OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4754(B)

Applications of Advanced Mathematics (C4)

Section B: Comprehension

INSERT

Thursday **16 JUNE 2005** Afternoon Up to 1 hour

INSTRUCTIONS TO CANDIDATES

- This insert contains the text for use with the questions.

This insert consists of 8 printed pages.

Communicating with other civilisations

Background

From time immemorial, people have looked into the night sky and wondered whether there are other civilisations out there. During the last hundred years, it has become a theoretical possibility that, if such civilisations do exist, we could communicate with them.

This article looks at some ideas as to how such communication might get started.

First contact

Imagine then that we have reason to think that a planet orbiting a star some light years away might be home to intelligent life. We have no idea what form that life might take, let alone how their society might work.

Natural curiosity means that we would try to make contact with them by sending some sort of message. There would be two requirements.

(i) The message would have to be such that it could be understood anywhere, and so free of any human or Earth-related influence.

(ii) We would want to know whether the message had been received and understood, so it would have to invite a reply.

Many people believe that mathematics is the only universal culture-free language on Earth. So it would be appropriate to use it as a basis for first contact.

One suggestion is that we should transmit the first 5 (say) digits of π, 3.1415. This could be done using short pulses, with a longer one for the decimal point, as illustrated in Fig. 1.

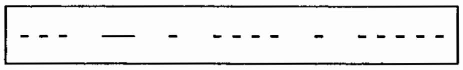

Fig. 1

We would then await a reply consisting of the next 5 digits, 92653.

Will they understand π?

The number we call π arises because it is the ratio of the circumference to the diameter of a circle. It also occurs in many other aspects of mathematics.

An important point is that π requires no units. It is one length divided by another so that, provided the lengths are measured in the same units, the units cancel out. You get the same answer whether you measure the circumference and diameter in metres, feet, miles or anything else. Thus π is a pure number; it is dimensionless.

However, the digits we associate with π are dependent on our number system which uses base 10. If we used base 8, the first 6 digits of π would be 3.11037 since

$$\pi \approx 3 + \frac{1}{8} + \frac{1}{8^2} + \frac{0}{8^3} + \frac{3}{8^4} + \frac{7}{8^5},$$

compared with the base 10 equivalent of

$$\pi \approx 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5}.$$

The reason we use base 10 is probably because we have a total of 10 fingers and thumbs. So another civilisation might well use a different number base. However, it is perhaps reasonable to assume that, if they are intelligent enough to communicate with us, they also have the sense to try out different number bases.

By communicating in base 10, we are telling the other civilisation that the number 10 is for some reason important in our culture.

A reply comes back

We would not, of course, send π just once. The other civilisation would almost certainly miss it! So we would keep on sending it until either we received a reply or we decided to give up. The nearest star is 4.3 light years away, so the soonest we could possibly get a reply would be 8.6 years. If, however, the star of interest was 50 light years away (not far in astronomical terms), the conversation would be even slower, once every 100 years.

The reply would almost certainly consist of two parts: the answer to our question (92653) and a question of their own. This might perhaps be the signal given in Fig. 2, representing the first 5 digits of e, the base of natural logarithms.

```
- -   —   - - - - - - -   -   - - - - - - - -   - -
```

Fig. 2

Receiving this message would be one of the most exciting events in human history. Undoubtedly it would set off a lively debate about what to send next. It is reasonable to conjecture that three groups would be particularly interested.

- Military personnel would want to assess what the outcome would be if we were to find ourselves at war with the other civilisation.
- Others would want to find out about their culture.
- Scientists would hope to learn from them.

Military questions

The military would find themselves in considerable difficulty. They would want to assess the weapons capability of the other civilisation, and so would need to devise a sequence of different numbers (or questions) each of which related to a different stage of weapons development.

However, sending out such a sequence of numbers would be fraught with danger. The other civilisation might well work out the precise purpose of the questions and so deduce our own

[Turn over

level of military sophistication without giving away any information from their side. It could well be seen as sending out a hostile message with the possibility that the other civilisation would then break off communication.

So it is quite possible that direct military involvement would be seen as just too risky.

Cultural questions

The golden ratio

A number that would certainly be considered is the *golden ratio*. The sides of the rectangle in Fig. 3 are in the ratio 1.618... to 1. This is called the golden ratio and the number 1.618... is denoted by ϕ (the Greek letter phi). Such a rectangle is called a *golden rectangle*.

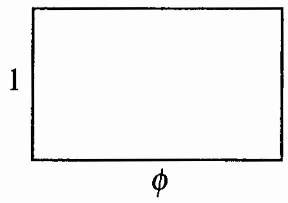

Fig. 3

The golden ratio has considerable cultural significance. For thousands of years a golden rectangle, like the one in Fig. 3, has been regarded as more pleasing to the eye than any other rectangle. So ϕ is related to our artistic sense.

