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Friday 18 January 2013 – Afternoon

A2 GCE MATHEMATICS (MEI)

4754/01B Applications of Advanced Mathematics (C4) Paper B: Comprehension

INSERT

Duration: Up to 1 hour

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Taxicab geometry

Introduction

Fig. 1 shows part of the road map of an imaginary town called Newtown.
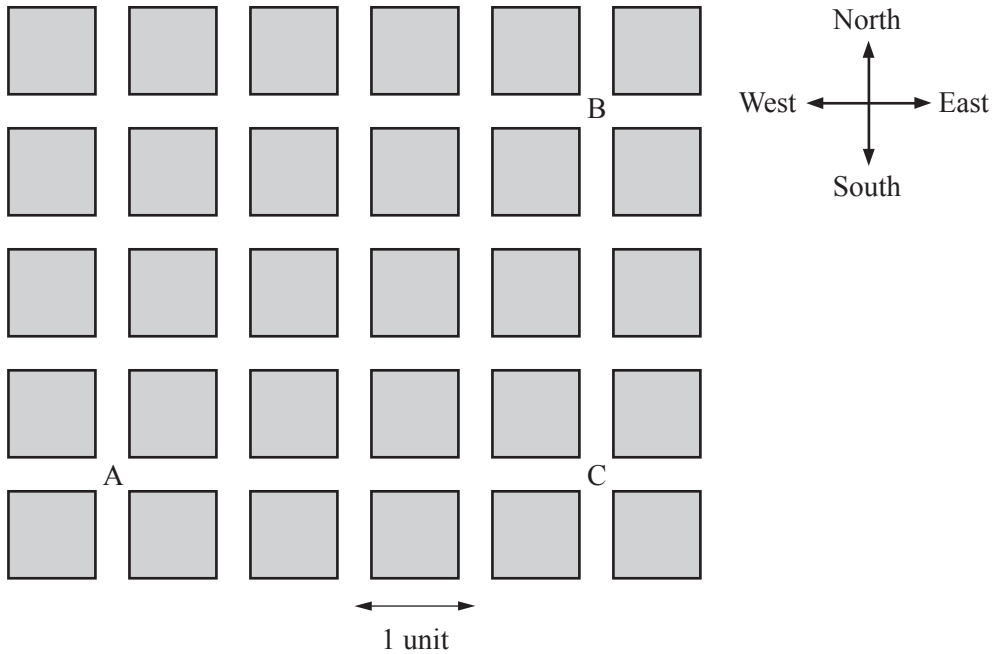

Fig. 1

Newtown's buildings are grouped in equal-sized square blocks. The roads between the blocks run in north-south and east-west directions and traffic can travel along every road in both directions.   5

Imagine you want to take a taxi from point A to point B. If the taxi travelled east from A to C and then north from C to B, the total distance travelled would be 7 units. Many other routes from A to B are also 7 units in length but no route is shorter. This shortest distance is called the *taxicab distance* from A to B and the related mathematics is called *taxicab geometry*.

This article introduces some of the mathematics of taxicab geometry.   10

Introducing the notation

Fig. 2 shows part of the road map of Newtown and one particular bus route with bus stops at positions M and N. Imagine you are at position L and you wish to catch a bus at one of these bus stops. Which is closer?
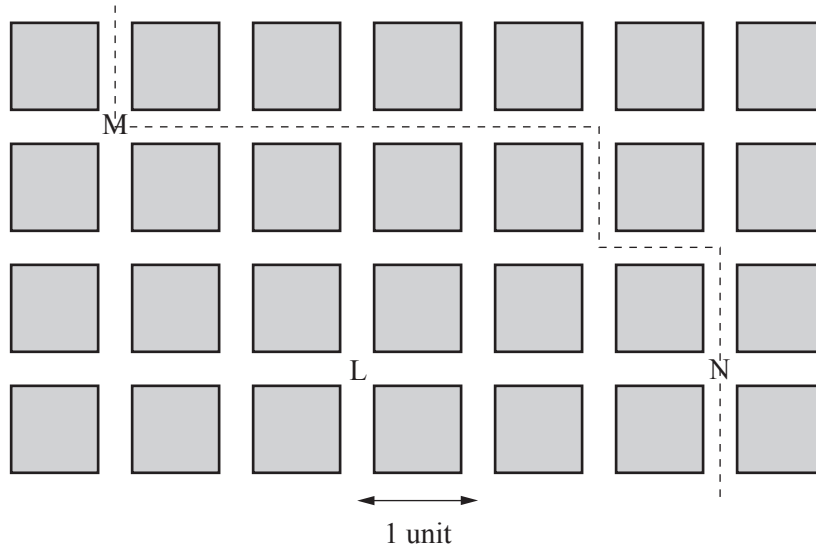

