frac{4}{2}(e^{2x} + e^{-2x}) + \frac{6}{2}$ $\equiv \cosh 4x - 4\cosh 2x + 3$	B1	$\sinh x = \frac{1}{2}(e^x - e^{-x})$ seen or implied
		M1	For attempt to expand $(\dots)^4$
		M1	by binomial theorem <i>OR</i> otherwise
		A1	For grouping terms for $\cosh 4x$ <i>or</i> $\cosh 2x$
		A1	<i>OR</i> using e^{4x} <i>or</i> e^{2x} expressions from RHS
		4	For correct expression AG
	SR may be done wholly from RHS to LHS	M1 M1 B1 A1	Evidence of factorising required for 2nd M1
(ii)	METHOD 1 $\cosh 4x - 3\cosh 2x + 1 = 0$		
	$\Rightarrow (8\sinh^4 x + 4\cosh 2x - 3) - 3\cosh 2x + 1 = 0$	M1	For using (i) and $\cosh 2x \equiv \pm 1 \pm 2\sinh^2 x$
	$\Rightarrow 8\sinh^4 x + 2\sinh^2 x - 1 = 0$	A1	For correct equation
	$\Rightarrow (4\sinh^2 x - 1)(2\sinh^2 x + 1) = 0 \Rightarrow \sinh x = \pm \frac{1}{2}$	M1	For solving their quartic for $\sinh x$
		A1	For correct $\sinh x$ (ignore other roots)
	$\Rightarrow x = \ln\left(\pm \frac{1}{2} + \frac{1}{2}\sqrt{5}\right) = \pm \ln\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)$	A1✓	5 For correct answers (and no more) f.t. from their value(s) for $\sinh x$
	SR Similar scheme for $8\cosh^4 x - 14\cosh^2 x + 5 = 0 \Rightarrow \cosh x = \frac{1}{2}\sqrt{5} \Rightarrow x = \pm \ln\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)$		
	METHOD 2 $\cosh 4x - 3\cosh 2x + 1 = 0$		
	$\Rightarrow (2\cosh^2 2x - 1) - 3\cosh 2x + 1 = 0$	M1	For using $\cosh 4x \equiv \pm 2\cosh^2 2x \pm 1$
	$\Rightarrow 2\cosh^2 2x - 3\cosh 2x = 0$	A1	For correct equation
	$\Rightarrow \cosh 2x = \frac{3}{2} \Rightarrow x = \frac{1}{2}\ln\left(\frac{3}{2} \pm \frac{1}{2}\sqrt{5}\right)$	M1	For solving for $\cosh 2x$
		A1	For correct $\cosh 2x$ (ignore others)
	$= \pm \frac{1}{2}\ln\left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right)$	A1✓	For correct answers (and no more) f.t. from value(s) for $\cosh 2x$
	METHOD 3 Put all into exponentials	M1	For changing to $e^{\pm kx}$
	$\Rightarrow e^{4x} - 3e^{2x} + 2 - 3e^{-2x} + e^{-4x} = 0$	A1	For correct equation
	$\Rightarrow (e^{4x} + 1)(e^{4x} - 3e^{2x} + 1) = 0$	M1	For solving for e^{2x}
		A1	For correct e^{2x} (ignore others)
	$\Rightarrow e^{2x} = \frac{1}{2}(3 \pm \sqrt{5}) \Rightarrow x = \frac{1}{2}\ln\left(\frac{3}{2} \pm \frac{1}{2}\sqrt{5}\right)$	A1✓	For correct answers (and no more) f.t. from value(s) for e^{2x}
		9	
5 (i)	$x_{n+1} = x_n - \frac{x_n^3 - 5x_n + 3}{3x_n^2 - 5} = \frac{2x_n^3 - 3}{3x_n^2 - 5}$	M1	For attempt at N-R formula
		A1	For correct N-R expression
		A1	3 For correct answer (necessary details needed) AG
			Allow omission of suffixes
(ii)	$F'(x) = \frac{6x^2(3x^2 - 5) - 6x(2x^3 - 3)}{(3x^2 - 5)^2} = \frac{6x(x^3 - 5x + 3)}{(3x^2 - 5)^2}$	M1	For using quotient <i>OR</i> product rule to find $F'(x)$
		M1	For factorising numerator to show $k(x^3 - 5x + 3)$
	$F'(\alpha) = \frac{6\alpha(\alpha^3 - 5\alpha + 3)}{(3\alpha^2 - 5)^2} = 0 \text{ since } \alpha^3 - 5\alpha + 3 = 0$	A1	3 For correct explanation of AG
(iii)	$x_1 = 2 \Rightarrow 1.85714, 1.83479, 1.83424, 1.83424$ $(\alpha =) 1.8342$	B1	First iterate correct to at least 4 d.p. <i>OR</i> $\frac{13}{7}$
		B1	For 2 equal iterates to at least 4 d.p.
		B1	3 For correct α to 4 d.p.
	SR For starting value leading to another root allow up to B1 B1 B0		Allow answer rounding to 1.8342
			SR If not N-R, B0 B0 B0
		9	