To derive the number ϕ, look at the rectangle ABCD in Fig. 4. It is divided into two parts: a square EBCF at one end, and a smaller rectangle AEFD.

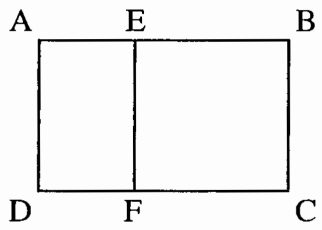

Fig. 4

If the ratio of the sides of the small rectangle AEFD is the same as that of ABCD, then ABCD is a golden rectangle. In this case,

$$\frac{AB}{AD} = \frac{AD}{AE} = \phi.$$

Thus, if the smaller side, AD, of the main rectangle is given a value of 1 unit, the longer side, AB, is ϕ units.

Since EBCF is a square, the length AE is $(\phi - 1)$ units, and so

$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

$$\Rightarrow \quad \phi = \frac{1 \pm \sqrt{5}}{2}.$$

Since ϕ must be positive, it follows that the golden ratio is $\dfrac{1+\sqrt{5}}{2}$ or 1.61803… .

Notice that if the longer side of a golden rectangle is assigned a length of 1 unit, the length of the shorter side is $\dfrac{1}{\phi} = \dfrac{\sqrt{5}-1}{2}$ units.

The number $\dfrac{1+\sqrt{5}}{2}$ crops up in many other places in mathematics, including the Fibonacci sequence. This is usually written

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ \ldots\ .$$

The first two terms are both given the value 1. After that, each term is the sum of the previous two terms and so the sequence can be defined iteratively by

$$a_{n+1} = a_n + a_{n-1} \quad \text{with } a_1 = 1 \text{ and } a_2 = 1.$$

The ratios of one term to the previous term form the sequence

$$\frac{1}{1} = 1,\ \frac{2}{1} = 2,\ \frac{3}{2} = 1.5,\ \frac{5}{3} = 1.666\ldots,\ \frac{8}{5} = 1.6,$$
$$\frac{13}{8} = 1.625,\ \frac{21}{13} = 1.615\ldots,\ \frac{34}{21} = 1.619\ldots,\ \ldots\ .$$

This sequence converges and it looks as though its limit has the same value as ϕ.

To prove that it does, imagine that you have taken so many terms that the ratio has settled down to a value, r.

Thus $$r = \frac{a_{n+1}}{a_n} \quad \text{and} \quad r = \frac{a_n}{a_{n-1}}.$$

Take the equation $a_{n+1} = a_n + a_{n-1}$ and divide through by a_n.

$$\frac{a_{n+1}}{a_n} = 1 + \frac{a_{n-1}}{a_n}$$

and so $$r = 1 + \frac{1}{r}$$
$$\Rightarrow \quad r^2 - r - 1 = 0$$
$$r = \frac{1+\sqrt{5}}{2} \quad \left(\text{or } \frac{1-\sqrt{5}}{2}\right).$$

The Fibonacci sequence occurs in nature, for example in connection with the numbers of petals on several types of flowers.

So the message conveyed to another civilisation by the number ϕ would not be unique. They might think we were telling them about our artistic sense, or about the flowers that grow on our planet or about something else.

Feigenbaum's number

A particularly interesting number to send is Feigenbaum's number. This was discovered in 1975 as a result of work on the (then) new subject of chaos.

A simple model for population growth is given by the *logistic equation*,

$$x_{n+1} = kx_n(1 - x_n),$$

where x_n is the population, on a scale of 0 to 1, at a certain time and x_{n+1} is the population one unit of time later. The starting point, x_0, is a number between 0 and 1. The number k represents the reproductivity of the species in question; it is a *parameter* of the model.

This is an iterative process and the outcome depends on the value of k. For $0 \leqslant k \leqslant 1$, the values of x get progressively smaller and converge to zero. The population dies out, whatever the starting value. Table 5(a) illustrates this in the case $k = 0.3$ with starting value $x_0 = 0.5$.

For values of k between 1 and 3, the values of x converge to a particular value which depends on the value of k but not on the starting point. Thus when $k = 2.2$, x converges to 0.545 454 … , as shown in Table 5(b). The population assumes a stable level.

$k = 0.3$	
n	x_n
0	0.5
1	0.075
2	0.020 812 …
3	0.006 113 …
4	0.001 822 …
5	0.000 545 …
6	0.000 163 …
7	0.000 049 …
8	0.000 014 …
9	0.000 004 …
10	0.000 001 …
11	0.000 000 …

Table 5(a)

$k = 2.2$	
n	x_n
0	0.5
1	0.55
2	0.544 5
3	0.545 643 …
4	0.545 416 …
5	0.545 462 …
6	0.545 453 …
7	0.545 454 …
8	0.545 454 …
9	0.545 454 …
10	0.545 454 …
11	0.545 454 …

Table 5(b)

For values of k a little bigger than 3, the value of x ends up oscillating between two outcomes. Thus when $k = 3.2$, the value of x ends up oscillating between 0.513… and 0.799… . The population goes up and down regularly.