Fig. 2

By Pythagoras's Theorem, the straight line distance, measured in units as shown in Fig. 2, from L to M is $2\sqrt{2}$. This is expressed using the notation $d(L,M) = 2\sqrt{2}$. Similarly, $d(L,N) = 3$. In terms of straight line distances, M is closer than N since $d(L,M) < d(L,N)$. [15]

For a pedestrian, who is constrained to walking along roads, it is the taxicab distance rather than the Pythagorean distance that is important. The taxicab distance from L to M is 4. This is expressed using the notation $t(L,M) = 4$. Similarly $t(L,N) = 3$. For a pedestrian at L, since $t(L,N) < t(L,M)$, N is closer than M. [20]

This is an example of a situation in which $d(L,M) < d(L,N)$ but $t(L,M) > t(L,N)$.

In general, if the Pythagorean distance between two points A and B is $d(A,B)$ then the taxicab distance satisfies the inequalities $d(A,B) \leqslant t(A,B) \leqslant \sqrt{2} \times d(A,B)$.

Minimum distance routes

In Fig. 1 the Pythagorean distance between the points A and B is 5. There is only one straight line segment from A to B; its length is 5.

However, this uniqueness property does not hold when considering the taxicab distance. In Fig. 1, the taxicab distance from A to B is 7. There are several routes from A to B which have this minimum distance; these are called *minimum distance routes*.

How many minimum distance routes are there from A to B?

In order to answer this question, the road grid is replaced by a coordinate system as shown in Fig. 3. The *x*-axis represents the west-east direction and the *y*-axis represents the south-north direction. Point A has coordinates $(0,0)$ and point B has coordinates $(4,3)$. The roads are shown by the grid lines.
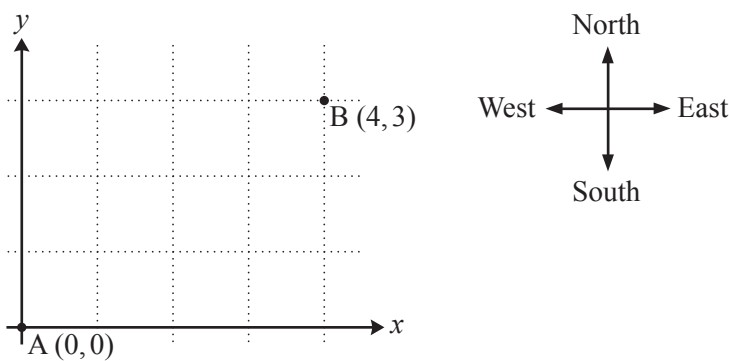

Fig. 3

Clearly, no minimum distance route from A to any point in the first quadrant will involve any motion in a westerly or southerly direction.

There is only one minimum distance route from A to any point on the *x*-axis or to any point on the *y*-axis.

There are two ways of reaching the point with coordinates $(1,1)$ along minimum distance routes as follows.

$$(0,0) \rightarrow (1,0) \rightarrow (1,1)$$

$$(0,0) \rightarrow (0,1) \rightarrow (1,1)$$

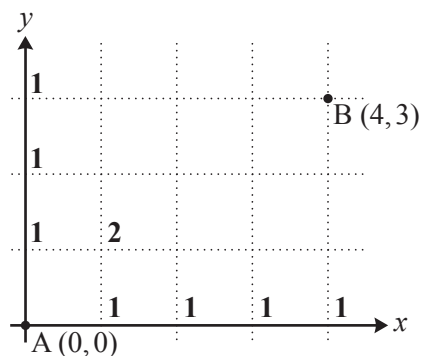The numbers of minimum distance routes from A to the points mentioned above are shown in Fig. 4.