<p>6 (i) $y = x^x \Rightarrow \ln y = x \ln x \Rightarrow \frac{1}{y} \frac{dy}{dx} = 1 + \ln x$</p> <p>$\frac{dy}{dx} = x^x (1 + \ln x) = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1}$</p>	<p>M1 For differentiating $\ln y$ OR $x \ln x$ w.r.t. x</p> <p>A1 For $(1 + \ln x)$ seen or implied</p> <p>A1 3 For correct x-value from fully correct working AG</p>
<p>(ii) $A > 0.2 \times 0.5^{0.5} + 0.2 \times 0.7^{0.7} + 0.1 \times 0.9^{0.9}$</p> <p>$\Rightarrow A > 0.3881(858) > 0.388$</p>	<p>M1 For areas of 3 lower rectangles</p> <p>A1 2 For lower bound rounding to AG</p>
<p>(iii) $A < 0.2 \times 0.7^{0.7} + 0.2 \times 0.9^{0.9} + 0.1 \times 1^1$</p> <p>$\Rightarrow A < 0.4377(177) < 0.438$</p>	<p>M1 For areas of 3 upper rectangles</p> <p>A1 2 For upper bound rounding to 0.438</p>
<p>(iv)</p> 	<p>M1 Consider rectangle of height $f(e^{-1})$</p> <p>A1 Use at least 1 lower rectangle, height $f(e^{-1})$</p> <p>B1 3 Use at least 1 upper rectangle, height $f(0)$</p> <p>SR If more than one rectangle is used for either bound, they must be shown correctly</p> <p style="text-align: center;">10</p>
<p>7 (i) $\cos 3\theta = \cos(-3\theta)$ OR $\cos \theta = \cos(-\theta)$ for all θ</p> <p>\Rightarrow equation is unchanged, so symmetrical about $\theta = 0$</p>	<p>M1 For a correct procedure for symmetry related to the equation OR to $\cos 3\theta$</p> <p>A1 2 For correct explanation relating to equation AG</p>
<p>(ii) $r = 0 \Rightarrow \cos 3\theta = -1$</p> <p>$\Rightarrow \theta = \pm \frac{1}{3}\pi, \pi$</p>	<p>M1 For obtaining equation for tangents</p> <p>A1 A1 for any 2 values</p> <p>A1 3 A1 for all, no extras (ignore outside range)</p>
<p>(iii) $\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \frac{1}{2}(1 + \cos 3\theta)^2 (d\theta)$</p> <p>$= \frac{1}{2} \int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} 1 + 2\cos 3\theta + \cos^2 3\theta d\theta$</p> <p>$= \frac{1}{2} \int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} 1 + 2\cos 3\theta + \frac{1}{2}(1 + \cos 6\theta) d\theta$</p> <p>$= \frac{1}{2} \left[\theta + \frac{2}{3}\sin 3\theta + \left(\frac{1}{2}\theta + \frac{1}{12}\sin 6\theta \right) \right]_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi}$</p> <p>$= \frac{1}{2}\pi$</p>	<p>B1 For correct integral with limits soi (limits may be $\left[0, \frac{1}{3}\pi\right]$ at any stage)</p> <p>M1* For multiplying out, giving at least 2 terms</p> <p>M1 For integration to $A\theta + B\sin 3\theta + C\sin 6\theta$ AEF</p> <p>M1 For completing integration and substituting</p> <p>(*dep) their limits into terms in $\frac{\cos}{\sin} n\theta$</p> <p>A1 5 For correct area WWW</p> <p style="text-align: center;">10</p>