For slightly larger values of k, the value of x ends up oscillating between 4 or 8 outcomes.

Fig. 6 shows the outcomes for values of k between 1 and 3.55.

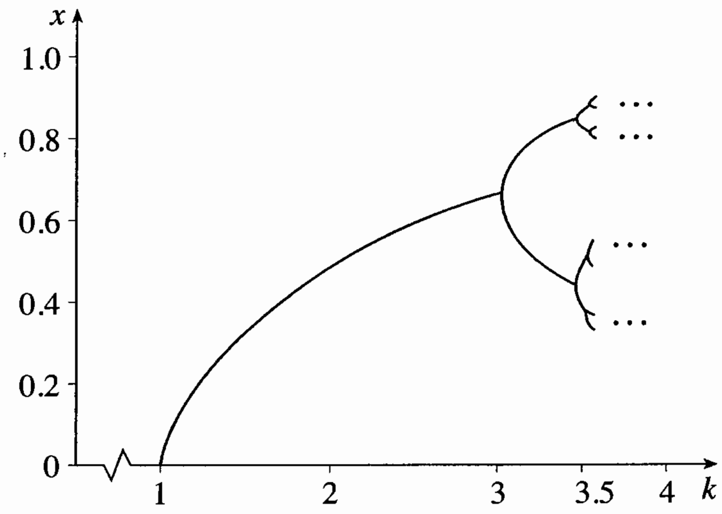

Fig. 6

Larger values of k than those illustrated in Fig. 6 produce even more outcomes, 16, 32 etc. The number of outcomes is always a power of 2.

For still larger values of k there is no pattern at all. The system goes into chaos.

When $k = 3$, there is a change of regime from one non-zero outcome to two. This is called a *point of bifurcation*.

The next point of bifurcation occurs when the number of outcomes changes from 2 to 4; it occurs at $k = 3.4485$ (to 4 decimal places). The one after that is at $k = 3.5437$ when the change is from 4 to 8, and so on.

Number of non-zero outcomes	**1**	**2**	**4**
Values of k	1 to 3	3 to 3.4485	3.4485 to 3.5437
Length of present interval	2	0.4485	0.0952
$\dfrac{\textbf{Previous interval}}{\textbf{Present interval}}$	–	$\dfrac{2}{0.4485} = 4.459\ldots$	$\dfrac{0.4485}{0.0952} = 4.711\ldots$

Table 7

Table 7 shows the intervals, in terms of the parameter k, for the first 3 regimes (excluding $0 \leqslant k \leqslant 1$ for which the population dies out). The final row gives the ratios of lengths of consecutive intervals.

What Feigenbaum discovered is that the sequence formed by the values of this ratio converges to a particular number, 4.669 201… . (Australian mathematicians have now computed it to 1000 significant figures.)

What is remarkable about Feigenbaum's number is that it arises not just in the iterative equation described above, but in a wide variety of equations modelling real-life situations which change from order into chaos. It is a universal constant.

For example, an entirely different iterative equation which generates Feigenbaum's number is

$$x_{n+1} = k\sin(\pi x_n).$$

The fact that 30 years ago we did not know about Feigenbaum's number, but do now, illustrates the point that it is a marker in our technological and cultural development. Its discovery depended on having the power of electronic calculation. The number and length of the calculations involved in its discovery were enormous.

So sending the first few digits of Feigenbaum's number to another civilisation could be taken as asking the question "Have you developed computers yet?" However, a positive response could also mean that those in the other civilisation have wonderfully good brains of their own, so good that they do not need computers.

Scientific questions

It would be a matter of great interest to scientists to find out whether quantities we believe to be constant really are: for example the velocity of light, $2.997\,924 \times 10^5$ kilometres per second. Even a small difference in this would require a fundamental re-think of the laws of physics.

Unfortunately this value uses units that are derived from conditions on Earth. The kilometre is approximately $\frac{1}{40\,000}$ of the circumference of the Earth, and the second is approximately $\frac{1}{60 \times 60 \times 24}$ of the time it takes the Earth to spin once on its axis (i.e. one day). Although both of these units are now defined more precisely using basic properties of matter, they retain essentially the same values and so would be meaningless to another civilisation.

It may be that the best that scientists could achieve would be certain ratios. For example

$$\frac{\text{Rest mass of proton}}{\text{Rest mass of electron}} = 1836.108.$$

The numbers scientists would send would almost certainly have been determined experimentally, to a known level of accuracy. If the reply came back with the same number but given to a much higher level of accuracy, it would be a good indication that the other civilisation is more technologically advanced than we are.

Thus it may be that the answers to science-based questions would tell us more about the civilisation's level of development than about science.

Conclusion

There are many other numbers that could be used in this context. No doubt a selection committee would be needed! The numbers mentioned in this article illustrate some of the principles which might guide the work of such a committee.