Fig. 4

The final step on a minimum distance route from A to the point $(2,1)$ must be either from $(2,0)$ to $(2,1)$ or from $(1,1)$ to $(2,1)$. There is 1 minimum distance route from A to $(2,0)$ and there are 2 minimum distance routes from A to $(1,1)$. Each of these routes can be continued to $(2,1)$ in only one way. Since all of these routes are different, the number of minimum distance routes from A to $(2,1)$ is 3.

45

This reasoning can be extended to other grid points. The notation $n(p,q)$ is used to denote the number of minimum distance routes from $(0,0)$ to (p,q), where p and q are non-negative integers. The following rules apply for $p \geqslant 1, q \geqslant 1$.

$$n(p,0) = 1$$
$$n(0,q) = 1$$
$$n(p,q) = n(p-1,q) + n(p,q-1)$$

50

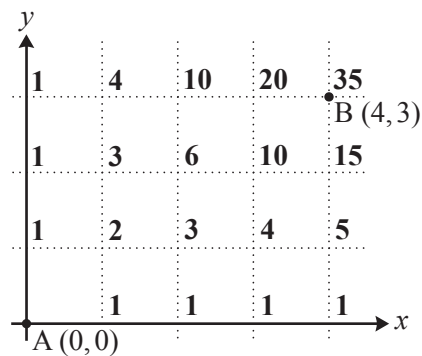These rules give the numbers of minimum distance routes shown in Fig. 5.

Fig. 5

So the answer to the question of how many minimum distance routes there are from $A(0,0)$ to $B(4,3)$ is 35.

55

Generalised taxicab geometry

The mathematical model of taxicab geometry described so far has been motivated by a system of roads and junctions. In this system there is a finite number of uniformly spaced parallel and perpendicular roads and all journeys start and end at junctions.

The mathematical ideas can be generalised by defining the taxicab distance for any two points in the x-y plane. In this generalised version, the points are not necessarily grid points.

60

Fig. 6 shows two points, $R(x_1, y_1)$ and $S(x_2, y_2)$, in the x-y plane.
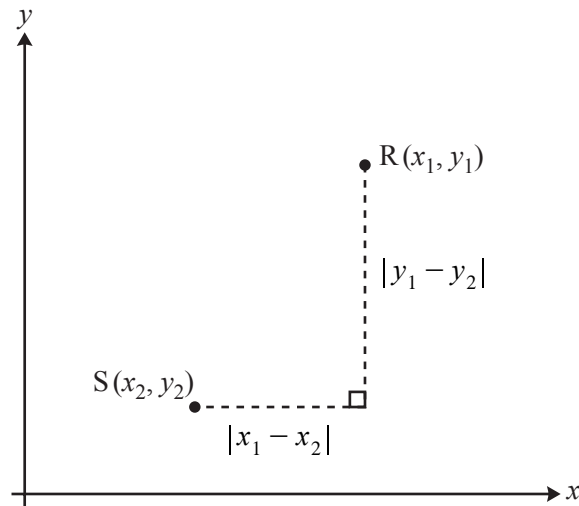
Fig. 6

The taxicab distance, $t(R, S)$, is defined as $|x_1 - x_2| + |y_1 - y_2|$. Thus the taxicab distance is still defined as the sum of the distances between the points in the x- and y-directions.

For example, the taxicab distance between the points with coordinates $(2.1, 1)$ and $(3.9, 4.3)$ is

65

$$|2.1 - 3.9| + |1 - 4.3|$$
$$= |-1.8| + |-3.3|$$
$$= 1.8 + 3.3$$
$$= 5.1.$$

Similarly the taxicab distance between $(-1.1, 1.4)$ and $(3.2, -0.8)$ is

70

$$|-1.1 - 3.2| + |1.4 - (-0.8)| = |-4.3| + |2.2| = 4.3 + 2.2 = 6.5.$$

This definition of distance produces some surprising geometric results, as will be seen below.