8 (i)	METHOD 1	M1	For appropriate use of $\sinh^2 \theta = \cosh^2 \theta - 1$
	$\sinh(\cosh^{-1} 2) =$		
	$\sinh \beta = \sqrt{\cosh^2 \beta - 1} = \sqrt{2^2 - 1} = \sqrt{3}$	A1	2 For correct verification to AG
	METHOD 2	M1	For attempted use of ln forms of $\sinh^{-1} x$ and $\cosh^{-1} x$
	$\sinh^{-1} \sqrt{3} = \ln(\sqrt{3} + 2), \cosh^{-1} 2 = \ln(2 + \sqrt{3})$		
	$\Rightarrow \sinh(\cosh^{-1} 2) = \sqrt{3}$	A1	For both ln expressions seen
	METHOD 3		
	$\cosh^{-1} 2 = \ln(2 + \sqrt{3})$	M1	For use of ln form of $\cosh^{-1} x$ and definition of $\sinh x$
	$\sinh(\cosh^{-1} 2) = \frac{1}{2} \left(e^{\ln(2+\sqrt{3})} - e^{-\ln(2+\sqrt{3})} \right)$	A1	For correct verification to AG
	$= \frac{1}{2} (2 + \sqrt{3} - (2 - \sqrt{3})) = \sqrt{3}$		SR Other similar methods may be used Note that $\ln(2 + \sqrt{3}) = -\ln(2 - \sqrt{3})$
(ii)	$I_n = \int_0^\beta \cosh^n x \, dx$	M1*	For attempt to integrate $\cosh x \cdot \cosh^{n-1} x$ by parts
	$= \left[\sinh x \cdot \cosh^{n-1} x \right]_0^\beta - \int_0^\beta \sinh^2 x \cdot (n-1) \cosh^{n-2} x \, dx$	A1	For correct first stage of integration (ignore limits)
	$= \sinh \beta \cdot \cosh^{n-1} \beta - (n-1) \int_0^\beta (\cosh^2 x - 1) \cosh^{n-2} x \, dx$	M1 (*dep)	For using $\sinh^2 x = \cosh^2 x - 1$
	$= 2^{n-1} \sqrt{3} - (n-1)(I_n - I_{n-2})$	A1	For $(n-1)(I_n - I_{n-2})$ correct
	$\Rightarrow n I_n = 2^{n-1} \sqrt{3} + (n-1) I_{n-2}$	B1	For $2^{n-1} \sqrt{3}$ correct at any stage
		A1	6 For correct result AG
(iii)	$I_1 = \int_0^\beta \cosh x \, dx = \sinh \beta = \sqrt{3}$	B1	For correct value
	$I_3 = \frac{1}{3} (2^2 \sqrt{3} + 2\sqrt{3}) = 2\sqrt{3}$	M1	For using (ii) with $n = 3$ OR $n = 5$
		A1	For $I_3 = \frac{1}{3} (2^2 \sqrt{3} + 2I_1)$
			OR $I_5 = \frac{1}{5} (2^4 \sqrt{3} + 4I_3)$
	$I_5 = \frac{1}{5} (2^4 \sqrt{3} + 8\sqrt{3}) = \frac{24}{5} \sqrt{3}$	A1	4 For correct value

OCR (Oxford Cambridge and RSA Examinations)
1 Hills Road
Cambridge
CB1 2EU

OCR Customer Contact Centre

14 – 19 Qualifications (General)

Telephone: 01223 553998

Facsimile: 01223 552627

Email: general.qualifications@ocr.org.uk

www.ocr.org.uk

For staff training purposes and as part of our quality assurance programme your call may be recorded or monitored

Oxford Cambridge and RSA Examinations
is a Company Limited by Guarantee
Registered in England
Registered Office; 1 Hills Road, Cambridge, CB1 2EU
Registered Company Number: 3484466
OCR is an exempt Charity

OCR (Oxford Cambridge and RSA Examinations)
Head office
Telephone: 01223 552552
Facsimile: 01223 552553