Fig. 7 shows a fixed point $C(2,3)$ and the locus of the point P satisfying $t(P,C) = 5$. The coordinates of every point $P(x,y)$ on this locus satisfy the equation $|x - 2| + |y - 3| = 5$.
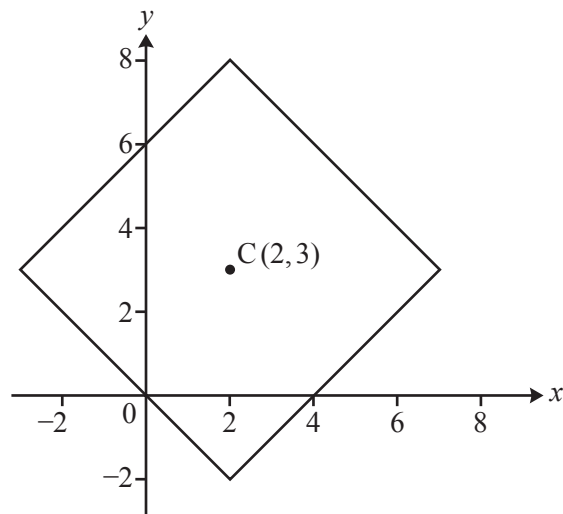

Fig. 7

Since all points are at a fixed taxicab distance from C, this is a taxicab 'circle' in this geometry! The circle has a taxicab 'radius' of 5.

Furthermore, adding a second taxicab circle with centre $(2,0)$ and radius 2 shows that in generalised taxicab geometry two different circles can have an infinite number of points in common!

Now consider the locus of a point Q which is 'equidistant' from two fixed points $A(0,0)$ and $B(8,6)$.

Fig. 8.1 shows the set of points Q satisfying $d(Q,A) = d(Q,B)$; this is the familiar perpendicular bisector of the line segment AB.

Fig. 8.2 shows the set of points Q satisfying $t(Q,A) = t(Q,B)$; so in generalised taxicab geometry the locus is quite different.
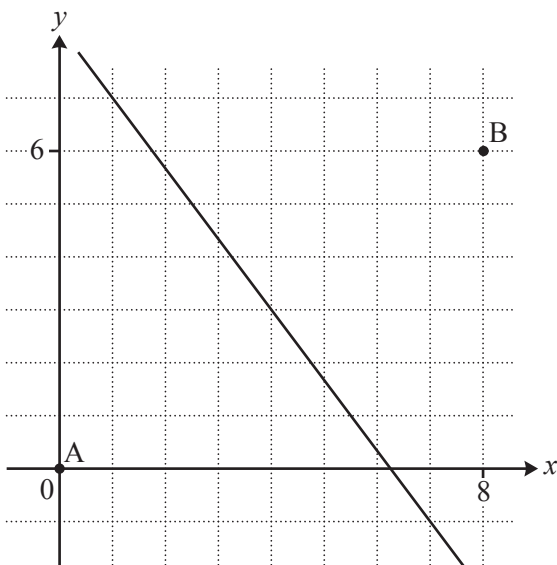
Fig. 8.1
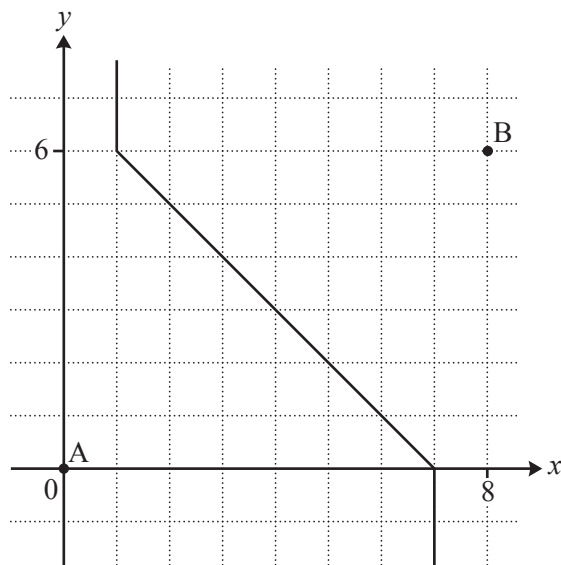

Fig. 8.2

Conclusion

In the natural world it is often appropriate to apply Pythagoras's Theorem to calculate the distance between two points. However, in urban geography, where there are obstacles such as buildings to be considered, taxicab geometry is often a more useful mathematical model.

In this article several simplifying assumptions have been made. For example, the imaginary town is laid out in a square grid, all roads are traversable in both directions and that the rate of progress along every route is uniform. Although these clearly do not exactly match any real cities, Fig. 9 below, a map of Manhattan in New York, suggests that, for some cities, some form of taxicab geometry can provide a good mathematical model.

Fig. 9

Line numbers: 85, 90